On slow decay of Peetre's K-functional

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XX ENCUENTROS DE ANÁLISIS REAL Y COMPLEJO
Cartagena, Mayo, 2022.

Dedicated to Fernando Cobos for his 65-th birthday

K-functional and Approximation theory

- Consider a couple (X, Y), where X is a quasi-Banach space and $Y \subset X$ is a quasi-semi-normed space which is continuously embedded into X.
- Peetre's K-functional is

$$K(x, t, X, Y) = \inf_{y \in Y} (\|x - y\|_X + t\|y\|_Y)$$

K-functional and Approximation theory

If Y is dense in X, then

$$K(x,0^+,X,Y) = \lim_{t\to 0^+} K(x,t,X,Y) = 0$$

and for each fixed $c,t>0$ the claim

$$K(x,t,X,Y) \geq c$$

is equivalent to the following property:

$$\text{If } y\in Y \text{ satisfies } \|x-y\|_X < c, \text{ then } \|y\|_Y \geq \frac{c-\|x-y\|_X}{t}$$

In particular, if $\|x-y\|_X < \frac{c}{2}$ with $y\in Y$, then $\|y\|_Y \geq \frac{c-\|x-y\|_X}{t} \geq \frac{c}{2t}$, which diverges to infinity when t tends to 0.

K-functional and Approximation theory

Property

If $S(X)$ denotes the unit sphere of X, the condition

$$K(S(X), t, X, Y) = \sup_{x \in S(X)} K(x, t, X, Y) > c \quad \textit{for all } t > 0$$

is equivalent to the existence of elements in X of small X-norm which are not approximable by elements of Y with small Y-norm.

For example, if we set $X = C[a, b]$ and $Y = C^{(m)}[a, b]$ with $m \geq 1$, the property holds since there are oscillating functions of small uniform norm that cannot be uniformly approximated by functions of small $C^{(m)}$-semi-norm.

K-functional and Approximation theory

Theorem (Central theorems in Approximation Theory)

Assume that the approximation scheme $(X, \{A_n\})$ satisfies Jackson's and Bernstein's inequalities with respect to the space $Y \hookrightarrow X$, which are given by

$$E(y, A_n) \leq Cn^{-r}\|y\|_Y \text{ for all } y \in Y \text{ and all } n \geq 0$$

and

$$\|a\|_Y \leq Cn^r\|a\|_X \text{ for all } a \in A_n,$$

respectively, then the approximation spaces

$$A_q^\alpha(X, \{A_n\}) = \{x \in X : \{2^{\alpha k}E(x, A_{2^k})\}_{k=0}^\infty \in \ell_q\}$$

are completely characterized as interpolation spaces by the formula

$$A_q^\alpha(X, \{A_n\}) = (X, Y)_{\alpha/r,q} \text{ for all } 0 < \alpha < r \text{ and all } 0 < q \leq \infty.$$

K-functional and Approximation theory

Recall that $(X, Y)_{\theta,q}$ denotes the real interpolation space

$$(X, Y)_{\theta,q} = \{x \in X : \rho_{\theta,q}(x) = \|t^{-\left(\theta+\frac{1}{q}\right)} K(x, t, X, Y)\|_{L^q(0,\infty)} < \infty\},$$

which, in case that $Y \hookrightarrow X$, can be renormalized with the following equivalent quasi-norm:

$$\|x\|_{\theta,q} = \|\{2^{\theta k} K(x, \frac{1}{2^k}, X, Y)\}_{k=0}^{\infty}\|_{\ell_q}.$$

It follows that, when $(X, \{A_n\})$ satisfies Jackson's and Bernstein's inequalities with respect to a subspace Y, the rates of convergence to zero of the sequence of best approximation errors $E(x, A_n)$ and the sequence of evaluations of Peetre's K-functional $K(x, t, X, Y)$ at points $t_n = 1/n$ seem to have similar roles and, in particular, serve to describe the very same subspaces of X. Thus, it is an interesting question to know under which conditions on the couple (X, Y), with $Y \hookrightarrow X$, the K-functional $K(\cdot, \cdot, X, Y)$ approaches to zero slowly, not only in presence of an approximation scheme but also for the general case.

