Desigualdades para el funcional de Wills de un cuerpo convexo.

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Joint work with M.A. Hernández Cifre and J. Yepes Nicolás

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Universidad de Zaragoza

28 de mayo de 2022

CONVEX BODIES.
THE WILLS FUNCTIONAL.

Convex bodies

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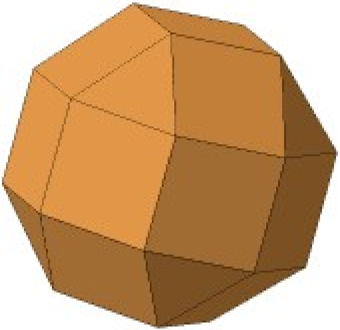

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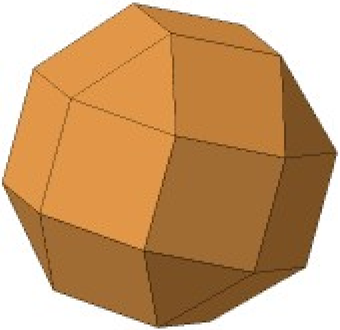

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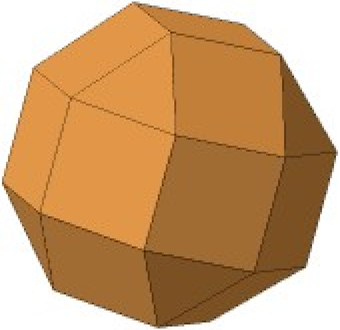

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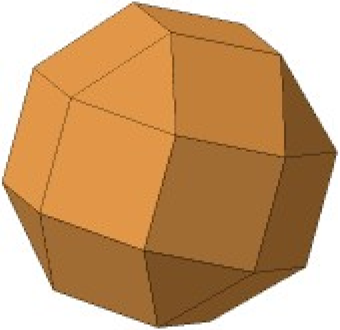
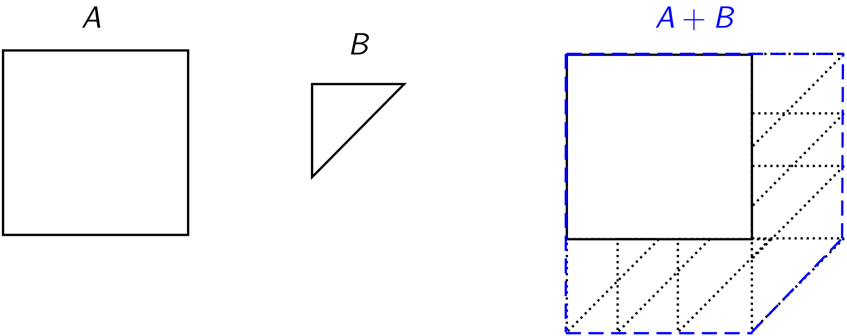


- We will not assume that K has non-empty interior.

The Minkowski sum

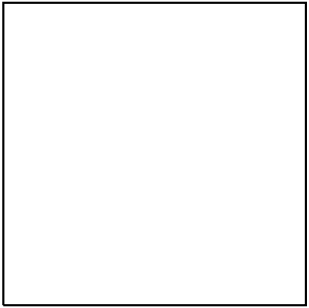
The Minkowski sum of two sets is defined as

$$A + B = \{x + y \,:\, x \in A, y \in B\} = \bigcup_{x \in A} (x + B)$$



The Steiner polynomial

- $|K + tB_2^n|_n = \sum_{i=0}^{n} \binom{n}{i} W_i(K) t^i$ (Steiner's polynomial)



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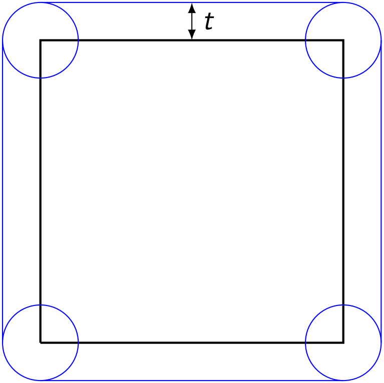

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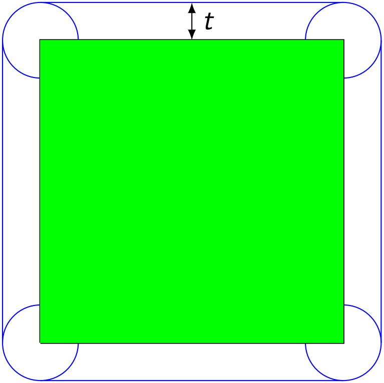

