# Decay of correlations for Gibbs states of local Hamiltonians

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Joint work with Andreas Bluhm (U. Copenhagen) and Ángela Capel (U. Tübingen)

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# Quantum systems in a nutshell

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- The **expectation** of an observable $T$ with respect to a state $\rho$ is $\mathrm{Tr}[\rho T]$.

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- Every state $\Phi$ on $\mathfrak{A}_{AC} = \mathfrak{A}_A \otimes \mathfrak{A}_C$ induces a state (**marginal**) on each subsystem

$$\Phi_A : \mathfrak{A}_A \longrightarrow \mathbb{C}\,, \; Q \mapsto \Phi(Q \otimes \mathbf{1}_C)$$

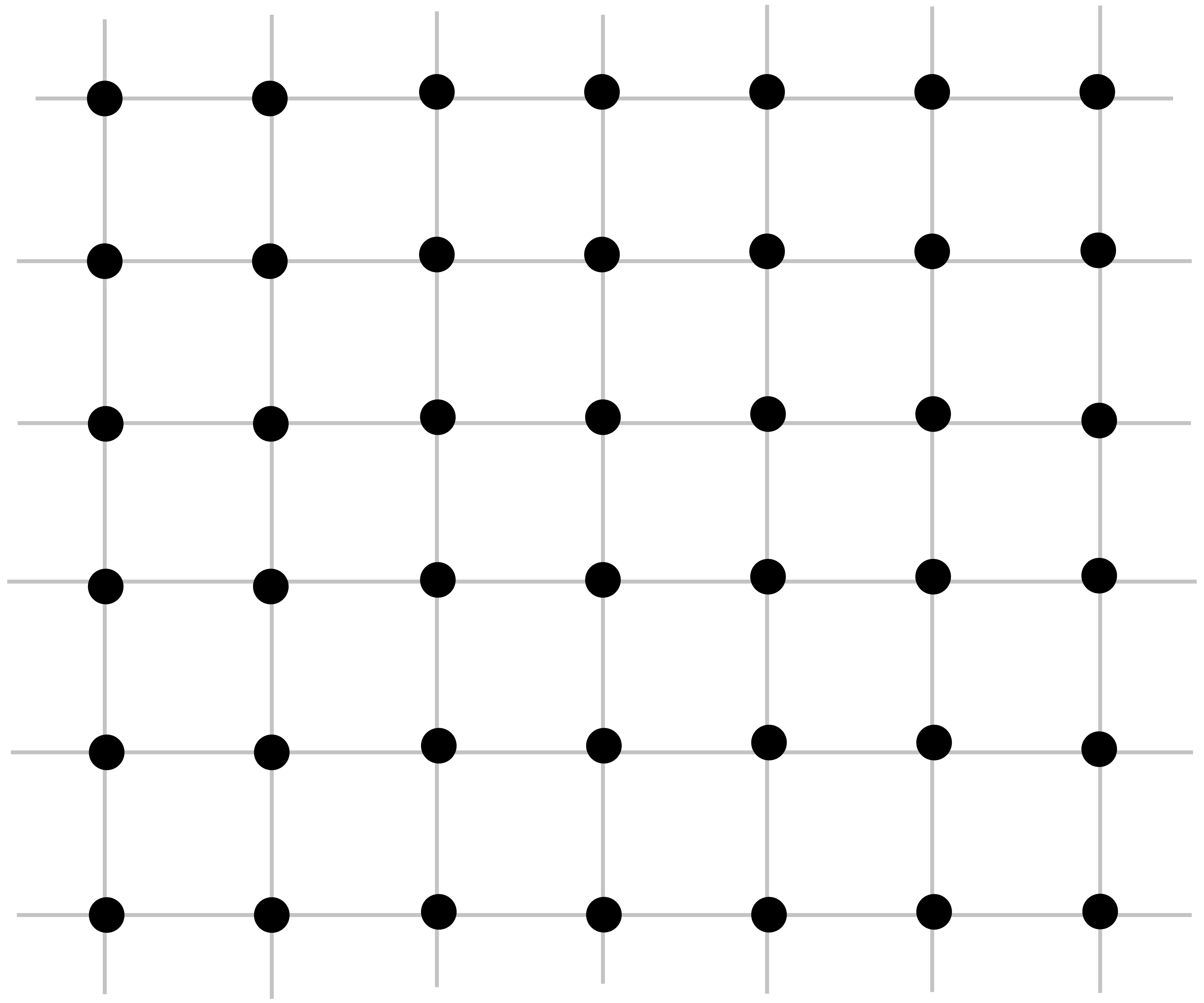
If $\rho$ and $\rho_A$ is the density operators of $\Phi$ and $\Phi_A$, respectively, then $\rho_A = (\mathrm{Id}_A \otimes \mathrm{Tr}_C)(\rho)$ .

1. **Quantum spin systems**

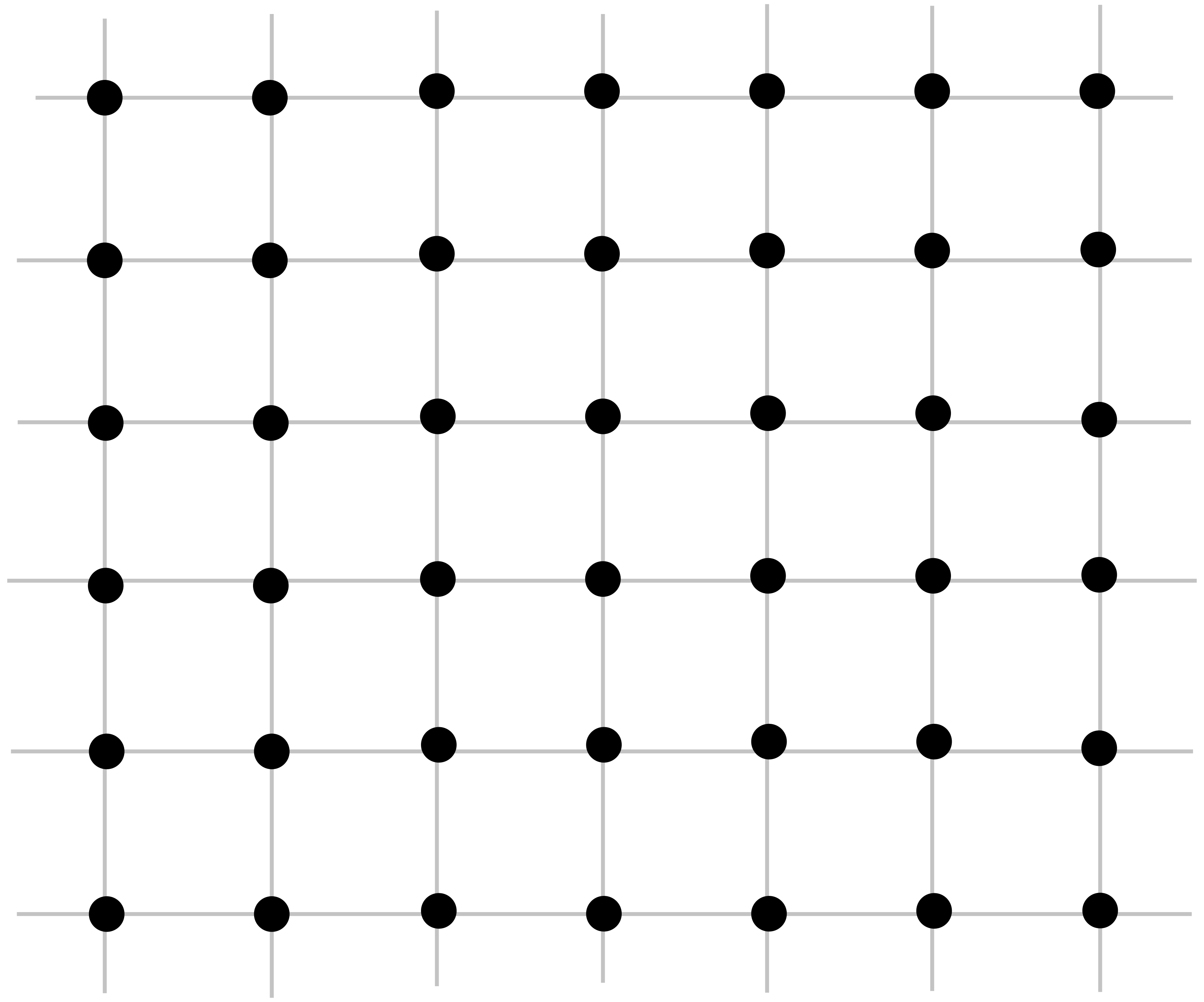
2. **Operator correlation function**

3. **Mutual Information**

4. **Approximate recoverability**



Regular lattice $\mathbb{Z}^D$



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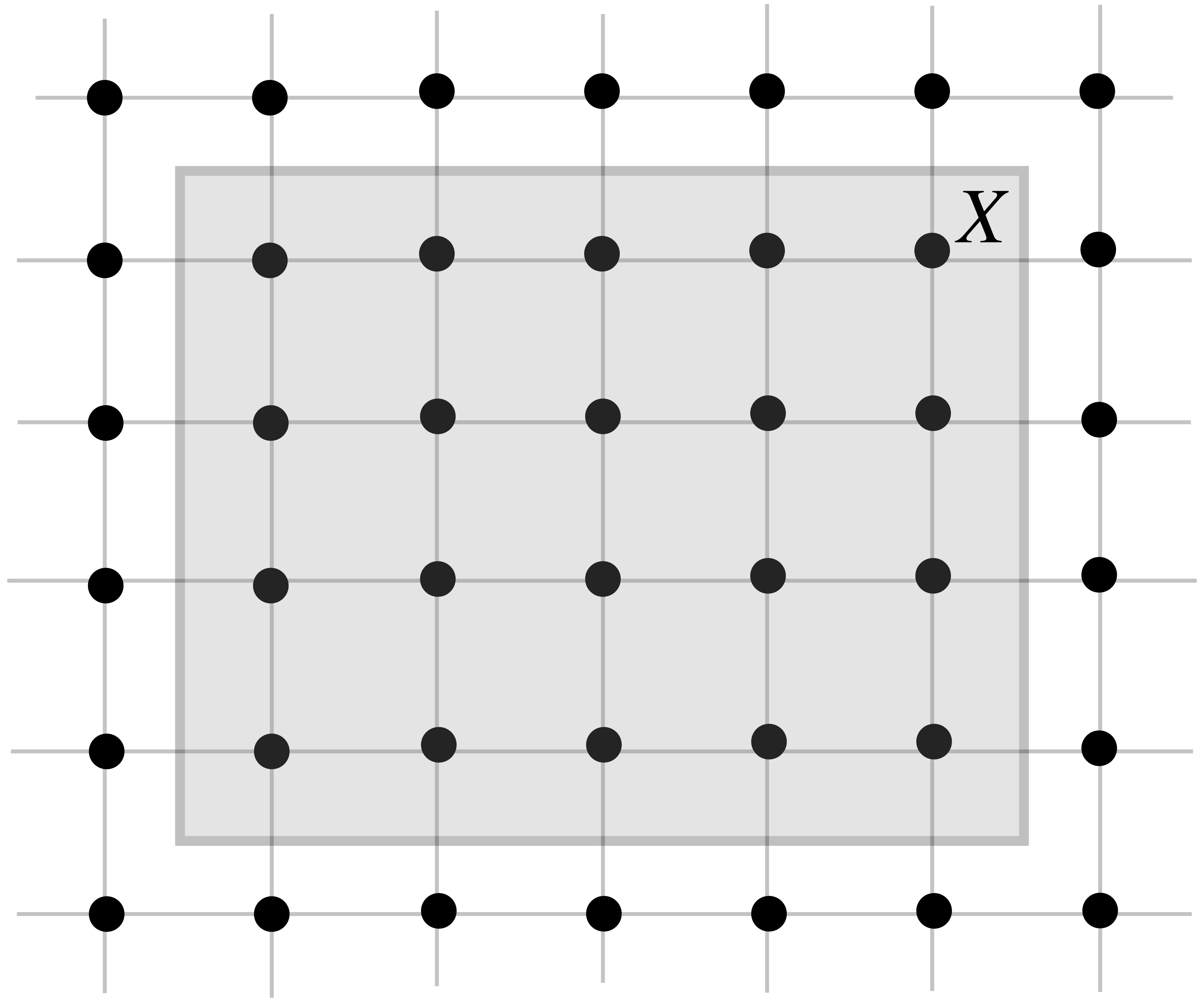