Characterizations of slowly decaying K-functionals

Theorem

Let (X,Y) be a couple, where $(X,\|\cdot\|_X)$ is a quasi-Banach space and $Y\subset X$ is a quasi-semi-normed space, $(Y,\|\cdot\|_Y)$ which is continuously embedded into X. The following are equivalent claims:

- (a) *$K(S(X),t,X,Y)>c$ for all $t>0$ and a certain constant $c>0$.*
- (b) *For every non-increasing sequences $\{\varepsilon_n\},\{t_n\}\in c_0$, there are elements $x\in X$ such that*
$$K(x,t_n,X,Y)\neq \mathbf{O}(\varepsilon_n)$$

Definition

We say that the K-functional $K(\cdot,\cdot,X,Y)$ slowly decays to zero if either condition (a) or (b) of Theorem above holds true. Note that this is equivalent to the existence of elements in X of small X-norm which are not approximable by elements of Y with small Y-norm.

Characterizations of slowly decaying K-functionals

$(a) \Rightarrow (b)$. Let us assume, on the contrary, that $K(x, t_n, X, Y) = \mathbf{O}(\varepsilon_n)$ for all $x \in X$ and certain sequences $\{\varepsilon_n\}, \{t_n\} \in c_0$. This can be reformulated as

$$X = \bigcup_{m=1}^{\infty} \Gamma_m,$$

where

$$\Gamma_m = \{x \in X : K(x, t_n, X, Y) \leq m\varepsilon_n \text{ for all } n \in \mathbb{N}\}.$$

Now, Γ_m is a closed subset of X for all m and the Baire category theorem implies that Γ_{m_0} has nonempty interior for some $m_0 \in \mathbb{N}$. On the other hand, $\Gamma_m = -\Gamma_m$ since $K(x, t, X, Y) = K(-x, t, X, Y)$ for all $x \in X$ and $t \geq 0$.

Characterizations of slowly decaying K-functionals

Furthermore, if $C > 1$ is the quasi-norm constant of X and Y, then

$$\mathbf{conv}(\Gamma_m) \subseteq \Gamma_{Cm},$$

since, if $x, y \in \Gamma_m$ and $\lambda \in [0, 1]$, then

$$\begin{aligned} K(\lambda x + (1-\lambda)y, t_n, X, Y) &\leq C(K(\lambda x, t_n, X, Y) + K((1-\lambda)y, t_n, X, Y)) \\ &= C(\lambda K(x, t_n, X, Y) + (1-\lambda)K(y, t_n, X, Y)) \\ &\leq C(\lambda m\varepsilon_n + (1-\lambda)m\varepsilon_n) \\ &= Cm\varepsilon_n \end{aligned}$$

Characterizations of slowly decaying K-functionals

Thus, if $B_X(x_0, r) = \{x \in X : \|x_0 - x\|_X < r\} \subseteq \Gamma_{m_0}$, then $\mathbf{conv}(B_X(x_0, r) \cup B_X(-x_0, r)) \subseteq \Gamma_{Cm_0}$. In particular,

$$\frac{1}{2}(B_X(x_0, r) + B_X(-x_0, r)) \subseteq \Gamma_{Cm_0}.$$

Now, it is clear that

$$B_X(0, r) \subseteq \frac{1}{2}(B_X(x_0, r) + B_X(-x_0, r)),$$

Thus,

$$B_X(0, r) \subseteq \Gamma_{Cm_0}.$$

This means that, if $x \in X \setminus \{0\}$, then

$$K(\frac{rx}{\|x\|}, t_n, X, Y) \leq Cm_0\varepsilon_n \text{ for all } n = 1, 2, \cdots .$$

Hence

$$K(x, t_n, X, Y) \leq \frac{\|x\|}{r} Cm_0\varepsilon_n \text{ for all } n = 1, 2, \cdots .$$

Characterizations of slowly decaying K-functionals

On the other hand, (a) implies that, for each n, there is $x_n \in S(X)$ such that

$$K(x_n, t_n, X, Y) > c,$$

so that

$$c < K(x_n, t_n, X, Y) \leq \frac{1}{r} C m_0 \varepsilon_n \text{ for all } n = 1, 2, \cdots ,$$

which is impossible, since ε_n converges to 0 and $c > 0$.

This proves $(a) \Rightarrow (b)$.

Characterizations of slowly decaying K-functionals

Let us demonstrate the other implication. Assume that (a) does not hold. Then there are non-increasing sequences $\{t_n\}, \{c_n\} \in c_0$ such that

$$K(S(X), t_n, X, Y) \leq c_n \text{ for all } n \in \mathbb{N}$$

In particular, if $x \in X$ is not the null vector, then

$$K(\frac{x}{\|x\|_X}, t_n, X, Y) \leq K(S(X), t_n, X, Y) \leq c_n \text{ for all } n \in \mathbb{N},$$

so that

$$K(x, t_n, X, Y) = \|x\|_X K(\frac{x}{\|x\|_X}, t_n, X, Y) \leq \|x\|_X c_n \text{ for all } n \in \mathbb{N},$$

and $K(x, t_n, X, Y) = \mathbf{O}(c_n)$ for all $x \in X$.