- $A(K + tB_2^2) = A$

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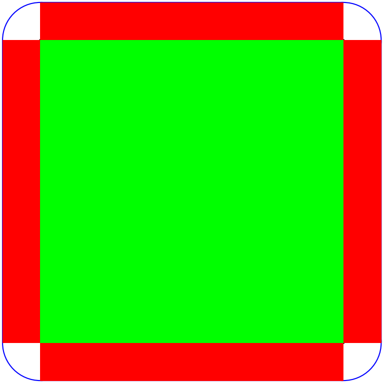

- $A(K + tB_2^2) = A + Lt$

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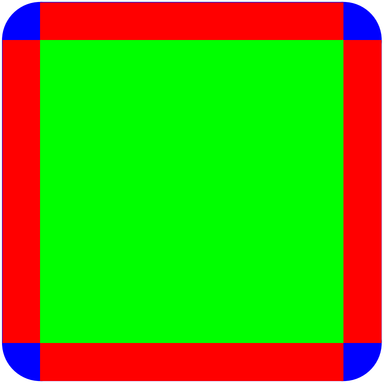

- $A(K + tB_2^2) = A + Lt + \pi t^2.$

The quermaßintegrals

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- $W_n(K) = |B_2^n|_n$
- $W_i(K) = \frac{|B_2^n|_n}{|B_2^{n-i}|_{n-i}} \int_{G_{n,n-i}} |P_F(K)|_{n-i} d\nu(F)$ (Kubota's formula)

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- In such case $W_i = 0$ for $0 \leq i \leq n-k-1$ and

$$\frac{\binom{k}{i}}{|B_2^i|_i} W_i^{(k)}(K) = \frac{\binom{n}{n-k+i}}{|B_2^{n-k+i}|_{n-k+i}} W_{n-k+i}(K) \quad 0 \leq i \leq k$$

Intrinsic volumes and the Wills functional

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- For any $\lambda > 0$

$$\mathcal{W}(\lambda K) = 1 + \sum_{i=1}^{n} \lambda^i V_i(K).$$

LOG-CONCAVE FUNCTIONS

Log-concave functions

$f : \mathbb{R}^n \to [0, \infty)$ is log-concave if

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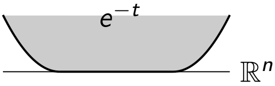
If f is integrable in $\mathbb{R}^n$ then $\frac{f(x)}{\|f\|_\infty} = e^{-v(x)}$ with $v : \mathbb{R}^n \to [0, \infty]$ convex and

$$\int_{\mathbb{R}^n} \frac{f(x)}{\|f\|_\infty} dx = \int_0^\infty e^{-t}|K_t|dt = \int_L e^{-t} dt dx,$$

where

$$K_t = \{x \in \mathbb{R}^n \ : \ f(x) \geq e^{-t}\|f\|_\infty\}$$

$$\begin{aligned} L &= \operatorname{epi}(v) = \{(x,t) \in \mathbb{R}^{n+1} \ : \ v(x) \leq t\} \\ &= \{(x,t) \in \mathbb{R}^{n+1} \ : \ f(x) \geq e^{-t}\|f\|_\infty\} \end{aligned}$$



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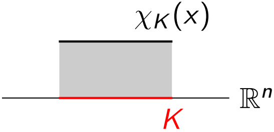
Given a convex body $K \subseteq \mathbb{R}$

- χ_K is log-concave and $|K| = \int_{\mathbb{R}^n} \chi_K(x)dx$



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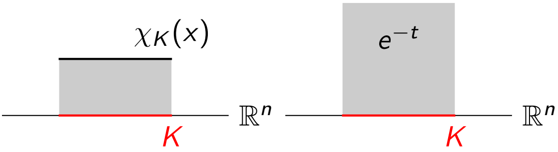

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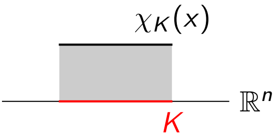
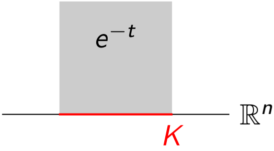
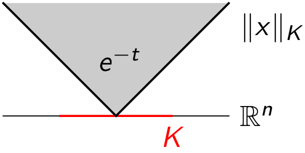




- If $0 \in \mathrm{int}K$ $e^{-\|x\|_K}$ is log-concave and,
  $|K| = \frac{1}{n!}\int_{\mathbb{R}^n} e^{-\|x\|_K}dx = \frac{1}{n!}\int_{\mathrm{epi}(\|\cdot\|_K)} e^{-t}dtdx$



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- Many geometric inequalities have been extended to the setting of log-concave functions, recovering the geometric inequalities when applied to $\chi_K(x)$ or $e^{-\|x\|_K}$.