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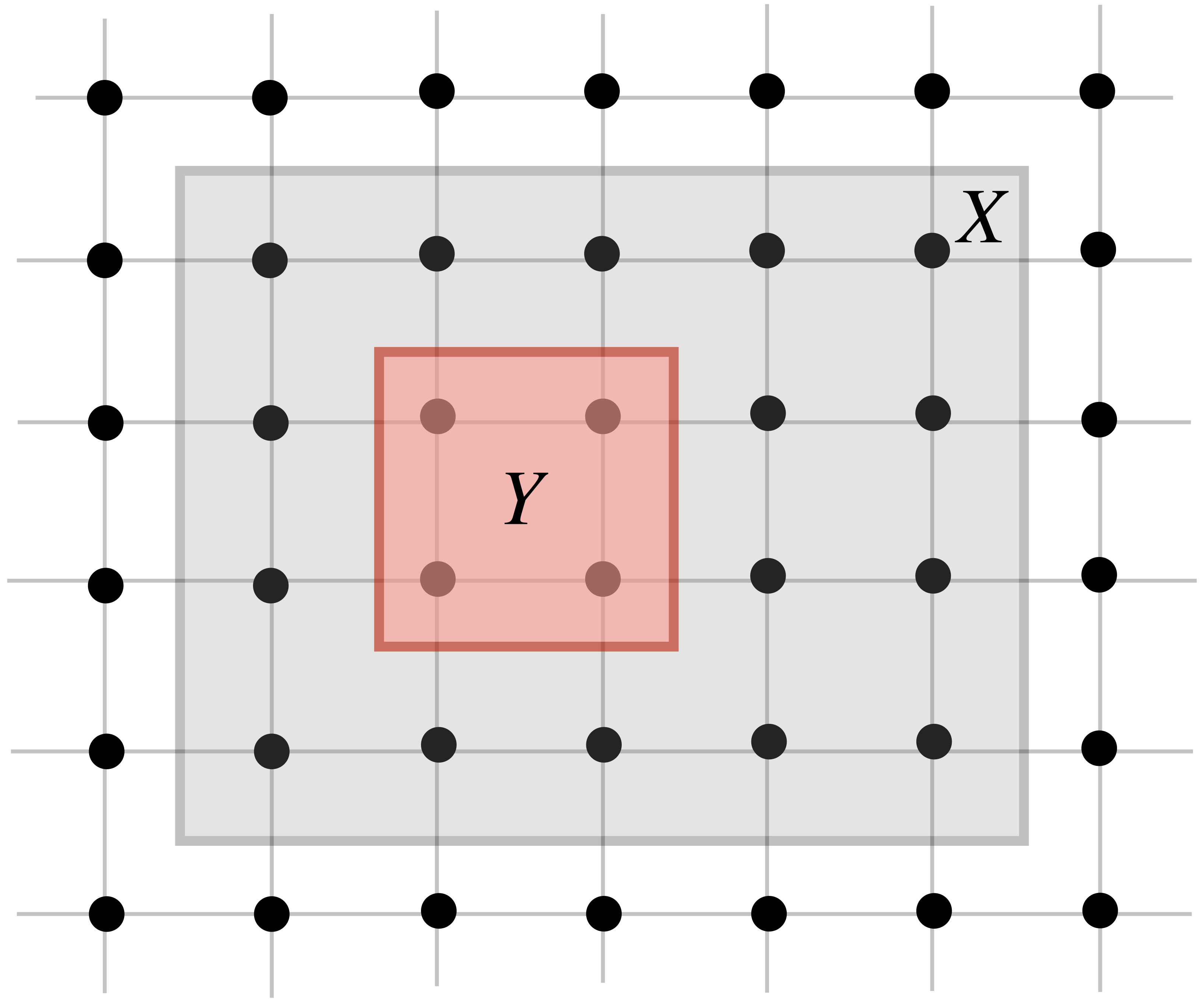


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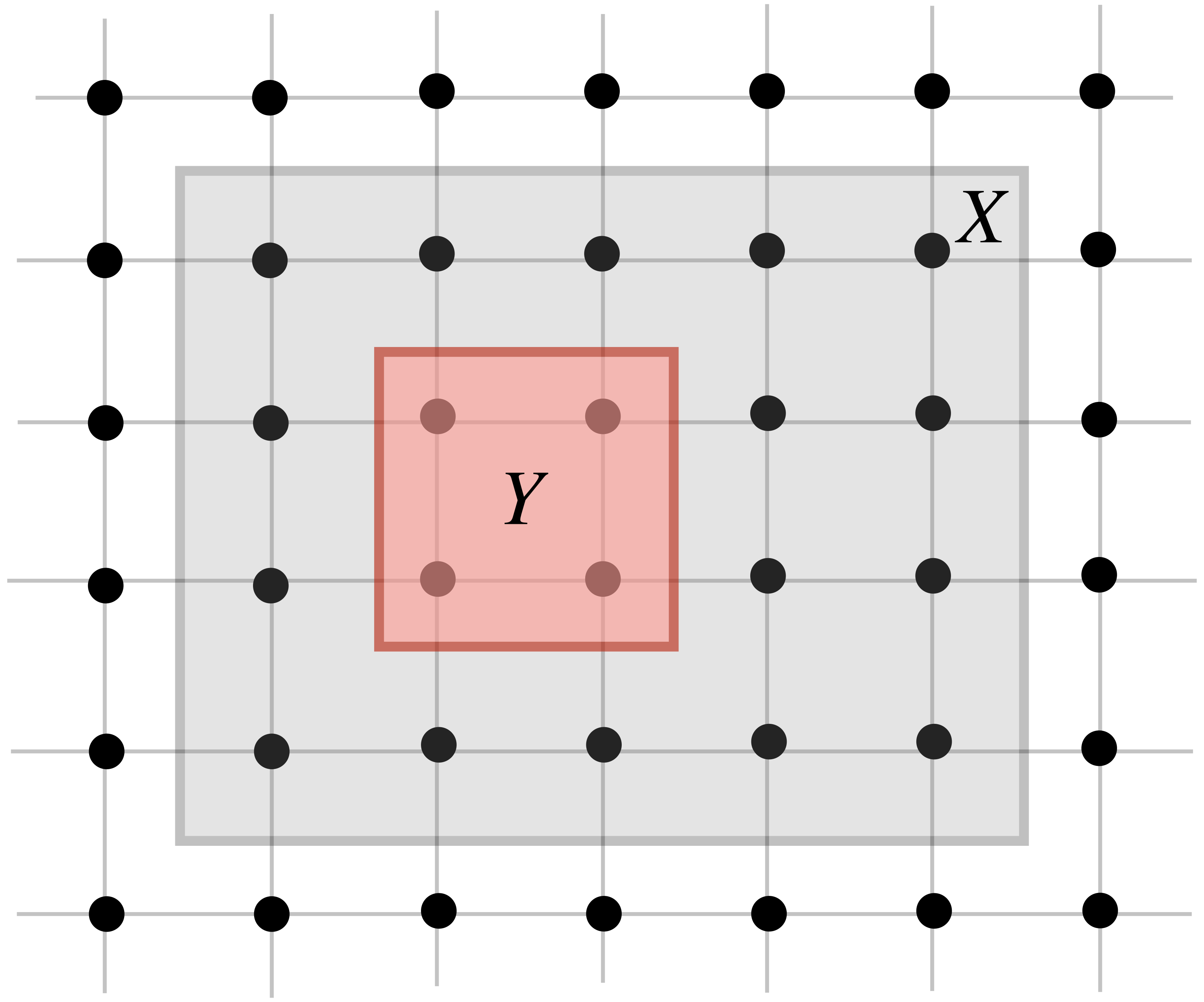
If $Y \subset X \subset \mathbb{Z}^D$ are finite subsets, there is a canonical linear isometry

$$\mathfrak{A}_Y \hookrightarrow \mathfrak{A}_X \qquad Q \mapsto Q \otimes \mathbf{1}_{X \setminus Y}$$

(supported in Y)

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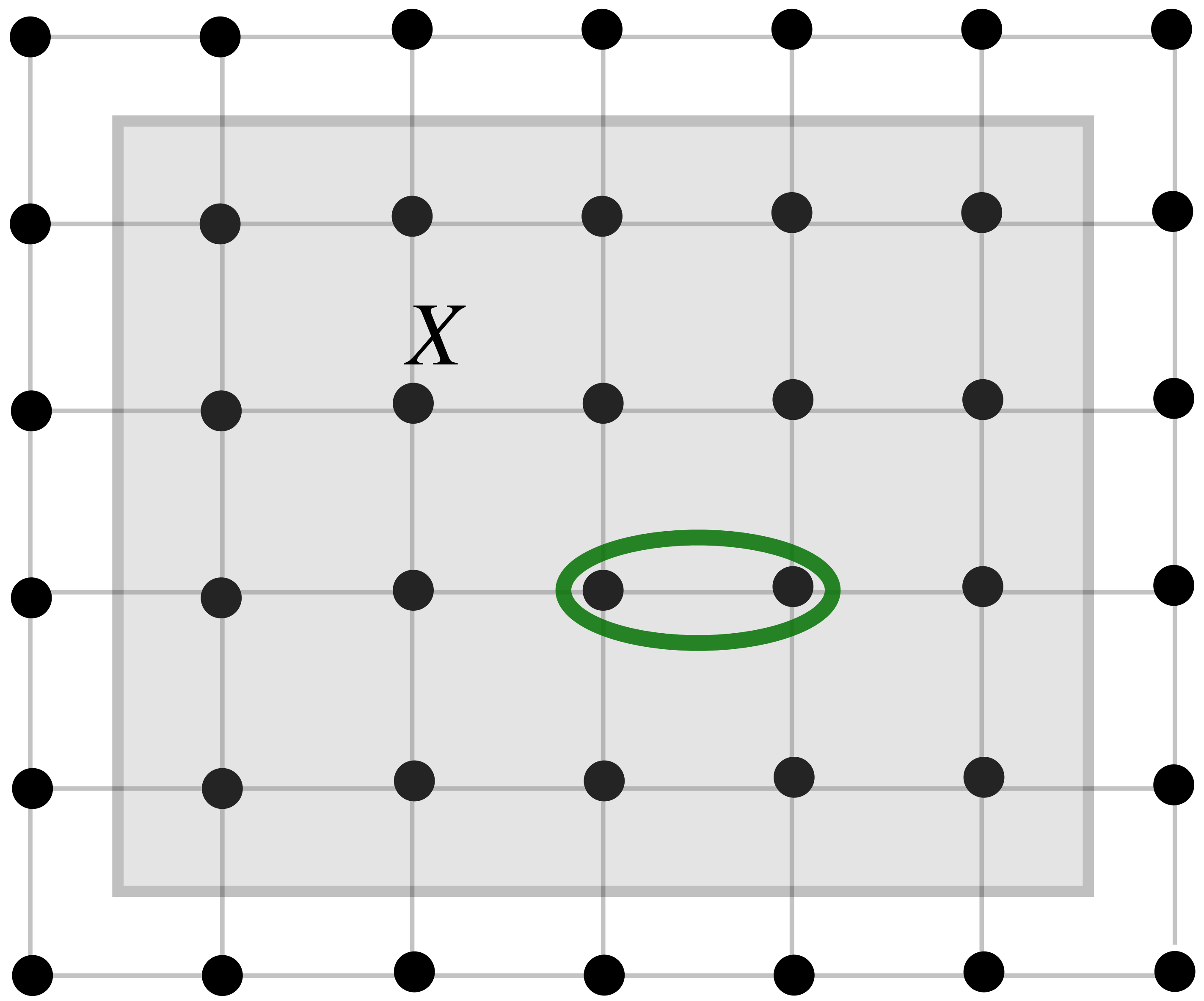
This allows to consider the algebra of local observables, and the algebra of quasi-local observables (its completion)