This proves $(b) \Rightarrow (a)$.

The Quasi-Banach setting

Theorem

Assume that $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are quasi-Banach spaces. Then

(a) *If $Y \hookrightarrow X$, $Y \neq X$. Then $K(S(X), t, X, Y) > c$ for all $t > 0$ and a certain constant $c > 0$.*

(b) *If X and Y are both p-normed spaces, $Y \hookrightarrow X$, then either $K(S(X), t, X, Y) = 1$ for all $t > 0$, or $X = Y$.*

The Quasi-Banach setting

It follows from Aoki-Rolewicz Theorem that X and Y can be renormed with equivalent quasi-norms in such a way that they become p-normed spaces for a certain $p \in]0, 1]$. Thus we only need to demonstrate part (b) of the Theorem.

The Quasi-Banach setting

Assume that X, Y are quasi-Banach p-normed spaces and $Y \hookrightarrow X$. Let us prove that:

- $K(S(X), t, X, Y) < 1$ for a certain $t > 0$, then $X = Y$.

Assume, on the contrary, that $K(S(X), t_0, X, Y) = c < 1$ and $X \neq Y$. Take $\rho \in]0, 1[$ such that $c < \rho^{1/p} < 1$.

The Quasi-Banach setting

Then $K(\frac{x}{\|x\|_X}, t_0, X, Y) < \rho^{1/p}$ for every $x \in X$, $x \neq 0$. Hence every element x of X which is different from zero satisfies

$$K(x, t_0, X, Y) < \rho^{1/p}\|x\|_X,$$

which implies that $\|x - y_0\|_X + t_0\|y_0\|_Y < \rho^{1/p}\|x\|_X$ for a certain $y_0 \in Y$. Thus, if we set $x_0 = x - y_0$, we have that

$$\begin{cases} x = x_0 + y_0 \text{ with } x_0 \in X, \text{ and } y_0 \in Y \\ \|x_0\|_X < \rho^{1/p}\|x\|_X \\ \|y_0\|_Y < t_0^{-1}\rho^{1/p}\|x\|_X \end{cases} \tag{0.1}$$

The Quasi-Banach setting

Let us take $x \in X \setminus Y$ and apply the argument above to this concrete element. We can repeat the argument just applying it to x_0 (if $x_0 = 0$ then $x = y_0 \in Y$, which contradicts our assumption). Hence, there are elements $x_1 \in X$ and $y_1 \in Y$ such that

$$\begin{cases} x_0 = x_1 + y_1 \text{ with } x_1 \in X, \text{ and } y_1 \in Y \\ \|x_1\|_X < \rho^{1/p}\|x_0\|_X < (\rho^{1/p})^2\|x\|_X \\ \|y_1\|_Y < t_0^{-1}(\rho^{1/p})^2\|x\|_X \end{cases}$$

Moreover, $x = x_0 + y_0 = x_1 + y_1 + y_0$. Again $x_1 \neq 0$ since $x \notin Y$.

The Quasi-Banach setting

We can repeat the argument m times to get a decomposition $x = x_m + y_m + \cdots + y_0$ with $x_m \in X$, $x_m \neq 0$, $y_k \in Y$ for all $0 \leq k \leq m$ and

$$\begin{cases} \|x_m\|_X < (\rho^{1/p})^{m+1}\|x\|_X \\ \|y_k\|_Y < t_0^{-1}(\rho^{1/p})^{k+1}\|x\|_X \text{ for all } 0 \leq k \leq m. \end{cases}$$

Let us set $z_m = x - x_m = y_0 + \cdots + y_m$. Then

$$\|x - z_m\|_X = \|x_m\|_X < (\rho^{1/p})^{m+1}\|x\|_X \to 0 \text{ for } m \to \infty.$$

and x is the limit of z_m in the norm of X.

The Quasi-Banach setting

On the other hand, if $n > m$, then

$$\|z_n - z_m\|_Y^p = \|y_{m+1} + \cdots + y_n\|_Y^p \leq \sum_{k=m+1}^{n} \|y_k\|_Y^p \leq t_0^{-p} \|x\|_X^p \sum_{k=m+1}^{n} \rho^{k+1},$$

which converges to 0 for $n, m \to \infty$. Hence $\{z_m\}$ is a Cauchy sequence in Y and its limit belongs to Y since Y is topologically complete. This implies $x \in Y$, which contradicts our assumptions. Thus, we have demonstrated, for p-normed quasi-Banach spaces X and Y satisfying $Y \hookrightarrow X$, that if $K(S(X), t_0, X, Y) < 1$ for a certain $t_0 > 0$, then $X = Y$. In particular, if $X \neq Y$ then $K(S(X), t, X, Y) = 1$ for all $t > 0$.