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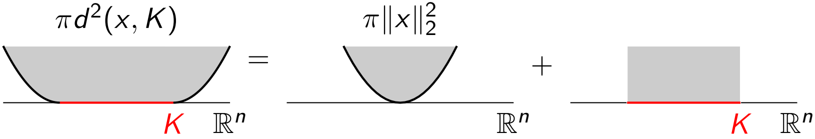
- First observation (Hadwiger, 1975)

$$\int_{\mathbb{R}^n} e^{-\pi d^2(x,K)}dx = \int_0^\infty e^{-t}\left|K + \sqrt{\frac{t}{\pi}}B_2^n\right| dt$$

$$= \int_0^\infty e^{-t}\sum_{i=0}^n \binom{n}{i} W_i(K)\left(\frac{t}{\pi}\right)^{\frac{i}{2}} dt = \sum_{i=0}^n V_i(K) = \mathcal{W}(K).$$

- Second observation (A-G, Hernández Cifre, Yepes, 2021): $e^{-\pi d^2(x,K)}$ is log-concave and

$$e^{-\pi d^2(x,K)} = (e^{-\pi\|\cdot\|_2^2} \star \chi_K)(x)$$



SOME INEQUALITIES

Brunn-Minkowski inequality

Brunn-Minkowski inequality (1896)

For any convex bodies $K, L \subseteq \mathbb{R}^n$ and any $\lambda \in [0,1]$

- $|K + L|^{\frac{1}{n}} \geq |K|^{\frac{1}{n}} + |L|^{\frac{1}{n}}$
- $|(1-\lambda)K + \lambda L|^{\frac{1}{n}} \geq (1-\lambda)|K|^{\frac{1}{n}} + \lambda |L|^{\frac{1}{n}}$
- $|(1-\lambda)K + \lambda L| \geq |K|^{1-\lambda}|L|^{\lambda}$

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- $|(1-\lambda)K + \lambda L| \geq |K|^{1-\lambda}|L|^{\lambda}$

The three forms of the inequality are equivalent.

Prékopa-Leindler inequality

Prékopa-Leindler inequality (1971)

Let $f, g, h : \mathbb{R}^n \to [0, \infty)$ be measurable functions and let $\lambda \in [0, 1]$. Assume that for any $x, y \in \mathbb{R}^n$

$$h((1-\lambda)x + \lambda y) \geq f(x)^{1-\lambda} g(y)^{\lambda}.$$

Then,

$$\int_{\mathbb{R}^n} h(z)dz \geq \left(\int_{\mathbb{R}^n} f(x)dx \right)^{1-\lambda} \left(\int_{\mathbb{R}^n} g(y)dy \right)^{\lambda}.$$

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Given $K, L \subseteq \mathbb{R}^n$ convex bodies and $\lambda \in [0, 1]$, taking

$$f(x) = \chi_K(x), \qquad g(y) = \chi_L(y), \qquad h(z) = \chi_{(1-\lambda)K+\lambda L}(z),$$

we obtain Brunn-Minkowski inequality

$$|(1-\lambda)K + \lambda L| \geq |K|^{1-\lambda} |L|^{\lambda}.$$

Brunn-Minkwoski type inequality for the Wills functional

Given $K, L \subseteq \mathbb{R}^n$ convex bodies and $\lambda \in [0,1]$, taking

$$f(x) = e^{-\pi d^2(x,K)}, \qquad g(y) = e^{-\pi d^2(y,L)}, \qquad h(z) = e^{-\pi d^2(z,(1-\lambda)K+\lambda L)},$$

we obtain

Theorem (A-G, Hernández Cifre, Yepes (2021))

Let $K, L \subseteq \mathbb{R}^n$ be convex bodies and $\lambda \in (0,1)$

$$\mathcal{W}((1-\lambda)K + \lambda L) \geq \mathcal{W}(K)^{1-\lambda}\mathcal{W}(L)^{\lambda}$$

