EXTENSION PROCEDURES FOR LATTICE LIPSCHITZ MAPPINGS

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Abstract. We present McShane and Whitney extension formulas for a new class of Lipschitz operators on Banach lattices, that we call lattice Lipschitz maps. Using them, and based on the (approximate) determination of the set of eigenvectors of the map, we find the (approximate) diagonal representation of these maps. Our work on such extension/representation formulas is intended to follow current research on the design of machine learning algorithms based on the extension of Lipschitz functions.

1. Introduction

The diagonal representation of symmetric linear operators is a classical and powerful tool in the mathematical treatment of many problems in all fields of scientific and technical activity. Following this general idea, it has been a common interest of many researchers in mathematics to extend the ideas that allow to obtain such diagonal representations to other classes of maps, such as bilinear operators (see for example [4, 5, 6, 12] and references therein), and Lipschitz operators ([1, 7, 10]). We are interested in developing new techniques to extend Lipschitz operators on finite-dimensional normed spaces from subsets of eigenvectors to the whole space to obtain explicit representation formulas for them, based on the previous computation of subsets of eigenvectors of the operator.

In the next sections, we introduce a new class of Lipschitz-type operators on Banach lattices which we call lattice Lipschitz maps. We also provide a complementary methodology for the extension of such maps from a set of vectors in the space that approximately satisfy eigenvector equations. Of course, the reason is that the set of eigenvectors is known to be a convenient set to start extending a linear operator in a Euclidean space. In this case the computations to obtain the pointwise evaluation of the map are just additions and multiplications by scalars, once the values of the operator in this set are known. However, for more general maps this rule would not work, even when these maps have a reasonably good linear approximation.

This opens the door to the project of defining more general extension rules while preserving that, essentially, the operator under consideration can be "reconstructed" simply by knowing its "fixed points" (except for a multiplicative constant). Such points are easily described by a geometric argument allowed by

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Euclidean geometry: an (almost) eigenvector of an operator T is an element x of the space for which the projection of $T(x)$ onto the subspace generated by x (almost) coincides with its multiplication by an scalar. Consequently, a useful extension rule is one that derives some benefit from this easy description, in terms of facilitating its mathematical representation. This is the motivation of this second part of the paper, and the reason why we propose the procedures explained here. We thus continue the investigation we started in [6, 7]. Specifically, in [7] several of these extension procedures are presented, and illustrative examples from physics and other research fields are given.

Once the set of approximate eigenvectors is fixed, we use it to provide a lattice structure to the Euclidean space from which we start. Using the related order we can use the extension formulas for the class of lattice Lipschitz operators that we have previously studied. The final result gives two extension formulas that can be used to represent approximately any Lipschitz operator satisfying a certain (pointwise) lattice order requirement. Our construction relies on the fundamental fact that, for the case of "diagonalizable" operators on Euclidean spaces, our two extensions (McShane-type and Whitney-type formulas) coincide with the original map when applied to the eigenvector set of the operator. It is proved in Theorem 2.14 (see also Corollary 2.15 for the linear diagonalizable operators).

Our results are motivated by the potential use of Lipschitz extension formulas in Machine Learning algorithms. It has been shown that Lipschitz extensions can be used to predict value functions in Reinforcement Learning, for example [14, 15]. Although the use of the theory of Lipschitz functions goes back to the origin of Artificial Intelligence, recently some new results have pointed out the power of this mathematical setting to provide competitive algorithms in different fields of Machine Learning (for example, [16, 17, 18, 22]). Hence our interest in showing a step-by-step methodology for the application of our results, together with some examples, which will be done in the last section of the paper.

Through the paper we will use standard concepts and notation. Let $\mathbb{R}^+$ be the set of non-negative real numbers and let D be a set. A metric on D is a function $d : D \times D \to \mathbb{R}^+$ such that for $a, b, c \in D$,

(1) $d(a, b) = 0$ if and only if $a = b$, and
(2) $d(a, b) \leq d(a, c) + d(c, b)$.

Let (M, ρ) be another metric space. A function $T : D \to M$ that satisfies that there exists a constant $K > 0$ such that

$$\rho\big(T(a), T(b)\big) \leq K d(a, b), \quad a, b \in D,$$

is a Lipschitz function. The Lipschitz constant of T is the infimum of all constants K which satisfy this inequality.

A real-valued Lipschitz function $T : B \to \mathbb{R}$ (where $\mathbb{R}$ is endowed with the Euclidean norm), defined on an arbitrary subset B of D can be extended to the whole space D preserving the Lipschitz constant K, see [11, 13]. This classical result is called the McShane-Whitney Theorem (from 1934): if B is a subspace of a metric space (D, d) and $T : B \to \mathbb{R}$ is a Lipschitz function with Lipschitz constant K, it is always possible to obtain an extension of T to D preserving the Lipschitz constant. In other words, there is the Lipschitz function $\widetilde{F} : M \to \mathbb{R}$ that extend T —that is, $\tilde{T}(a) = T(a)$ for all $a \in B$—, with Lipschitz constant K. This extension can be explicitly computed. There are two classical formulas that provide $\tilde{T}$, that are

$$T^M(b) := \sup_{a\in B}\{T(a) - K\, d(b,a)\}, \quad b \in D,$$

(we call it the McShane extension of T), and

$$T^W(b) := \inf_{a\in B}\{T(a) + K\, d(b,a)\}, \quad b \in D.$$

(that we call the Whitney extension of T). A comprehensive study of lipschtz functions and its extensions can be seen on the book by Cobzas, Miculescu and Nicolae [3].