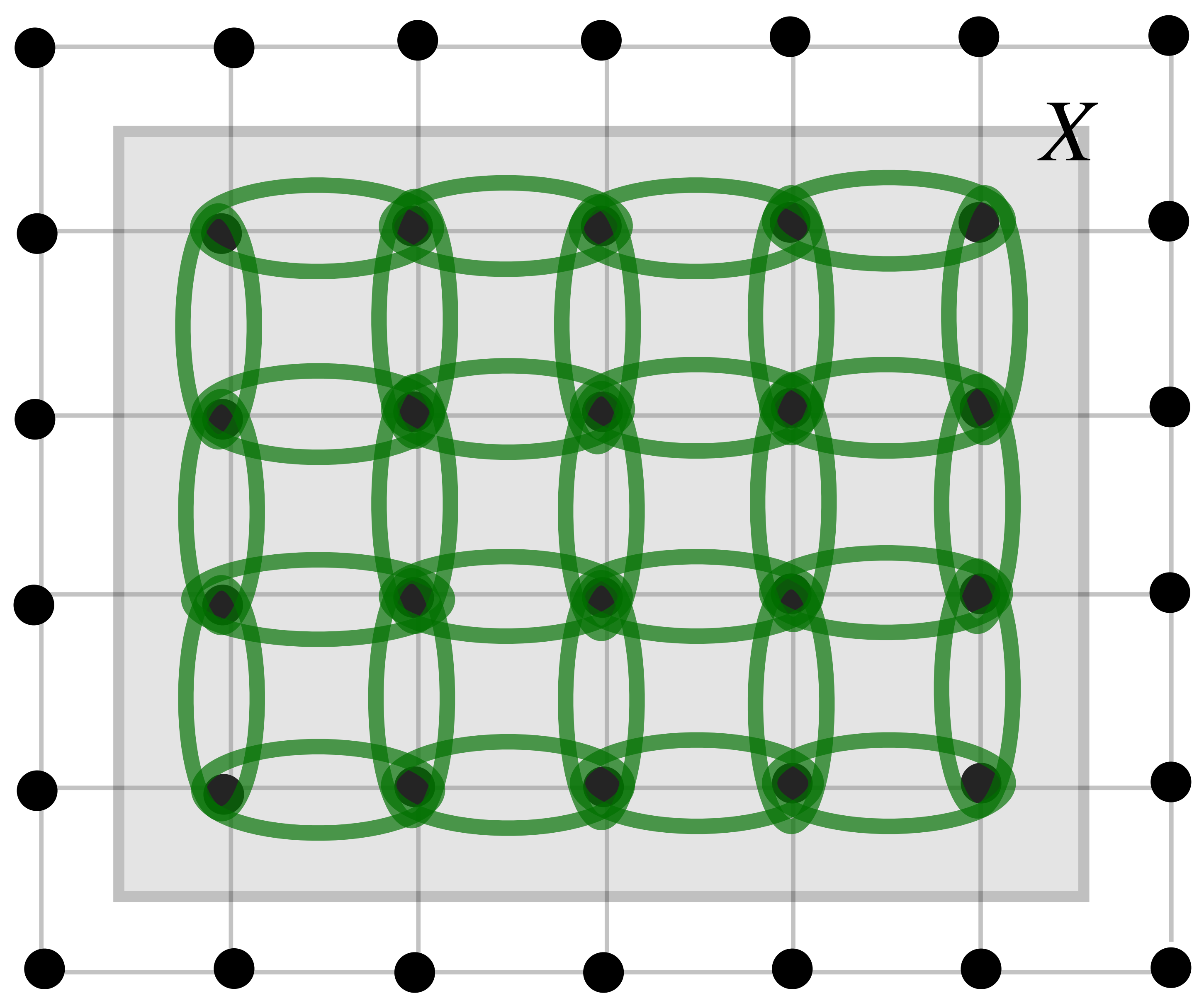
$$\mathfrak{A}_{loc} = \bigcup_{X \text{ finite}} \mathfrak{A}_X \qquad \mathfrak{A}_{\mathbb{Z}^D} = \overline{\mathfrak{A}_{loc}}$$




For every pair of neighboring sites $\langle x, y \rangle \subset \mathbb{Z}^D$ let us fix a **local interaction**

$$h_{\langle x,y \rangle} : \mathbb{C}^d \otimes \mathbb{C}^d \longrightarrow \mathbb{C}^d \otimes \mathbb{C}^d$$

$h_{\langle x,y \rangle}$ self-adjoint , $\|h_{\langle x,y \rangle}\| \leq 1$

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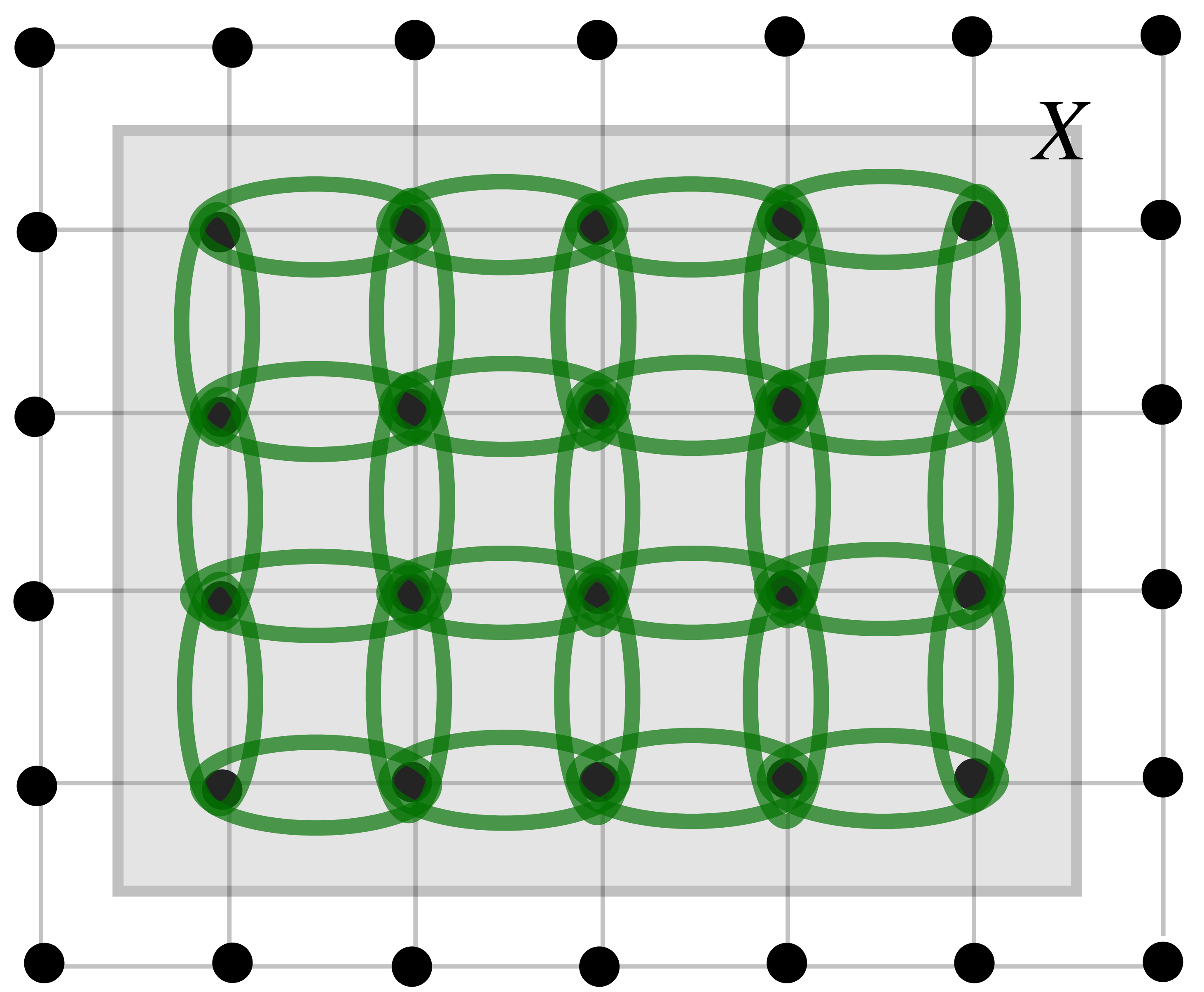
$$H_X = \sum_{\langle x,y\rangle \subset X} h_{\langle x,y\rangle} \otimes \mathrm{Id}_{rest}$$

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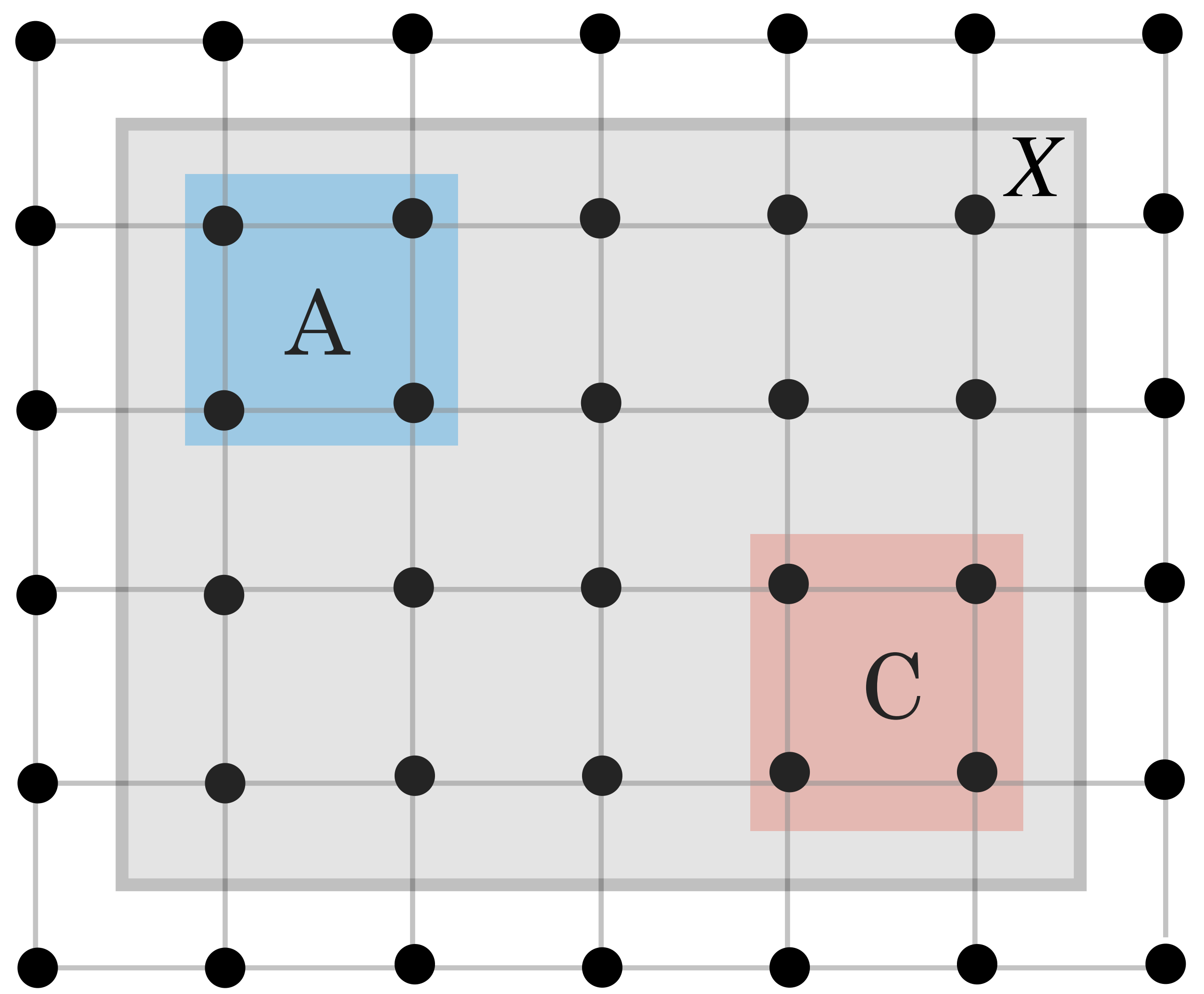
and the **Gibbs or thermal state**: for each $\beta > 0$ (a.k.a. inverse temperature $\beta = \frac{1}{Tk_B}$ )

$$\psi_X : \mathfrak{A}_X \longrightarrow \mathbb{C}$$

$$\psi_X(Q) := \mathrm{Tr}_X(\sigma^X Q) \quad , \quad Q \in \mathfrak{A}_X$$

$$\sigma^X = \sigma^X(\beta) := \frac{e^{-\beta H_X}}{\mathrm{Tr}(e^{-\beta H_X})}$$

Minimizes the Gibbs Free energy: $\rho \mapsto F_{\beta,X}(\rho)$

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**Gibbs or thermal state** at (inverse) temperature $\beta > 0$

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**Motivation:** Describe the correlation properties of Gibbs states between distant subregions $A$ and $C$