The Quasi-Banach setting

In the case we consider a couple (X, Y) where $(X, \|\cdot\|_X)$ is a quasi-Banach space and $Y \subset X$ is a quasi-semi-normed space $(Y, \|\cdot\|_Y)$ which is continuously included into X, we can no longer guarantee the slow decay of the associated K-functional. For example, in 1995 Z. Ditzian, V.H. Hristov, K.G. Ivanov, proved that, if $0 < p < 1$ and $r \in \mathbb{N}$ is a positive natural number, then

$$K(f, t, L_p, W_p^r) = 0 \text{ for all } f \in L_p[0,1],$$

when we deal the Sobolev space W_p^r with the quasi-seminorm $\|g\|_{p,r} = \|g^{(r)}\|_{L_p[0,1]}$, so that

$$K(f, t, L_p, W_p^r) = \inf_{g^{(r-1)} \text{ is absolutely continuous}} (\|f - g\|_{L_p[0,1]} + t\|g^{(r)}\|_{L_p[0,1]}).$$

Lions' Problem

Lions's problem (formulated in the 1960's) was to prove that different parameters (θ, q) produce different interpolation spaces $(X, Y)_{\theta,q}$.

We can use the Theorem above, in conjunction to reiteration theorems, to (partially) solve the problem in quasi-Banach setting.

Lions' Problem

Theorem

Let (X, Y) *be a couple of quasi-Banach spaces. Assume that* $Y \hookrightarrow X$*, that* $Y \neq X$ *and that* Y *is not closed in* X*. Then*

$$Y \hookrightarrow (X, Y)_{\theta,q} \hookrightarrow X, \text{ with strict inclusions, for } 0 < \theta < 1 \text{ and } 0 < q \leq \infty.$$

Lions' Problem

Theorem

Let (X, Y) be a couple of infinite dimensional quasi-Banach spaces. Assume that $Y \hookrightarrow X$, that $Y \neq X$ and that Y is not closed in X.

(a) *Assume that $0 < \theta_0, \theta_1 < 1$, $\theta_0 \neq \theta_1$, and $0 < p, q \leq \infty$. Then*

$$(X, Y)_{\theta_0,p} \neq (X, Y)_{\theta_1,q}$$

(b) *Let $\theta \in]0, 1[$ and assume that $0 < p, q \leq \infty$ are such that $(X, Y)_{\theta,p} \neq (X, Y)_{\theta,q}$ and that $r_1, r_2 \in [p, q]$, $r_1 \neq r_2$. Then*

$$(X, Y)_{\theta,r_1} \neq (X, Y)_{\theta,r_2}$$

(c) *If $0 < \theta < 1$ and $0 < p \leq \infty$, then $(X, Y)_{\theta,p}$ is an infinite-codimensional subspace of X and Y is an infinite-codimensional subspace of $(X, Y)_{\theta,p}$.*

Lions' Problem

For the case that $Y \subset X$ is a quasi-semi-normed space, continuously embedded into X, we have that:

Theorem

Assume that (X, Y) is a couple, where $(X, \|\cdot\|_X)$ is a quasi-Banach space and $Y \subset X$ is a quasi-semi-normed space $(Y, \|\cdot\|_Y)$ which is continuously embedded into X. If the interpolation space $(X, Y)_{\theta_0, p_0}$ is strictly embedded into X for some choice of $0 < \theta_0 < 1$ and $0 < p_0 \leq \infty$, then

$$K(S(X), t, X, Y) > c \text{ for all } t > 0 \text{ and a certain } c > 0.$$

In particular, the strict inclusion of just one interpolation space $(X, Y)_{\theta_0, p_0}$ into X implies that all interpolation spaces $(X, Y)_{\theta, p}$ are strictly embedded into X, where $0 < \theta < 1$ and $0 < p \leq \infty$.

The talk was based on:

- J.M. ALMIRA, P. FERNÁNDEZ-MARTÍNEZ, On slow decay of Peetre'ss K-functional,Journal of Mathematical Analysis and Applications, **494**(2), 15 February 2021, 124653.
- S. JANSON, P. NILSSON, J. PEETRE, M. ZAFRAN, Notes on Wolff's note on interpolation spaces, Proc. London Math. Soc. (3), 48 (1984) 283-299.
- J. STAFNEY, Analytic interpolation of certain multiplier spaces, Pacific J. Math., **32** (1970) 241-248.

Thanks for your kind attention and
congratulations to Fernando for his 65-th birthday!