In this paper we will modify these definitions for the pointwise evaluation of functions $T : E \to E$, where E is a function space; related extension formulas will be obtained in this case.

Let us recall the definition of Banach lattice. Essentially, this is a vector lattice —with an order relation $\le$— and a complete norm $\|\cdot\|$ that satisfies that $\|x\| \le \|y\|$ when $|x| \le |y|$, $x, y \in E$, where $|\cdot|$ is the modulus of the functions, that are computed using the order (and coincided with the modulus of the funcions when the lattice can be represented as a function space). We will use it when the set of elements in which the functions act are finite, so we will consider finite dimensional Banach lattices. Once a basis is fixed, we can associate to it the order in the space, that coincides with the coordinates order (that is, for two vectors of the space $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$, $x \le y$ if and only if $x_i \le y_i$ for $i = 1, ..., n$.

2. Lattice Lipschitz operators and lattice extensions for Lipschitz maps

In this section we define our main extension formula for a particular class of Lipschitz maps. In the case we consider, the finite dimensional normed space E is enriched with an order relation, that is defined by a cone associated to a basis with some characteristic properties. Let us fix first how we consider a finite dimensional space as a function space to define an order on it. The set of (approximated) eigenvectors of the map will be fundamental to provide our (approximated) representation formula.

2.1. **Lattice Lipschitz operators.** We will define a new extension rule for Lipschitz maps on Euclidean spaces that satisfy that it coincide with the usual representation of diagonalizable linear and non-linear operators (Theorem 2.14 and Corollary 2.15).

Fix $E = \mathbb{R}^n$ and let $\mathcal{B} = \{x_1, \dots, x_n\}$ be a basis for $\mathbb{R}^n$. Consider, as usual, the adittion and scalar multiplication defined by its components. We will consider the lattice order on $\mathbb{R}^n$ provided by the basis, that is if $x = (\alpha_1, ..., \alpha_n)$ and $y = (\beta_1, ..., \beta_n)$ are representations of two vectors of E given by its coordinates in $\mathcal{B}$, we define

$$x \le y \quad \text{if and only if} \quad \alpha_i \le \beta_i, \quad \textit{for every } i = 1, ..., n.$$

This order is the one given by by the positive cone

$$C = \big\{x = \sum_{i=1}^{n} \alpha_i x_i : \ \alpha_i \ge 0 \text{ for } i = 1, \dots n\big\},$$

so $x \le y$ if and only if $y - x \in C$. It is easy to see that the sumprema, infima and modulus in this space is the one given by its components, so

$$\left| \sum_{i=1}^{n} \alpha_i x_i \right| = \sum_{i=1}^{n} |\alpha_i| x_i$$

Recall that each vector $x = \sum_{i=1}^{n} \alpha_i x_i = (\alpha_1, \dots, \alpha_n)$ of the space E can be considered as a function $f_x : \Omega = \{1, \dots, n\} \to \mathbb{R}$ in the usual way, $f_x(w) = \alpha_w$ and we will refer to f_x also as x. Thus, E can be seen as a space of functions and it defines an order in E as the pointwise order in the space of functions.For $x, y \in E$,

$$x \le y \quad \text{if and only if} \quad x(w) \le y(w), \quad \textit{for every } w \in \Omega.$$

Observe that this order coincides with the one provided by its components. The multiplications of two elements of E will also be considered as pointwise multiplication.

In what follows we will consider the geometry of E induced by the 2-norm of the function f_x when integrating with the counting measure c, that is,

$$|||x|||_2 := \left(\int_{\{1,\dots,n\}} |f(k)|^2 dc(k) \right)^{1/2} = \left(\sum_{i=1}^{n} |\alpha_i|^2 \right)^{1/2}.$$

Obviously this norm is compatible with the order provided by the function structure of the space $L^2(c)$ (it is in fact the order in this space), so we have a Banach lattice. Note that this norm can be written as an integral of a function $\langle x, \cdot \rangle$ in the space $L^2(\nu)$ for a suitable measure ν (for example the one given by the addition of Dirac's deltas on biorthogonal functionals for a basis $\mathcal{B}$ of E, see [6, 7]).

In what follows, we will asume that we have a basis $\mathcal{B}$ of E and we consider the order defined above.

Definition 2.1. *We say that the operator $T : E \to E$ is lattice Lipschitz if there is a bounded function $K : E \to \mathbb{R}$ such that for every $x, y \in E$,*

$$|T(x) - T(y)|(w) \le K(w)|x - y|(w), \quad w \in \Omega.$$

This definition is based on the idea of the point wise evaluation of the functions, writing it in terms of the order of the space E, as

$$|T(x) - T(y)| \le K|x - y|.$$

It can be seen as a particular case of the one given in [25], with E as a Banach algebra, considering E as a cone metric space on E itself with the cone metric $\rho : E \times E \to E$

$$\rho(x, y)(w) = |x - y|(w), \quad w \in \Omega$$

for $x, y \in E$. Recall that a cone metric on a set X (see for example [25]) is a map $\rho : X \times X \to Y$, where Y is a Banach lattice, and satisfies that for all $x_1, x_2, x_3 \in X$,

(1) $0 < \rho(x_1, x_2)$ if $x_1 \ne x_2$, and $\rho(x_1, x_1) = 0$,
(2) $\rho(x_1, x_2) = \rho(x_2, x_1)$,
(3) $\rho(x_1, x_2) \le \rho(x_1, x_3) + \rho(x_3, x_2)$.