1. Setting

**2. Operator correlation function**

3. Mutual Information

4. Approximate recoverability

# Operator Correlation Function

$$\sigma^X = \frac{e^{-\beta H_X}}{\mathrm{Tr}(e^{-\beta H_X})}$$

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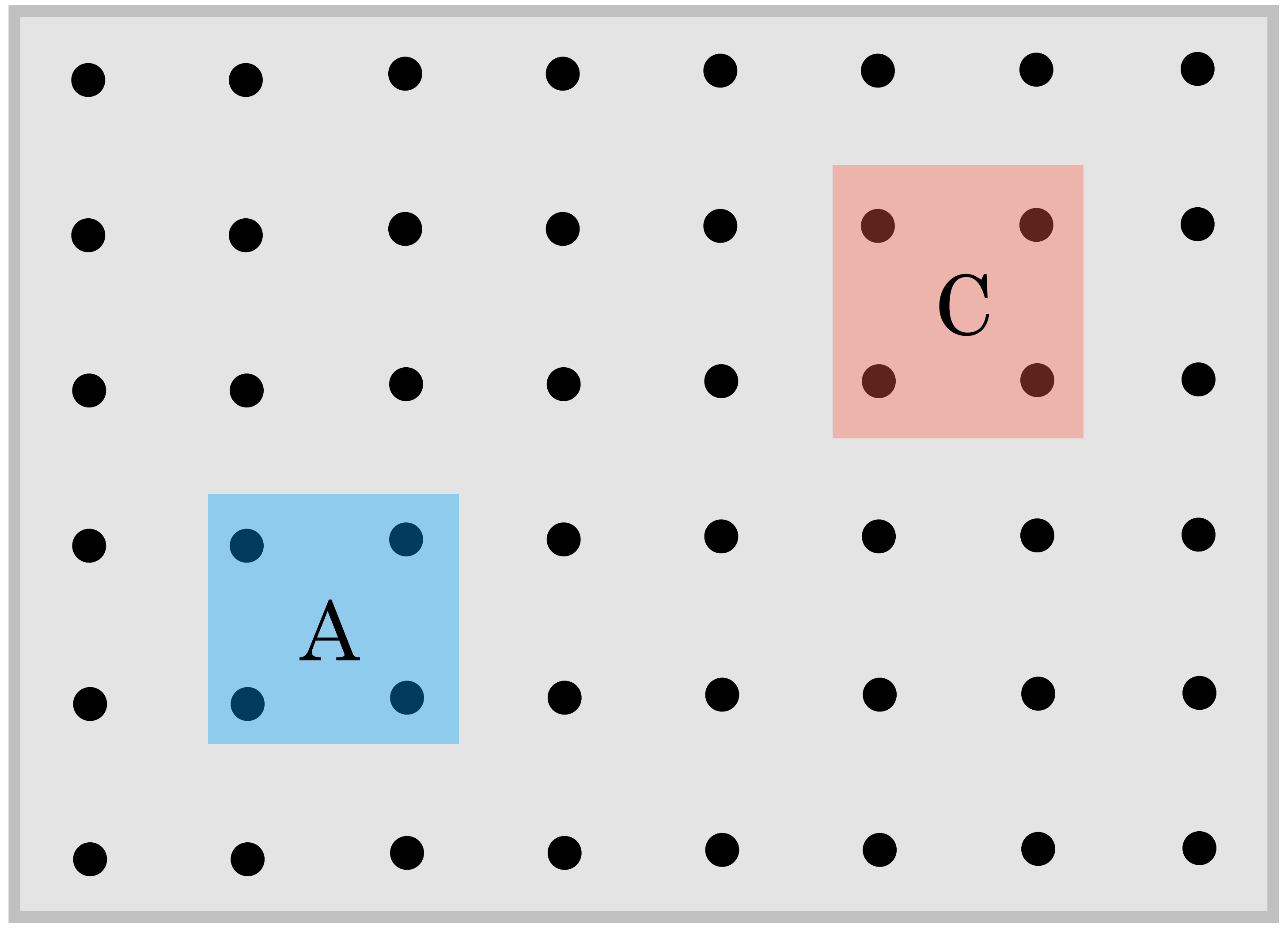


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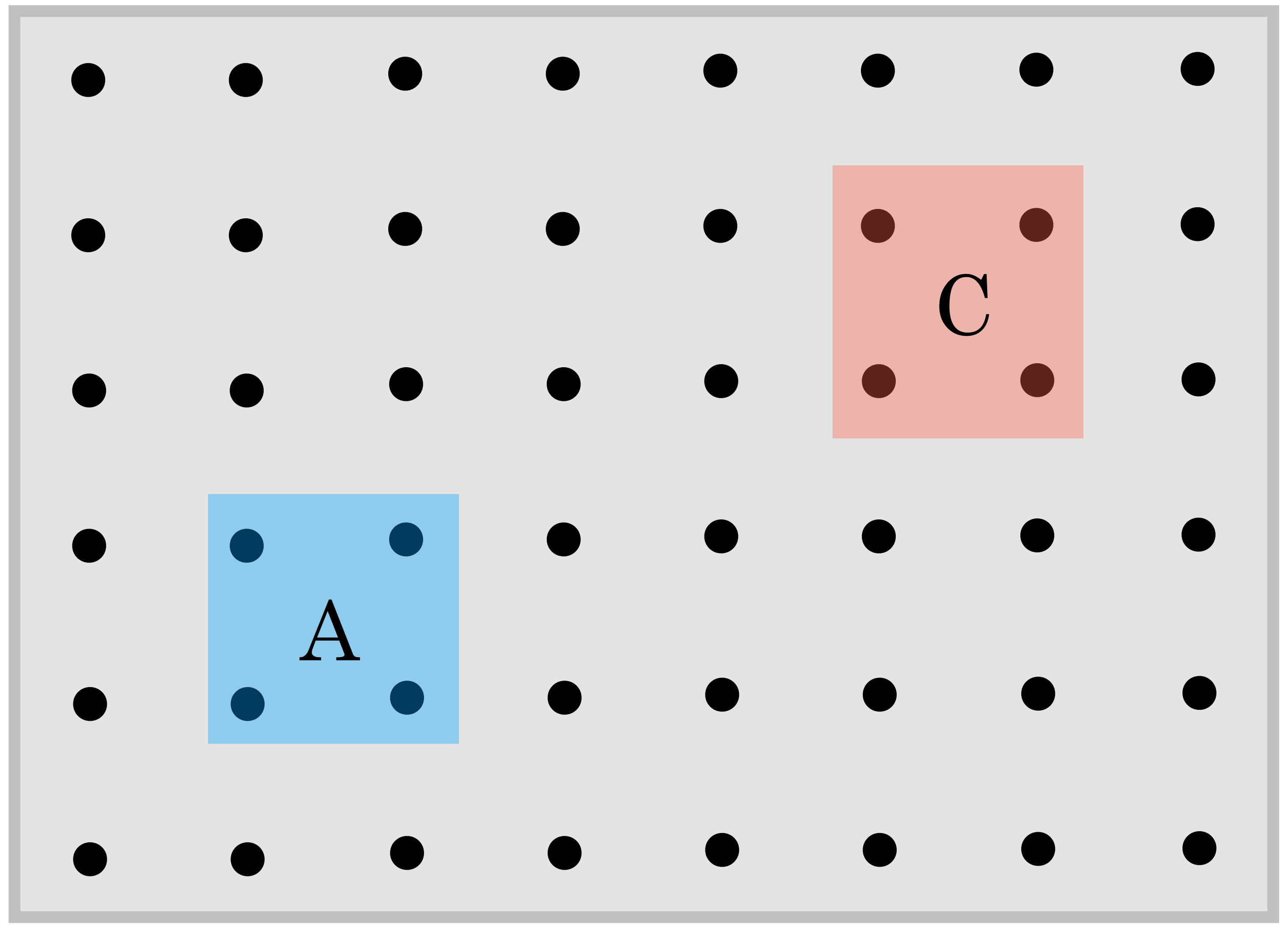


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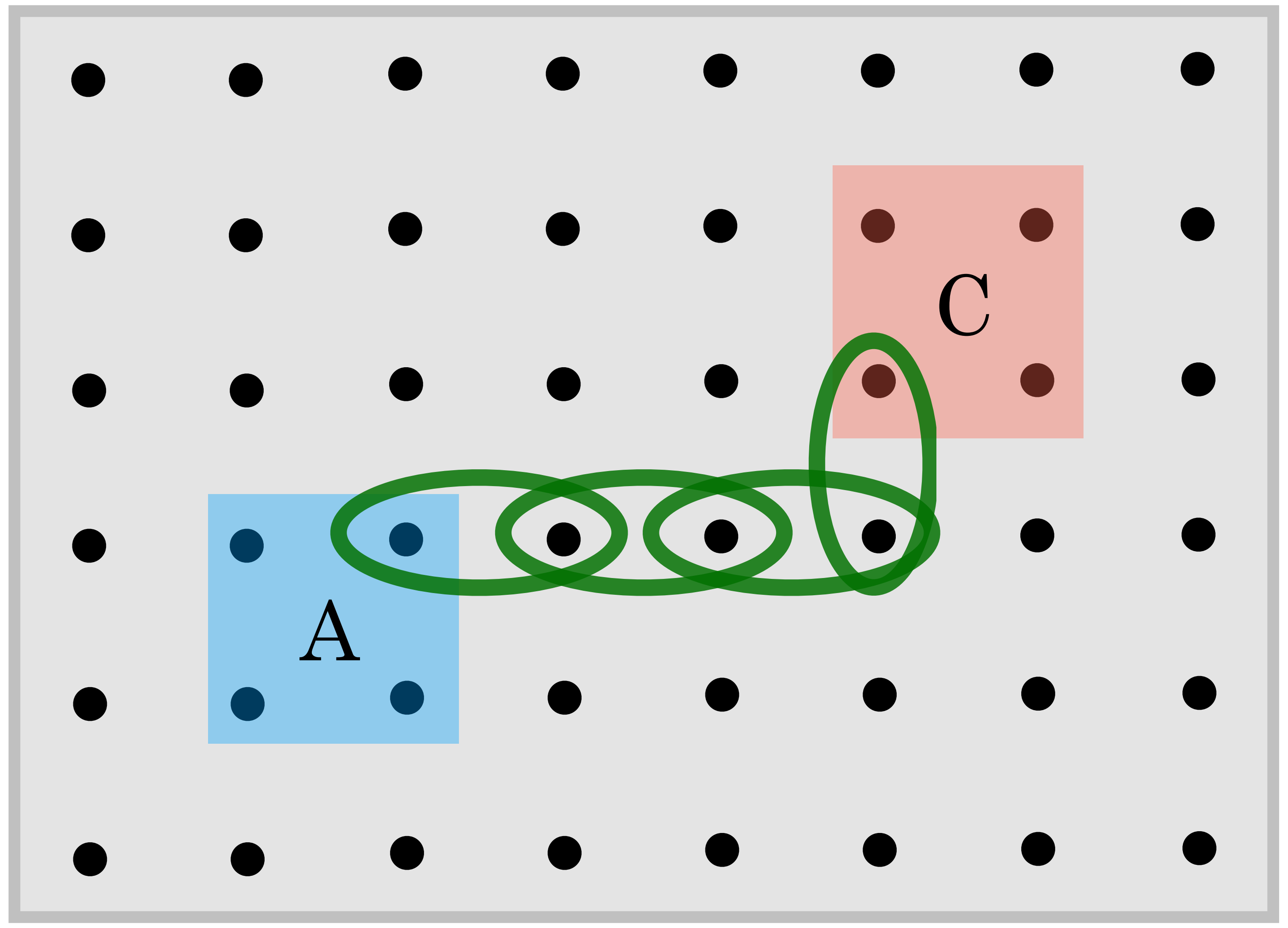


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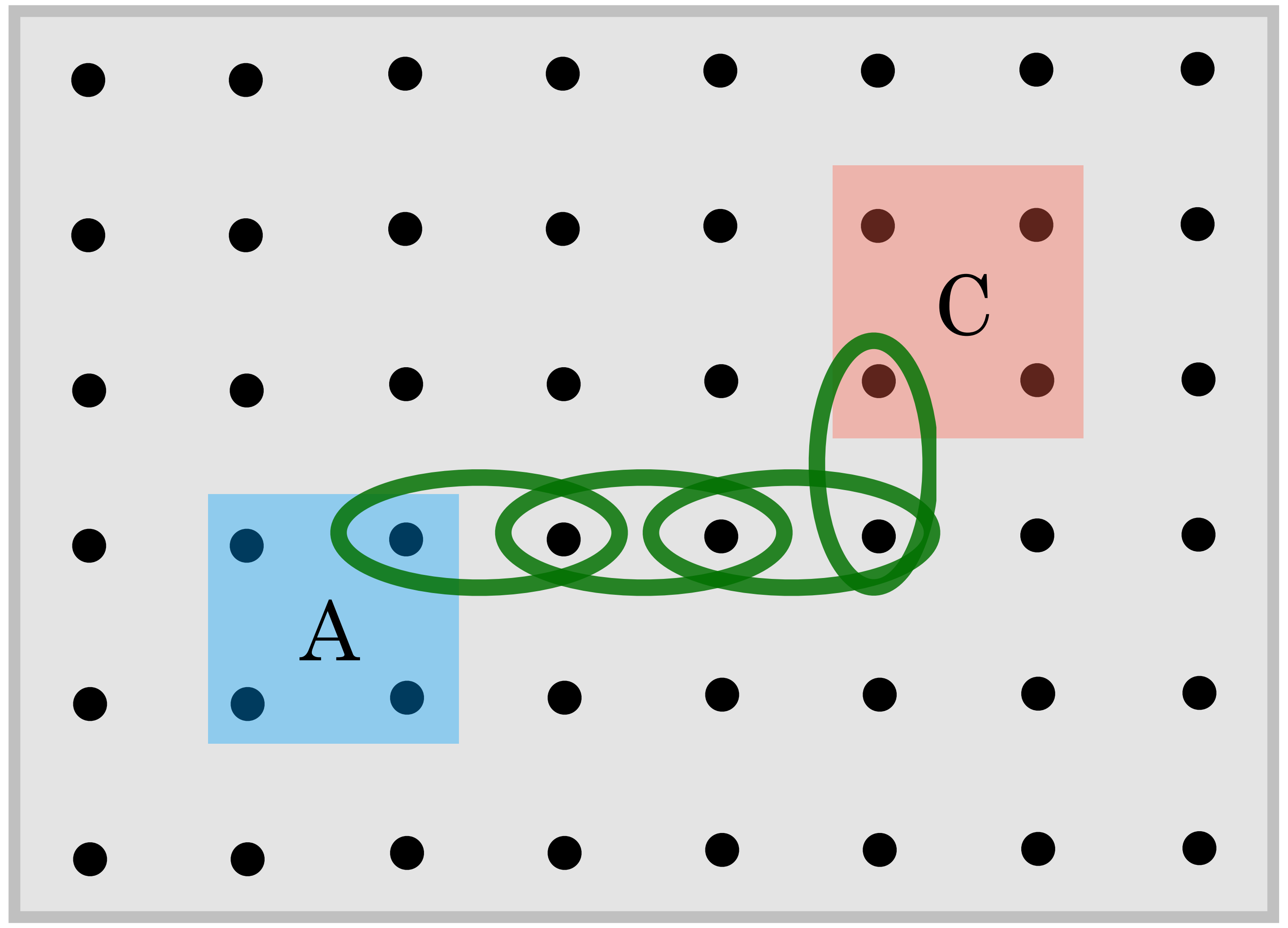