Brunn-Minkwoski type inequality for the Wills functional

Taking

$$f(x) = e^{-\pi(1-\lambda)\|x\|_2^2}, \quad g(y) = \chi_{\frac{K}{\lambda}}(y), \quad h(z) = e^{-\pi\|\cdot\|_2^2} \star \chi_K(z) = e^{-\pi d^2(z,K)}$$

we obtain

Theorem (A-G, Hernández Cifre, Yepes (2021))

Let $K \subseteq \mathbb{R}^n$ be convex bodies and $\lambda \in (0,1)$

$$\mathcal{W}(K) \geq \left(\frac{1}{\lambda^{\lambda}(1-\lambda)^{\frac{1-\lambda}{2}}} \right)^n |K|^{\lambda}.$$

Rogers-Shephard inequality

Rogers-Shephard inequality (1957)

$$|K - K| \leq \binom{2n}{n} |K|$$

with equality if and only if K is a simplex.

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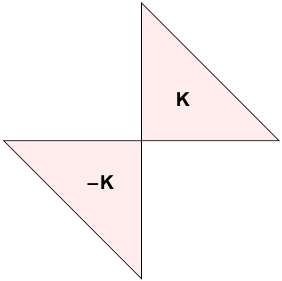

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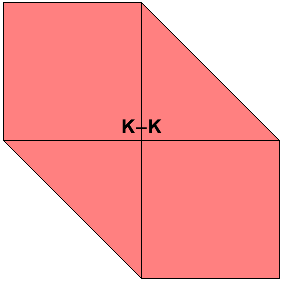

$$\binom{2 \cdot 2}{2} = 6$$

Rogers-Shephard inequality

Theorem (A-G, González, Jiménez, Villa (2016))

Let $f : \mathbb{R}^n \to [0, \infty)$ be an integrable log-concave function and $\overline{f}(x) = f(-x)$. Then

$$\int_{\mathbb{R}^n} f \star \overline{f}(x)dx \leq \binom{2n}{n} \|f\|_\infty \int_{\mathbb{R}^n} f(x)dx.$$

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Taking $f(x)=\chi_K(x)$ we obtain Rogers-Shephard inequality

$$|K-K|\leq\binom{2n}{n}|K|.$$

Rogers-Shephard inequality for the Wills functional

Taking $f(x) = e^{-\pi d^2(x,K)} = e^{-\pi\|\cdot\|_2^2} \star \chi_K(x)$ we have that

$$f \star \overline{f} \quad = \quad (e^{-\pi\|\cdot\|_2^2} \star \chi_K(\cdot)) \star (e^{-\pi\|\cdot\|_2^2} \star \chi_{-K}(\cdot))$$

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Taking $f(x) = e^{-\pi d^2(x,K)} = e^{-\pi\|\cdot\|_2^2} \star \chi_K(x)$ we have that

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and we obtain

Theorem (A-G, Hernández Cifre, Yepes (2021))

Let $K \subseteq \mathbb{R}^n$ be a convex body. Then

$$\mathcal{W}\left(\frac{K-K}{\sqrt{2}}\right) \leq \frac{\binom{2n}{n}}{2^{\frac{n}{2}}} \mathcal{W}(K).$$

...and many more

Let $K, L \subseteq \mathbb{R}^n$ be convex bodies and $\lambda \in [0,1]$. Assume that $0 \in K$ and let $H \in G_{n,k}$. Then

- $\mathcal{W}(K \cap L)\mathcal{W}\left(\frac{K-L}{2}\right) \leq 2^n \mathcal{W}(K)\mathcal{W}(L)$

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- If K is symmetric and in John's position

$$\mathcal{W}(\lambda K) \leq \mathcal{W}(\lambda[-1,1]^n)$$

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- $e^{V_1(K) - \pi R(K)^2} \leq \mathcal{W}(K) \leq e^{V_1(K)}$

...and many more

Let $K, L \subseteq \mathbb{R}^n$ be convex bodies and $\lambda \in [0,1]$. Assume that $0 \in K$ and let $H \in G_{n,k}$. Then

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- $e^{V_1(K) - \pi R(K)^2} \leq \mathcal{W}(K) \leq e^{V_1(K)}$
- $\mathcal{W}((1-\lambda)K + \lambda L)^{\frac{1}{n}} \geq \frac{1}{(n!)^{\frac{1}{n}}}((1-\lambda)\mathcal{W}(K)^{\frac{1}{n}} + \lambda \mathcal{W}(L)^{\frac{1}{n}})$

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