However, it has only been applied to fixed point theory and the idea of extending operators which satisfy the previous definition is, as far as we now, new.

Some other Lipschitz-type inequalities in ordered spaces can be found on Papageorgiou [27], Németh [26] or Li [28]. For the specific case of Banach function spaces, the recent paper [2] provides a concrete adaptation of such kind of inequalities for spaces of integrable functions. Ch.4.1 in [3] provides an overview on the topic.

Remark 2.2. *A relevant property of the lattice Lipschitz operators with Lipschitz function $K(w)$ is that they are, in particular, Lipschitz operators with norm less than or equal to $\sup_w |K(w)|$. Indeed, note that for $x, y \in E$, if we have that*

$$|T(x) - T(y)|(w) \leq K(w) \cdot |x - y|(w) \quad \text{for every point of the domain set } w,$$

we also have that, by the relation of the order and the norm on any Banach lattice,

$$\|T(x) - T(y)\| \leq \|K(w) \cdot |x - y|(w)\| \leq \sup_w |K(w)| \, \|x - y\|.$$

In what follows we will develop the extension results for this family of operators. It can be found in the scientific literature about Lipschitz extensions ideas that are in a sense similar. The fundamental extension methods for Lipschitz maps can be found in Cobzas [3]. In this sense, the already mentioned results of McShane and Whitney ([11, 13]), together with the celebrated Kirszbraun Theorem ([8])) form the classical core of the topic.

Let us define lattice-based extension formulas for such an operator T.

Definition 2.3. *Let $E_0 \subseteq E$. For a lattice Lipschitz operator $T : E_0 \to E$ with associated bounded function $K : \Omega \to \mathbb{R}$, we consider the formulas*

$$T^M(x)(w) := \bigvee \{T(z)(w) - K(w)|x - z|(w) : z \in E_0\}, \quad x \in E,$$

and

$$T^W(x)(w) := \bigwedge \{T(z)(w) + K(w)|x - z|(w) : z \in E_0\}, \quad x \in E.$$

Remark 2.4. *Previous expressions are well defined in the sense that the suprema and infima exists for any $w \in \Omega$ and T^M, T^W are operators from E to E. Indeed, given $x \in E$, let $y \in E_0$ be fixed and let $w \in \Omega$. Then, for any $z \in E_0$,*

$$\begin{aligned} T(z)(w) - T(y)(w) &\leq |T(z) - T(y)|(w) \leq K(w)|z - y|(w) \\ &\leq K(w)|z - x|(w) + K(w)|x - y|(w). \end{aligned}$$

It follows that

$$T(z)(w) - K(w)|x - z|(w) \leq T(y)(w) + K(w)|x - y|(w) =: M.$$

Since M does not depend on z, $\{T(z)(w) - K(w)|x - z|(w) : z \in E_0\}$ is a set of real numbers bounded from above by M, so its supremum exist. It shows that $T^M(x) \in E$, for the T^W case consider following remark.

Remark 2.5. *Notice that the extension formulas T^M and T^W are related by the equation*

$$\begin{aligned} T^W(x)(w) &= \bigwedge \{T(z)(w) + K(w)|x - z|(w) : z \in E_0\} \\ &= \bigwedge \{-((-T)(z)(w) - K(w)|x - z|(w)) : z \in E_0\} \\ &= -\bigvee \{(-T)(z)(w) - K(w)|x - z|(w) : z \in E_0\} = -(-T)^M(x)(w). \end{aligned}$$

Moreover, $-T : E_0 \to E$ is also a lattice Lipschitz operator with the same associated function K. This fact will allow us to prove some properties only to one of the formulas and apply the previous identity for the other case.

Observe that the only information of the space E required in both extension formulas is the order in E and the associated Lipschitz function, which is calculated using the order. Non information about metric or linear structure is required.

Proposition 2.6. *Let $E_0 \subseteq E$. Let $T : E \to E$ be a lattice Lipschitz operator with associated bounded function $K : \Omega \to \mathbb{R}$. Then T^M and T^W are suitable extension formulas from E_0 to E which preserve the lattice Lipschitz inequality with the same function K.*

Proof. First we show that T^M is an extension of T. For any $x \in E_0$ and $w \in \Omega$, clearly one has

$$T^M(x)(w) \geq T(x)(w) - K(w)|x - x|(w) = T(x)(w).$$

In addition, for every $y \in E_0$,

$$T(y)(w) - T(x)(w) \leq |T(y) - T(x)|(w) \leq K(w)|y - x|(w),$$

so $T(y)(w) - K(w)|x - s|(w) \leq T(x)(w)$ and, taking supremum, $T^M(x)(w) \leq T(x)(w)$. We conclude that $T(x) = T^M(x)$ for every $x \in E_0$.