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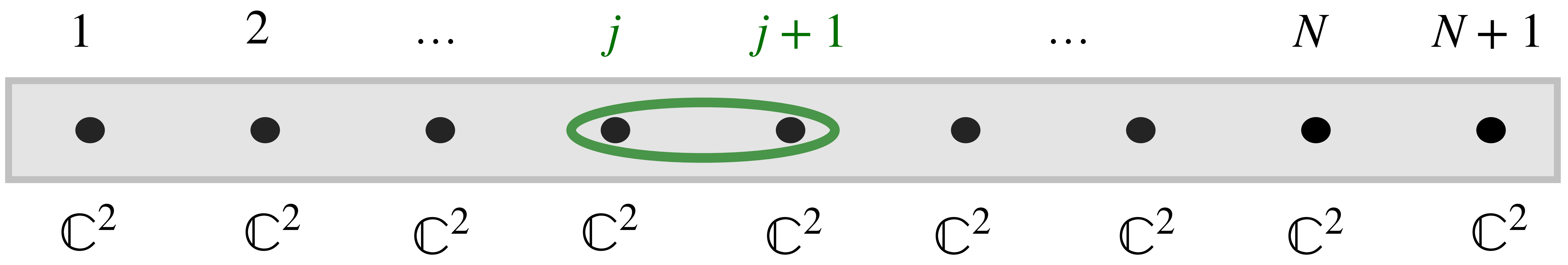
Interesting consequences if **correlations decay exponentially fast**.

# Example: Ising model 1D

$$X = [1\,, N+1] \subset \mathbb{Z}$$

$$\mathscr{H}_X \equiv (\mathbb{C}^2)^{\otimes X}$$

$$H_X \;=\; -\sum_{j=1}^{N} Z_j \otimes Z_{j+1}$$




Pauli $Z$-operator

$$Z : \mathbb{C}^2 \longrightarrow \mathbb{C}^2$$

$$|0\rangle \;\mapsto\; |0\rangle$$

$$|1\rangle \;\mapsto\; -|1\rangle$$

$$Z_j \otimes Z_{j+1} : \mathbb{C}^2 \otimes \mathbb{C}^2 \longrightarrow \mathbb{C}^2 \otimes \mathbb{C}^2$$

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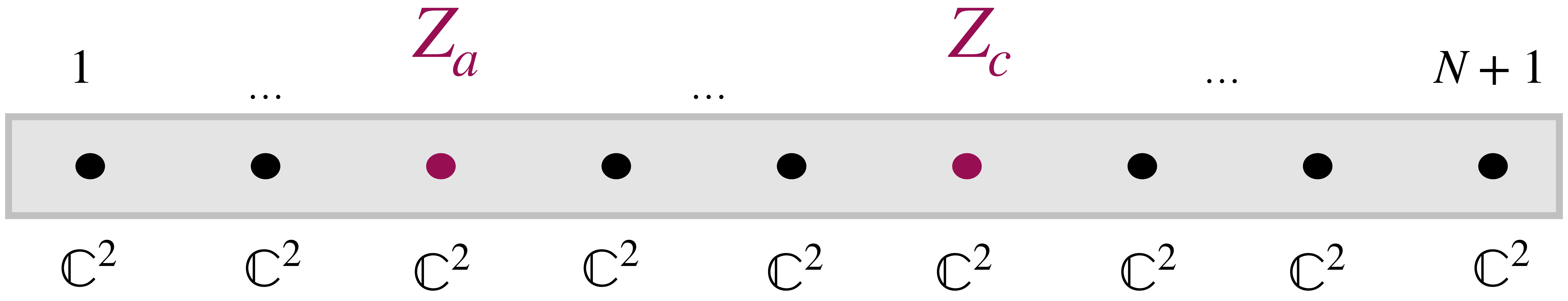
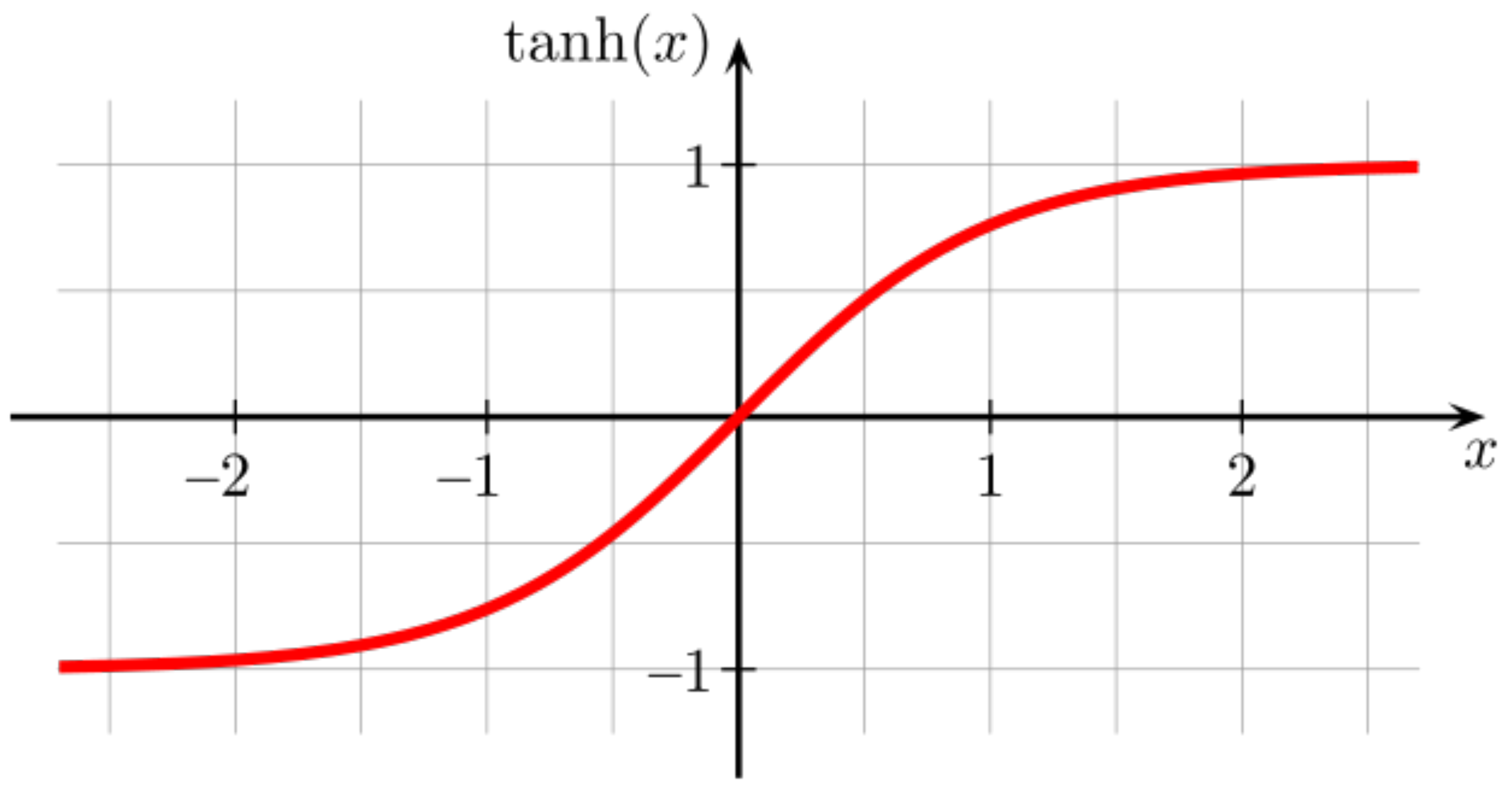



$$\sigma_\beta^X = \frac{e^{-\beta H_X}}{\mathrm{Tr}[e^{-\beta H_X}]} = \frac{1}{2^{N+1}} \sum_{S\subset\{1,\ldots,N\}} \tanh(\beta)^{\#S} \prod_{j\in S} Z_j \otimes Z_{j+1}$$

$$\left| \psi_X(Z_a Z_c) - \psi_X(Z_a)\,\psi_X(Z_c) \right| = (\tanh(\beta))^{|a-c|}$$




Source: wikipedia

**Correlations decay exponentially fast with** $\mathrm{dist}(a,c)$

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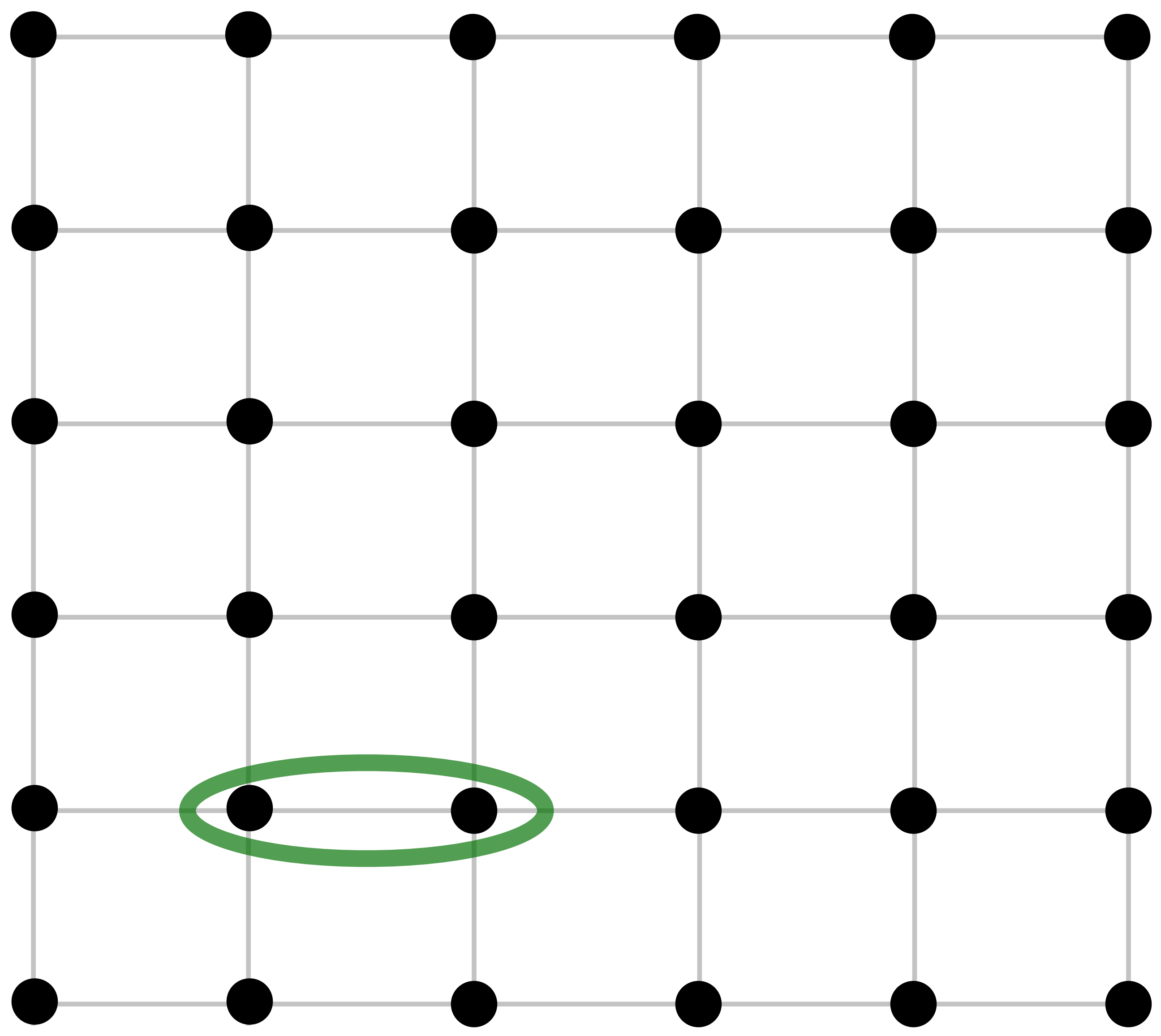