In order to justify that T^M verifies the lattice Lipschitz inequality with the same function K, let $x, y \in E$. For any $w \in \Omega$, if $z \in E_0$

$$\begin{aligned} T^M(x)(w) &\geq T(z)(w) - K(w)|x - z|(w) \\ &\geq T(z)(w) - K(w)|y - z|(w) - K(w)|x - y|(w). \end{aligned}$$

It follows that

$$\begin{aligned} T^M(x)(w) &\geq \bigvee\{T(z)(w) - K(w)|y - z|(w) - K(w)|x - y|(w) : z \in E_0\} \\ &= T^M(y)(w) - K(w)|x - y|(w), \end{aligned}$$

so $K(w)|x-y|(w) \geq T^M(y)(w) - T^M(x)(w)$. Interchanging the roles of x and y, one obtains $K(w)|y - x|(w) \geq T^M(x)(w) - T^M(y)(w)$, so

$$|T^M(x)(w) - T^M(y)(w)| \leq K(w)|y - x|(w).$$

The case of T^W, can be proved similarly or by applying the previous case to $-T$ as in Remark 2.5. □

The following result gives information about the extremal properties of the McShane and Whitney lattice extensions.

Proposition 2.7. *Let $E_0 \subseteq E$. Let $T : E_0 \to E$ be a lattice Lipschitz operator with associated bounded function $K : \Omega \to \mathbb{R}$. If $\hat{T} : E \to E$ is an extension of T which preserve the lattice Lipschitz inequality with the same function K, then $T^M \leq \hat{T} \leq T^W$.*

Proof. Let $x \in E$ and $w \in \Omega$. Since $\hat{T}$ is a lattice Lipschitz operator with associated function K, for any $z \in E_0$

$$-K(w)|x - z|(w) \leq \hat{T}(x)(w) - \hat{T}(z)(w) \leq K(w)|x - z|(w).$$

Notice that $\hat{T}(z)(w) = T(z)(w)$, so

$$T(z)(w) - K(w)|x - z|(w) \leq \hat{T}(x)(w) \leq T(z)(w) + K(w)|x - z|(w).$$

Now, tacking supremum on the left and infimum on the right, $T^M(x)(w) \leq \hat{T}(x)(w) \leq T^W(x)(w)$. □

Remark 2.8. *Last proposition also show that $T^M \leq T^W$, but we can also study how both extensions differ. We claim that*

$$0 \leq T^W(x) - T^M(x) \leq 2K \bigwedge\{|x-z| : z \in E_0\}$$

for any $x \in E$. Indeed, if $w \in \Omega$,

$$\begin{aligned} T^W(x) - T^M(x) = \\ &= \bigwedge\{T(z) + K|x-z| : z \in E_0\} - \bigvee\{T(z) - K|x-z| : z \in E_0\} \\ &= \bigwedge\{T(z) + K|x-z| : z \in E_0\} + \bigwedge\{-T(z) + K|x-z| : z \in E_0\} \\ &\leq \bigwedge\{T(z) + K|x-z| - T(z) + K|x-z| : z \in E_0\} \\ &= \bigwedge\{2K|x-z| : z \in E_0\} \end{aligned}$$

Definition 2.9. *We say that an operator $T : E \to E$ is* diagonal *with respect to a basis $\mathcal{B} = \{x_1, x_2, \ldots, x_n\}$ of $E = \mathbb{R}^n$ if there exists $f_i : \mathbb{R} \to \mathbb{R}$ real functions for $1 \leq i \leq n$ such that*

$$T\big(\sum_{i=1}^n \alpha_i x_i\big) = \sum_{i=1}^n f_i(\alpha_i) x_i, \qquad \text{for any } \alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}. \tag{1}$$

In this case, we call the f_i coordinate functions *of T.*

Observe that previous conditions can be rewritten as

$$T\big((\alpha_1, \alpha_2, \ldots, \alpha_n)\big) = \big(f_1(\alpha_1), f_2(\alpha_2), \ldots, f_n(\alpha_n)\big)$$

in the coordinates of the basis $\mathcal{B}$.

Next results characterizes the lattice Lipschitz condition on euclidean space.

Theorem 2.10. *Let $T : E \to E$ be an operator. Consider on E the order provided by the basis $\mathcal{B} = \{x_1, \ldots, x_n\}$. Then, T is a lattice Lipschitz function with associated function $K : \Omega \to \mathbb{R}$ if and only if T is diagonal respect to the basis $\mathcal{B}$ with coordinate functions begin real Lipschitz with Lipschitz constant $K(i)$.*

Proof. Suppose first that T is diagonal on the basis $\mathcal{B}$ and f_i are real Lipschitz function with Lipschitz constant $K(i)$. Let $x = \sum_{i=1}^n \alpha_i x_i$ and $y = \sum_{i=1}^n \beta_i x_i$ two elements of E, then

$$|T(x) - T(y)| = \sum_{i=1}^n |f_i(\alpha_i) - f_i(\beta_i)| x_i \leq \sum_{i=1}^n K_i |\alpha_i - \beta_i| x_i = K|x-y|.$$

For the converse, let T be a lattice Lipschitz function with associated function K. Consider $T_i : E \to \mathbb{R}$ the functions such that

$$T(x) = \sum_{i=1}^n T_i(x) x_i, \qquad x \in E. \tag{2}$$

Define now the real functions $f_j : \mathbb{R} \to \mathbb{R}$ for $1 \leq j \leq n$ as

$$f_j(\alpha) = T_j(\alpha x_j).$$

We claim that

$$T_j\big(\sum_{i=1}^n \alpha_i x_i\big) = f_j(\alpha_j). \tag{3}$$

To see this, fix $1 \leq j \leq n$, if $x = \sum_{i=1}^{n} \alpha_i x_i$, then

$$\begin{aligned} |T_j(x) - f_j(\alpha_j)| &= |T_j(x) - T_j(\alpha_j x_j)| = |T(x) - T(\alpha_j x_j)|(j) \\ &\leq K(j)|x - \alpha_j x_j|(j) = K(j)|\alpha_j - \alpha_j| = 0. \end{aligned}$$

In other words, the function T_j only depends on the j component of x. As a consequence of (2) and (3),

$$T\Big(\sum_{i=1}^{n} \alpha_i x_i\Big) = \sum_{j=1}^{n} T_j\Big(\sum_{i=1}^{n} \alpha_i x_i\Big) x_j = \sum_{j=1}^{n} f_j(\alpha_j) x_j.$$

To finish the proof, we shall show that each f_j is a (real) Lipschitz function. Let $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} |f_j(\alpha) - f_j(\beta)| &= |T_j(\alpha x_j) - T_j(\beta x_j)| = |T(\alpha x_j) - T(\beta x_j)|(j) \\ &\leq K(j)|\alpha x_j - \beta x_j|(j) = K(j)|\alpha - \beta|. \end{aligned}$$

□

Remark 2.11. *Consider on E the order provided by the basis $\mathcal{B} = \{x_1, \ldots, x_n\}$. For an operator $T : E \to E$ the following statements are equivalent.*

(1) T is a lattice Lipschitz operator with the order provided by $\mathcal{B}$.
(2) T is a diagonal operator with coordinate functions being real Lipschitz functions.
(3) T verifies that for any $1 \leq i \leq n$ there exists $K_i > 0$ such that

$$|\langle T(x) - T(y), x_i^* \rangle| \leq K_i |\langle x - y, x_i^* \rangle|$$

where $x_i^* \in E^*$ is the functional defined as $x_i^*(x_i) = 1$ and $x_i^*(x_j) = 0$ if $i \neq j$.

The following examples show the importance of the order in the space on the study of lattice Lipschitz or diagonal mappings.

Example 2.12. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be $f(x, y) = (y, x)$. Clearly it is not a lattice Lipschitz mapping (on the usual order of $\mathbb{R}^2$) because*

$$|f(1, 0) - f(0, 0)| = (0, 1) \not\leq K \cdot |(1, 0) - (0, 0)| = (K_1, 0),$$

for any $K = (K_1, K_2) \in \mathbb{R}^2$. But we claim that it is a lattice Lipschitz function on $\mathbb{R}^2$ when an appropriate order is considered. Indeed, let be $v_1 = (1, 1)$ and $v_2 = (1, -1)$, consider the basis $\mathcal{B} = \{v_1, v_2\}$ and the order provided by $\mathcal{B}$. Then,

$$f(\alpha v_1 + \beta v_2) = f(\alpha + \beta, \alpha - \beta) = (\alpha - \beta, \alpha + \beta) = f_1(\alpha) v_1 + f_2(\beta) v_2,$$

where $f_1(t) = t, f_2(t) = -t$, that are real Lipschitz functions. By Theorem 2.10, it is a lattice Lipschitz mapping.

Example 2.13. *Let on $\mathbb{R}^2$ with the usual order be the mapping $\phi : \mathbb{R}^2 \to \mathbb{R}^2$*

$$\phi(x, y) = \begin{cases} (2x^2 + 2y^2, 4xy) & \text{if } |x + y| \leq 2, |x - y| \leq 2 \\ ((x + y)^2 + 4, (x + y)^2 - 4) & \text{if } |x + y| \leq 2, |x - y| > 2 \\ (4 + (x - y)^2, 4 - (x - y)^2) & \text{if } |x + y| > 2, |x - y| \leq 2 \\ (8, 0) & \text{if } |x + y| > 2, |x - y| > 2. \end{cases}$$

Clearly it is not a lattice Lipschitz function, since

$$|\phi(0, 3) - \phi(0, 0)|(1) = 2 \not\leq K|(0, 3) - (0, 0)|(1) = 0$$

for any K. Let us see that it is a diagonal function when considering on $\mathbb{R}^2$ the order induced by the basis $\mathcal{B} = \{v_1, v_2\}$ with $v_1 = (1,1), v_2 = (1,-1)$. For $\alpha, \beta \in \mathbb{R}$, it is easy to see that

$$\phi(\alpha v_1 + \beta v_2) = \begin{cases} (4\alpha^2 + 4\beta^2, 4\alpha^2 - 4\beta^2) & \text{if } |\alpha| \le 1, |\beta| \le 1 \\ (4\alpha^2 + 4, 4\alpha^2 - 4) & \text{if } |\alpha| \le 1, |\beta| > 1 \\ (4 + 4\beta^2, 4 - 4\beta^2) & \text{if } |\alpha| > 1, |\beta| \le 1 \\ (8, 0) & \text{if } |\alpha| > 1, |\beta| > 1 \end{cases}$$

so $\phi(\alpha v_1 + \beta v_2) = f_1(\alpha)v_1 + f_2(\beta)v_2$ where $f_1(t) = f_2(t) = 4t^2$ for $|t| \le 1$ and 4 otherwise. Since ϕ is diagonal on the basis $\mathcal{B}$ with coordinate functions f_1, f_2 real Lipschitz functions with Lipschitz constant equal to 8, by Theorem 2.10 ϕ it is a lattice Lipschitz function with associate function $K(1) = K(2) = 8$.