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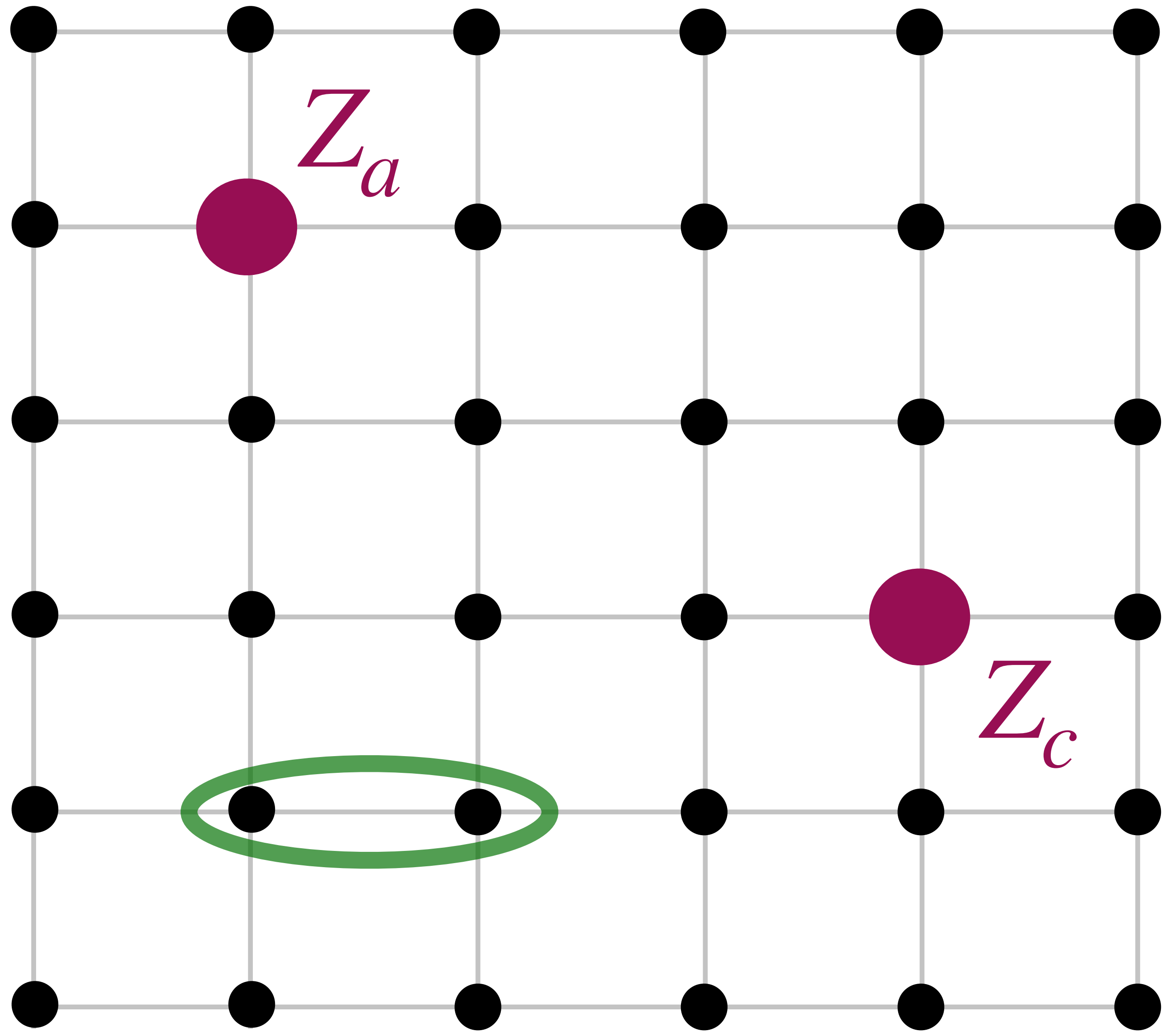


$$\left| \psi_X(Z_a Z_c) - \psi_X(Z_a)\, \psi_X(Z_c) \right| \leq K(\beta)\, e^{-\alpha(\beta)\, \|a-c\|} \quad , \quad 0 < \beta < \beta_c$$

**Thermal phase transition!!**

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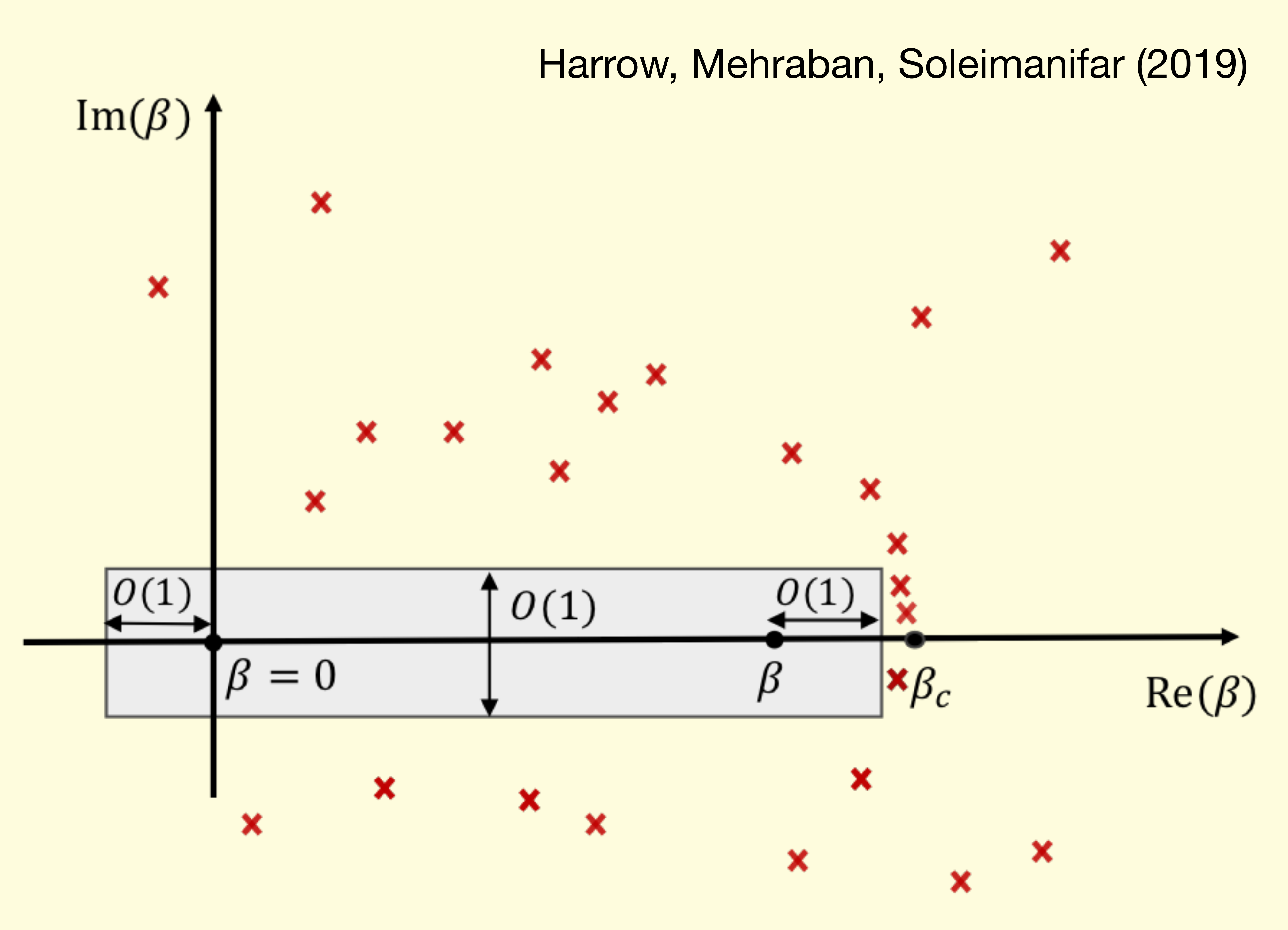


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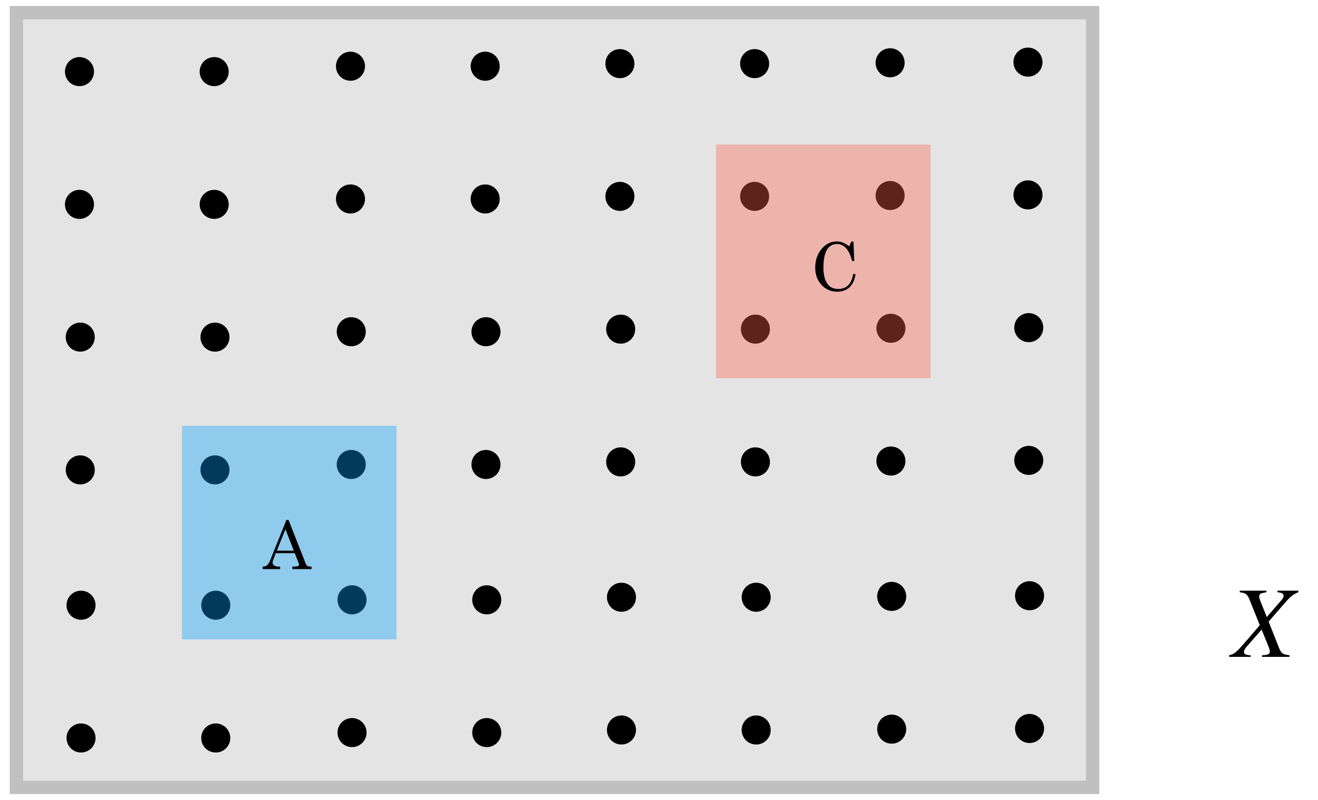
# Exponential Uniform Clustering - arbitrary dimension

$$\sigma^X = \frac{e^{-\beta H_X}}{\mathrm{Tr}(e^{-\beta H_X})} \qquad \psi_X(Q) := \mathrm{Tr}_X(\sigma^X Q) \quad , \quad Q \in \mathfrak{U}_X$$



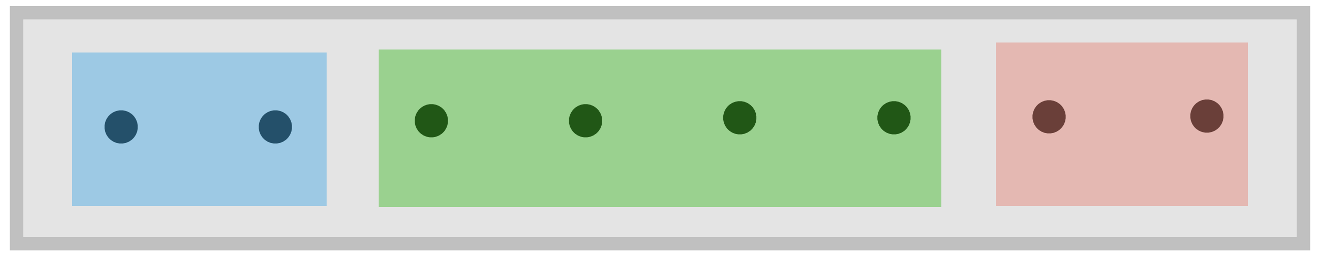

**Theorem (Kliesch, Gogolin, Kastoryano, Riera, Eisert, 2014) At any dimension** and **for high-temperatures**, correlations decay exponentially fast. More formally, there exists $\beta^*$ (depending on the lattice $\mathbb{Z}^D$) such that for every $0 < \beta < \beta^*$ there are constants $K(\beta), \alpha(\beta) > 0$ satisfying for every finite $X \subset \mathbb{Z}^D$ and any $A, C \subset X$ we have that

$$\mathrm{Corr}_{\sigma^X}(A : C) := \sup_{\|Q_A\|, \|Q_C\| \leq 1} \left| \psi_X(Q_A Q_C) - \psi_X(Q_A)\, \psi_X(Q_C) \right| \leq K(\beta) \min\{ |\partial A|, |\partial C| \}\, e^{-\alpha(\beta)\, \mathrm{dist}(A,C)}$$

**Can we improve this in 1D???**

# 1D Exponential Uniform Clustering condition

$I = ABC$



$$\sigma^I = \frac{e^{-\beta H_I}}{\mathrm{Tr}(e^{-\beta H_I})} \qquad \psi_I(Q) := \mathrm{Tr}(\sigma^I Q) \quad , \quad Q \in \mathfrak{A}_I$$

**Definition. Exponential Uniform Clustering at $\beta$ :** There exist $K(\beta), \alpha(\beta) > 0$ such that for every finite interval $I = ABC$ as in the picture

$$\mathrm{Corr}_{\sigma^I}(A : C) := \sup_{\|Q_A\|,\|Q_C\| \leq 1} \left| \psi_I(Q_A Q_C) - \psi_I(Q_A)\, \psi_I(Q_C) \right| \leq K(\beta)\, e^{-\alpha(\beta)\,|B|}$$

Proved by Araki (1969) for every $\beta > 0$?

Gibbs States of a One Dimensional Quantum Lattice

HUZIHIRO ARAKI*
I.H.E.S., 91 – Bures-sur-Yvette, France

Received June 20, 1969

**Abstract.** A one dimensional infinite quantum spin lattice with a finite range interaction is studied. The Gibbs state in the infinite volume limit is shown to exist as a primary state of a UHF algebra. The expectation value of any local observables in the state as well as the mean free energy depend analytically on the potential, showing no phase transition. The Gibbs state is an extremal KMS state.

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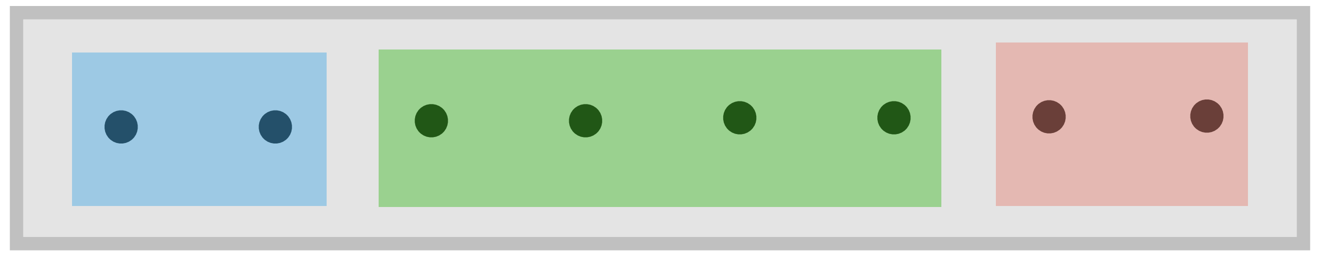

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# 1D Exponential Uniform Clustering condition

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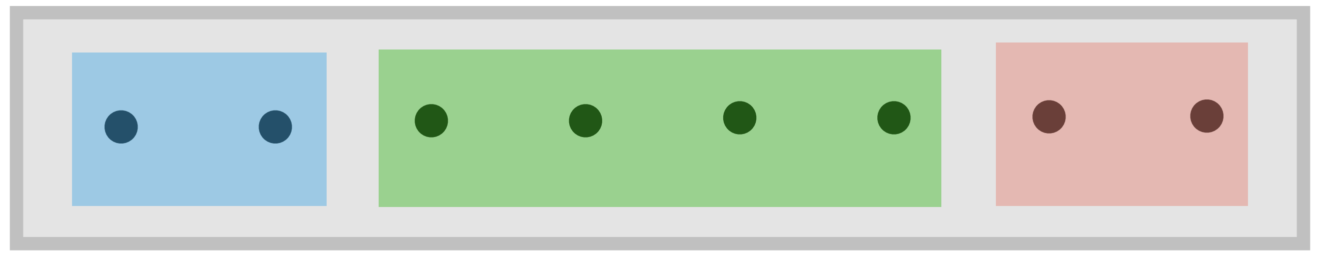

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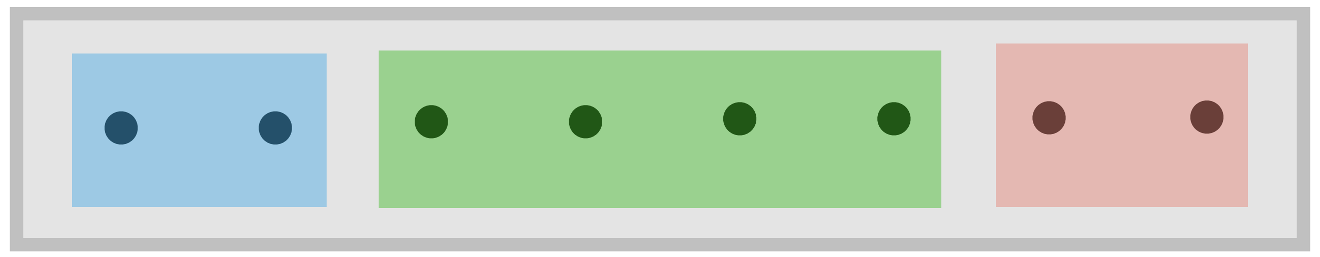

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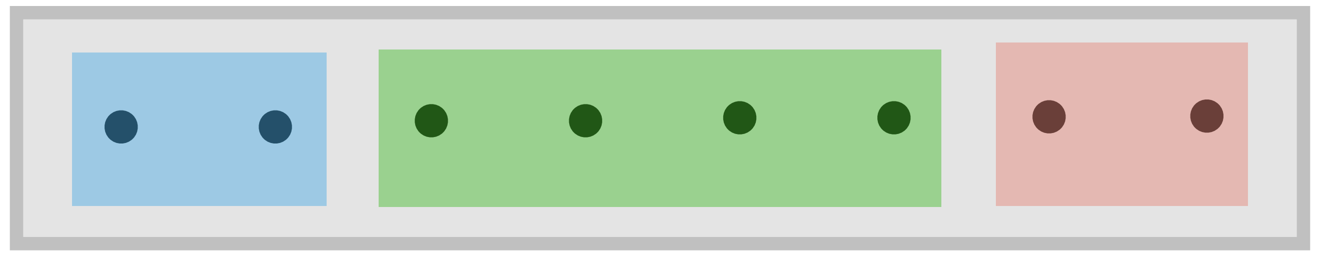

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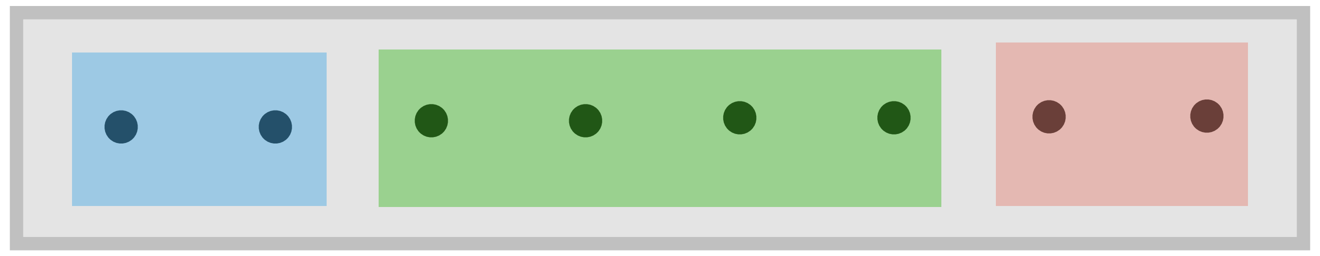

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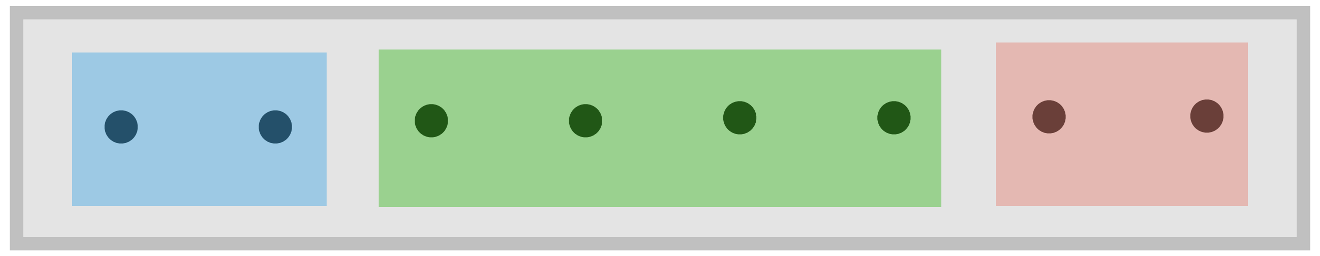

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1. Setting

2. Operator correlation function

**3. Mutual Information**

4. Approximate recoverability

# Mutual information

**Definition.** Let $\rho, \sigma$ be full-rank states on a finite-dimensional Hilbert space $\mathscr{H}$. Their Umegaki *relative entropy* id given by

$$D(\rho \| \sigma) = \mathrm{Tr}[\,\rho\,(\log \rho \;-\; \log \sigma)]$$

$$D(\rho \| \sigma) \geq 0 \quad , \qquad D(\rho \| \sigma) = 0 \;\Leftrightarrow\; \rho = \sigma$$

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$$\sigma_{AC} = \mathrm{Tr}_{X\backslash AC}[\sigma]$$

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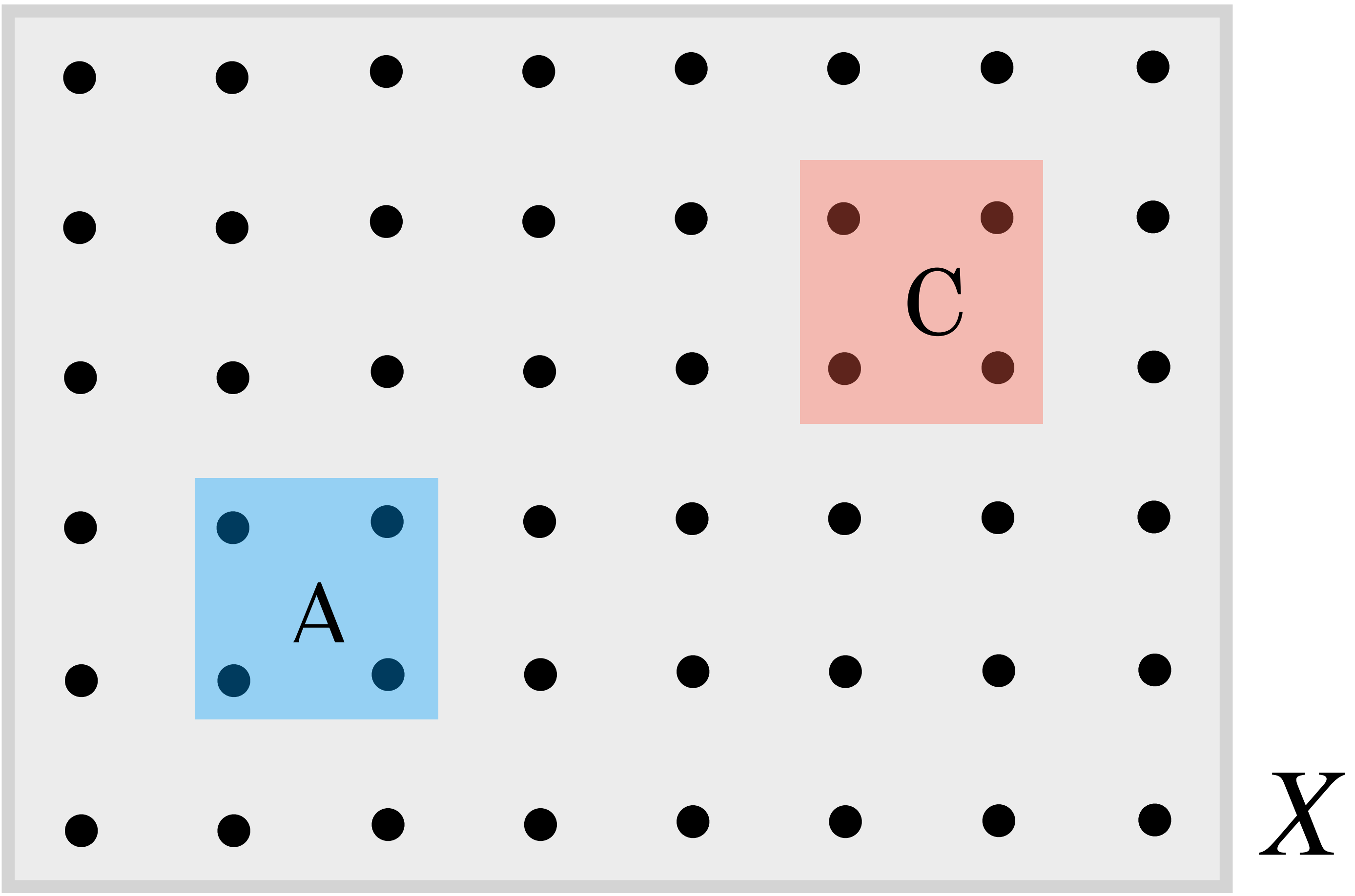