Theorem 2.14. *Let $E = \mathbb{R}^n$ and consider the order provided by a basis $\mathcal{B} = \{x_1, x_2, \ldots, x_n\}$ of E, and let $T : E \to E$ be a lattice Lipschitz function with associated function K and coordinate functions f_i (the decomposition (1) is possible because of Theorem 2.10). Consider the "axis" set $\Sigma = \{\alpha x_i : \ \alpha \in \mathbb{R}, 1 \le i \le n\}$. Then,*

(1) the Whitney and McShane extensions of $T|_\Sigma$ from Σ to E are

$$(T|_\Sigma)^M = T \quad \text{and} \quad (T|_\Sigma)^W = T.$$

If, in addition, $T(0) = 0$,

(2) the set of eigenvectors of T contains Σ, and

(3) if λ is an eigenvalue of T, $|\lambda| \le \sup_{w \in \Omega} K_w$.

Proof. (1) Let us consider now $x = \sum_{i=1}^n \alpha_i x_i$ and study the McShane extension of $T|_\Sigma$ at x.

$$T^M(x) = \bigvee_{s \in \Sigma} T(s) - K|x - s| = \bigvee_{i=1}^n \left(\bigvee_{\alpha \in \mathbb{R}} T(\alpha x_i) - K|x - \alpha x_i| \right).$$

Fix $1 \le w \le n$ and let $1 \le i \le n$. Observe that $x - \alpha x_i = (\alpha_i - \alpha)x_i + \sum_{j \ne i} \alpha_j x_j$, so if $w \ne i$

$$\left(\bigvee_{\alpha \in \mathbb{R}} T(\alpha x_i) - K|x - \alpha x_i| \right)(w) = \bigvee_{\alpha \in \mathbb{R}} f_w(0) - K_w|\alpha_w| = f_w(0) - K_w|\alpha_w|.$$

Since $f_w(0) - f_w(\alpha_w) \le |f_w(0) - f_w(\alpha_w)| \le K_w|\alpha_w|$, then $f_w(0) - K_w|\alpha_w| \le f_w(\alpha_w)$, so

$$\left(\bigvee_{\alpha \in \mathbb{R}} T(\alpha x_i) - K|x - \alpha x_i| \right)(w) \le f_w(\alpha_w). \tag{4}$$

For the case $w = i$,

$$\left(\bigvee_{\alpha \in \mathbb{R}} T(\alpha x_w) - K|x - \alpha x_w| \right)(w) = \bigvee_{\alpha \in \mathbb{R}} f_w(\alpha) - K_w|\alpha_w - \alpha|$$

Observe that $f_w(\alpha) - f_w(\alpha_w) \le |f_w(\alpha) - f_w(\alpha_w)| \le K_w|\alpha - \alpha_w|$ for any $\alpha \in \mathbb{R}$, so $f_w(\alpha) - K_w|\alpha_w - \alpha| \le f_w(\alpha_w)$, so the supremum is attained a $\alpha = \alpha_w$. It implies

$$\left(\bigvee_{\alpha \in \mathbb{R}} T(\alpha x_w) - K|x - \alpha x_w| \right)(w) = f_w(\alpha_w). \tag{5}$$

As a consequence of (5) and (4),

$$T^M(x)(w) = \bigvee_{i=1}^{n} \left(\bigvee_{\alpha \in \mathbb{R}} T(\alpha e_i) - K|x - \alpha e_i| \right)(w) = f_w(\alpha_w) = T(x)(w),$$

which proves that $T^M(x) = T(x)$.

The proof for the Whitney case is immediate by using previous case on $-T$.

(2) Clearly, $x = 0$ is an eigenvector of T. Observe that the condition $T(0) = 0$ implies that $f_i(0) = 0$ for all $1 \leq i \leq n$. Let αx_i in Σ with $\alpha \neq 0$,

$$T(\alpha x_i) = \sum_{j \neq i} f_j(0) x_j + f_i(\alpha) x_i = \frac{f_i(\alpha)}{\alpha} \alpha x_i,$$

so αx_i is an eigenvector of T with eigenvalue $\frac{f_i(\alpha)}{\alpha}$.

(3) If $T(x) = \lambda x$ (with $x \neq 0$), equation (1) implies that $f_i(\alpha_i) = \lambda \alpha_i$ for each i. Since $x \neq 0$, at least one α_{i_0} is not 0, so

$$|\lambda| = \left| \frac{f_{i_0}(\alpha_{i_0})}{\alpha_{i_0}} \right| = \left| \frac{f_{i_0}(\alpha_{i_0}) - f_{i_0}(0)}{\alpha_{i_0}} \right| \leq \left| \frac{K_{i_0}|\alpha_{i_0} - 0|}{\alpha_{i_0}} \right| = K_{i_0} \leq \sup_{w \in \Omega} K_w.$$

□

An immediate consequence of Theorem 2.14 is the following, in which the linear case is considered.

Corollary 2.15. *Let $T : E \to E$ be a linear diagonalizable operator with real eigenvalues $\lambda_1, ..., \lambda_n \in \mathbb{R}$. Let $\mathcal{B} = \{x_1, ..., x_n\}$ be a basis for E of eigenvectors of T. Consider the complete set of eigenvectors E_0 and recall that the order in the lattice is induced by the cone defined $\mathcal{B}$. Then*

(1) $T|_{E_0}$ is lattice Lipschitz with associated function $K(r) = |\lambda_r|$, $r \in \{1, ..., n\}$.

(2) Both T^M and T^W provide lattice Lipschitz extensions of $T|_{E_0}$ from E_0 to E preserving the associated function K such that

$$(T|_{E_0})^M = T \quad \text{and} \quad (T|_{E_0})^W = T.$$

Consequently, T is lattice Lipschitz with associated function K.

The coincidence of the lattice Lipschitz extension rule and the linear rule opens the door to a general procedure for extending Lipschitz maps with a "largue enough" set of eigenvectors. For the diagonalizable case, the minimum-maximum condition of the McShane and Whitney extensions (Proposition 2.7) can be considered, along with the extension behavior of diagonalizable mappings (Theorem 2.14) when applying these formulas to obtain the following results, which reveals a uniqueness property for the lattice extension of linear maps.

Corollary 2.16. *With the same hypothesis as in Theorem 2.14, T is the unique lattice Lipschitz operator with associated function K that extends $T|_{\Sigma}$.*

Corollary 2.17. *With the same hypothesis as in Theorem 2.14, if $T(0) = 0$ and E_0 is the set of eigenvectors of T, then the McShane and Whitney extensions of $T|_{E_0}$ are equal to T.*

Although we have shown that the extension formulas work as expected for the case of diagonalizable operators, recall that we are interested in the case of general lattice Lipschitz maps. Next we will show the error bounds for the lattice extension formulas at a point $x \in E$ with respect to the original operator T. This expression is valid for the Lipschitz functions of the lattice, and has to be used to control the error committed when the reconstruction of the function is done from the information we have about it in a subset.

Proposition 2.18. *Let $T : E \to E$ be a lattice Lipschitz operator with associated bounded function $K : \Omega \to \mathbb{R}$ and $E_0 \subseteq E$. Then, for any $x \in E$*

$$-2K \bigwedge\{|x-z| : z \in E_0\} \le (T|_{E_0})^M(x) - T(x) \le 0 \qquad \textit{and}$$

$$0 \le (T|_{E_0})^W(x) - T(x) \le 2K \bigwedge\{|x-z| : z \in E_0\}.$$

Proof. For $T : E \to E$ as in the statement of the proposition, consider $T|_{E_0}$. If $x \in E$, applying that T is a lattice Lipschitz operator in the whole E,

$$\begin{aligned}(T|_{E_0})^W(x) - T(x) &= \bigwedge\{T(z) + K|x-z| : z \in E_0\} - T(x)\\ &= \bigwedge\{T(z) - T(x) + K|x-z| : z \in E_0\}\\ &\le \bigwedge\{|T(z) - T(x)| + K|x-z| : z \in E_0\}\\ &\le \bigwedge\{K|z-x| + K|x-z| : z \in E_0\}\\ &= 2K \bigwedge\{|z-x| : z \in E_0\}.\end{aligned}$$

In addition,

$$\begin{aligned}(T|_{E_0})^W(x) - T(x) &= \bigwedge\{T(z) - T(x) + K|x-z| : z \in E_0\}\\ &\ge \bigwedge\{-|T(z) - T(x)| + K|x-z| : z \in E_0\}\\ &\ge \bigwedge\{-K|z-x| + K|x-z| : z \in E_0\} = 0.\end{aligned}$$

The bounds for the McShane case can be proved by applying Remark 2.5. □

Remark 2.19. *Previous proposition can also be proved as a consequence of Proposition 2.7 and Remark 2.8. Observe that the original map T is a suitable extension of $T|_{E_0}$ (with the same associate function). So it is clear that for $x \in E$, $T^M(x) \le T(x) \le T^W(x)$. Also note that $|T^W(x) - T^M(x)| \le 2K \bigwedge\{|x-z| : z \in E_0\}$, so $T(x)$ cannot differ from $T^M(x)$ and $T^W(x)$ more than $2K \bigwedge\{|x-z| : z \in E_0\}$.*

To finish this section, let us show a complete example that can be completely computed. We will show the set of eigenvectors, the associated lattice Lipschitz function, and the McShane and Whitney extensions of the map.

Example 2.20. *Consider the function $\phi : \mathbb{R}^2 \to \mathbb{R}^2$ given by*

$$\phi(x,y) = \left(\frac{x^2}{1+x^2}, \frac{y^2}{1+y^2}\right), \quad (x,y) \in \mathbb{R}^2.$$

The computation of the eigenvectors by means of the eigenvector equations provide the next formulas in terms of the eigenvalues λ; for $\lambda = 0$, the eigenvector is $(0,0)$. The spectrum of the map is given by the values of $\lambda \in [-1/2, 1/2]$. The eigenvectors associated to each of these values are given by the combination of values (x,y) for $\lambda \neq 0$ given by the formulas

$$x = 0, \quad x = \frac{1}{2\lambda}\left(1 + \sqrt{1-4\lambda^2}\right) \quad \textit{or} \quad x = \frac{1}{2\lambda}\left(1 - \sqrt{1-4\lambda^2}\right)$$

and

$$y = 0, \quad y = \frac{1}{2\lambda}\left(1 + \sqrt{1-4\lambda^2}\right) \quad \textit{or} \quad x = \frac{1}{2\lambda}\left(1 - \sqrt{1-4\lambda^2}\right).$$

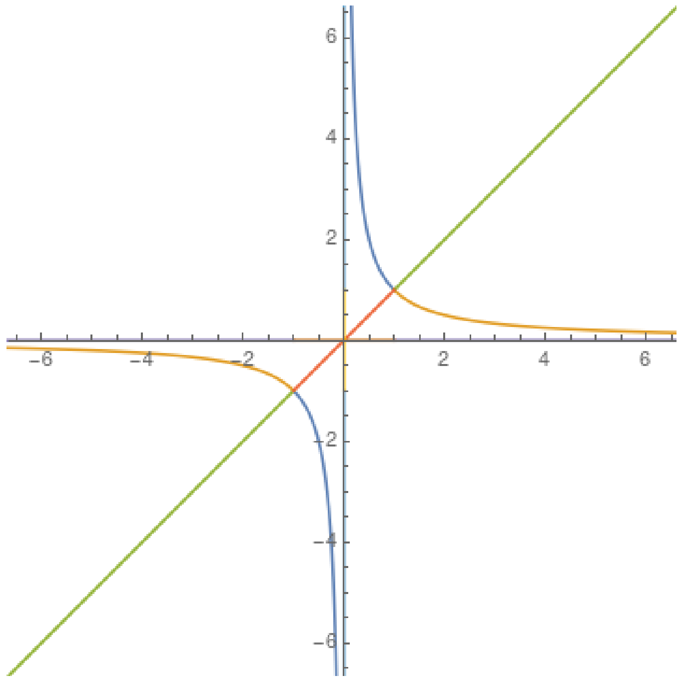

FIGURE 1. Representation of the set of eigenvectors for the function ϕ. All the points in the curves and lines represented are eigenvectors, including the axis OX and OY.

The representation of the points (x, y) that are obtained following this description is given in Figure 1.

In this case the lattice structure is given by a function space acting in a two-point measurable space $\{w_1, w_2\}$, and the evaluation of the functions at each of these points is one of each coordinates of the resulting vector. A standard optimization procedure leads to find the lattice Lipschitz function that plays the role of the Lipschitz norm, that is

$$K(w_1) = K(w_2) = \frac{3\sqrt{3}}{8}.$$

We consider as subset for constructing the McShane and Whitney extensions the set E_0 of all the eigenvectores described above. We write the Whitney formula. Using the parametrization provided by the eigenvalues λ and taking into account that the infimum can be computed separately for each coordinate, the formulas are, for every $(x_0, y_0) \in \mathbb{R}^2$,

$$P_1\big((\phi|_{E_0})^W(x_0, y_0)\big) = \bigwedge_{\lambda \in [-\frac{1}{2}, \frac{1}{2}]} \Big(\lambda\, x(\lambda) + \frac{3\sqrt{3}}{8} \cdot \big|x(\lambda) - x_0\big|\Big) \wedge \frac{3\sqrt{3}}{8} \cdot |x_0|$$

$$= \inf_{\lambda \in [-\frac{1}{2}, \frac{1}{2}]} \left\{ \frac{1}{2}\Big(1 \pm \sqrt{1 - 4\lambda^2}\Big) + \frac{3\sqrt{3}}{8} \cdot \left| \frac{1}{2\lambda}\Big(1 \pm \sqrt{1 - 4\lambda^2}\Big) - x_0 \right| \right\} \wedge \frac{3\sqrt{3}}{8} \cdot |x_0|,$$

and

$$P_2\big((\phi|_{E_0})^W(x_0, y_0)\big) = \bigwedge_{\lambda \in [-\frac{1}{2}, \frac{1}{2}]} \Big(\lambda\, y(\lambda) + \frac{3\sqrt{3}}{8} \cdot \big|y(\lambda) - y_0\big|\Big) \wedge \frac{3\sqrt{3}}{8} \cdot |y_0|$$

$$= \inf_{\lambda\in[-\frac{1}{2},\frac{1}{2}]} \left\{ \frac{1}{2}\Big(1 \pm \sqrt{1-4\lambda^2}\Big) + \frac{3\sqrt{3}}{8}\cdot\left|\frac{1}{2\lambda}\Big(1 \pm \sqrt{1-4\lambda^2}\Big) - y_0\right| \right\} \wedge \frac{3\sqrt{3}}{8}\cdot |y_0|,$$

where $(\phi|_{E_0})^W(x_0,y_0) = \big(P_1\big((\phi|_{E_0})^W(x_0,y_0)\big), P_2\big((\phi|_{E_0})^W(x_0,y_0)\big)\big)$.

As a consequence of Corollary 2.17, we know that this extension formula (and the one of McShane) coincides with the original function. Note, however, that we can also use these formulas if we take a (proper) subset of eigenvectors. In case that such a subset does not contain the axis, the coincidence of the extension with the function need not occur.

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