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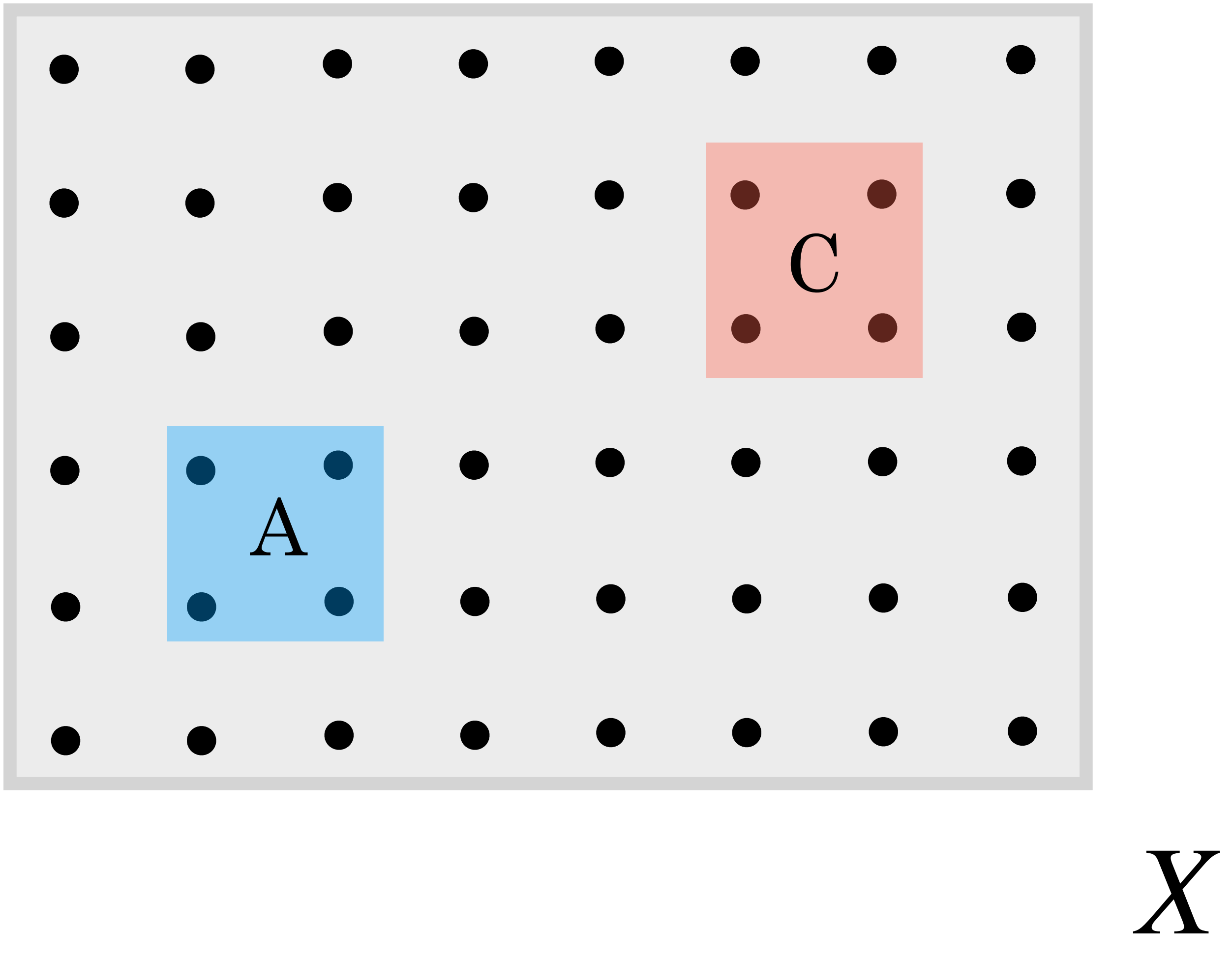

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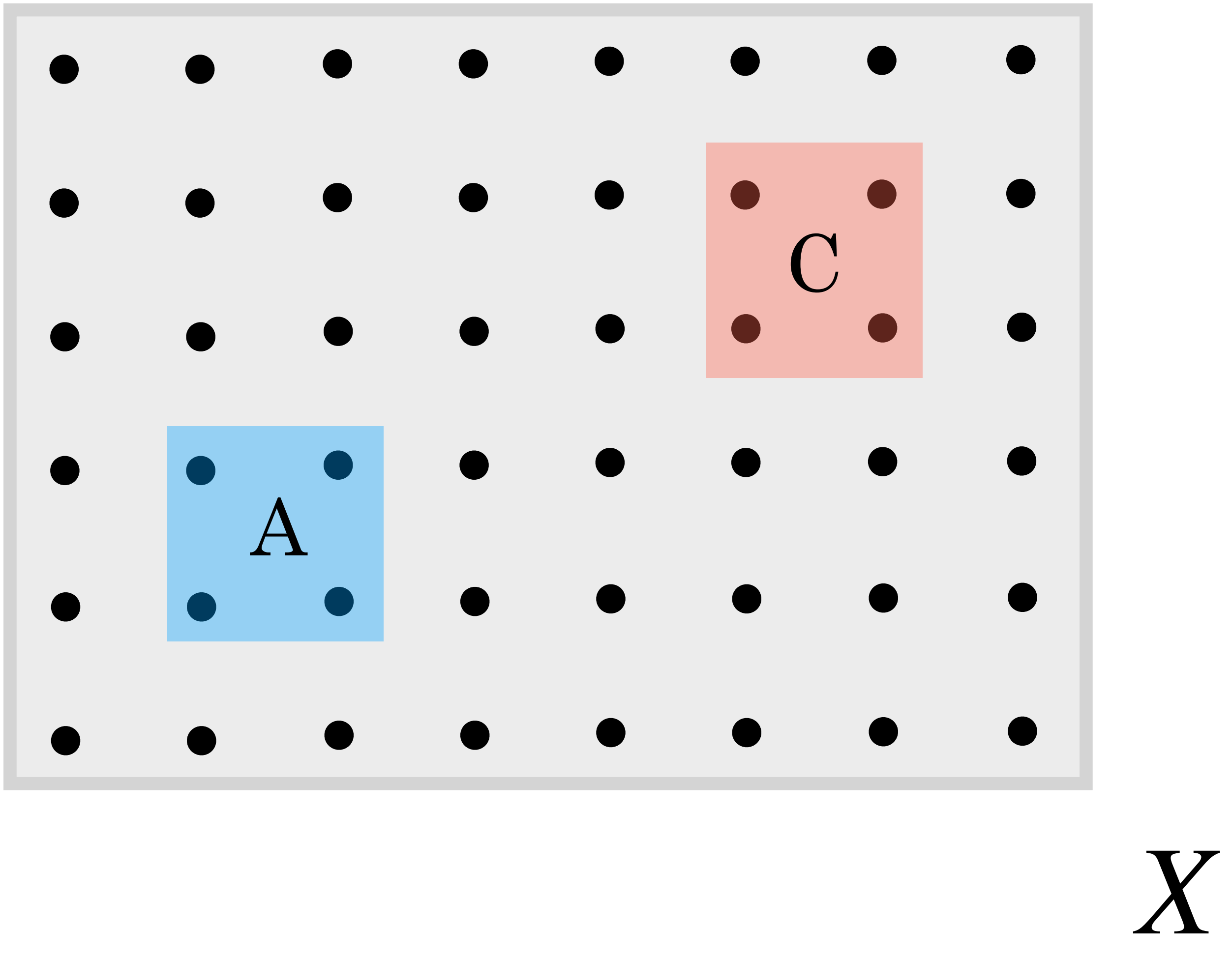

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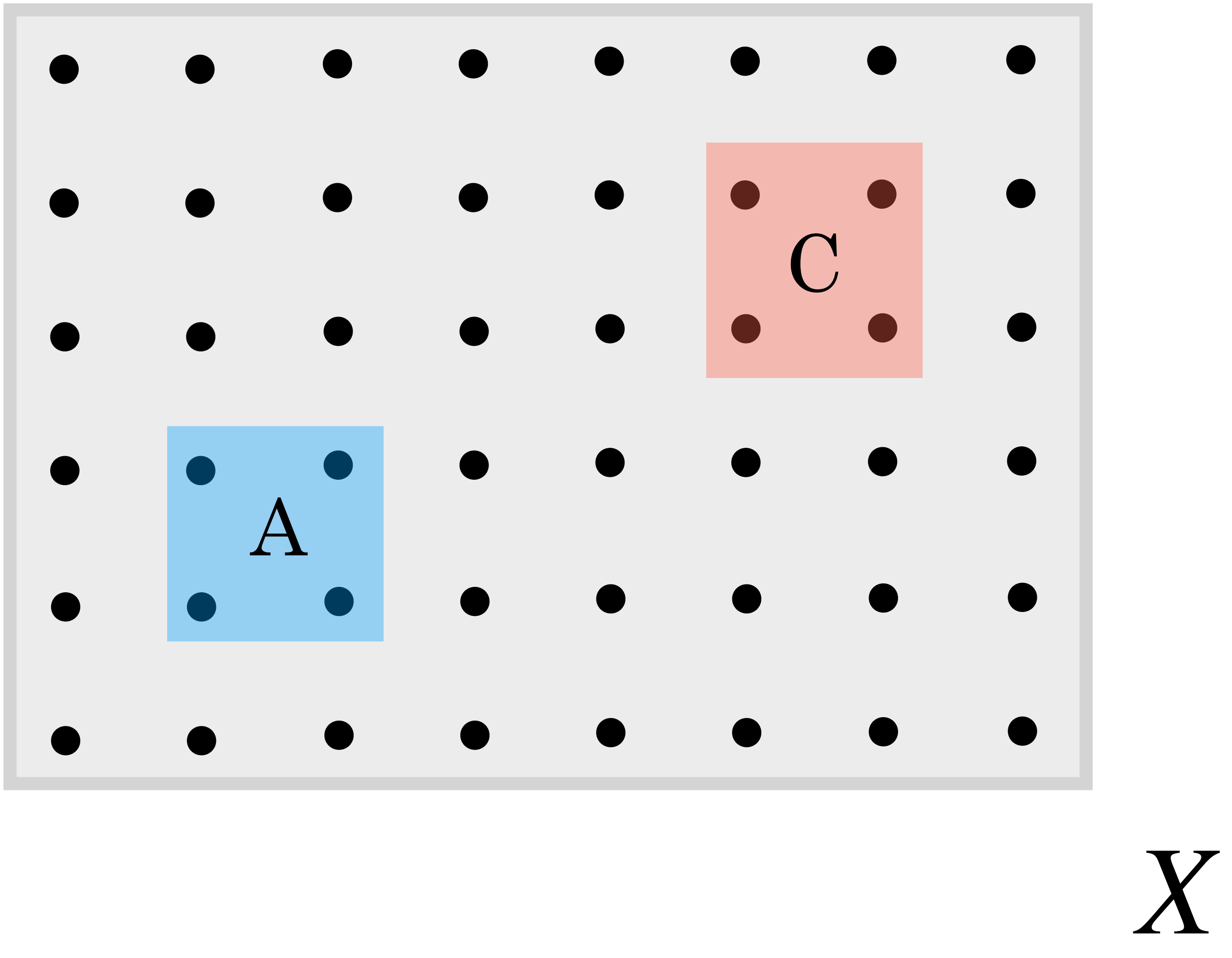
$$\le \|\sigma_{AC} - \sigma_A \otimes \sigma_C\|_1$$

Hölder ineq



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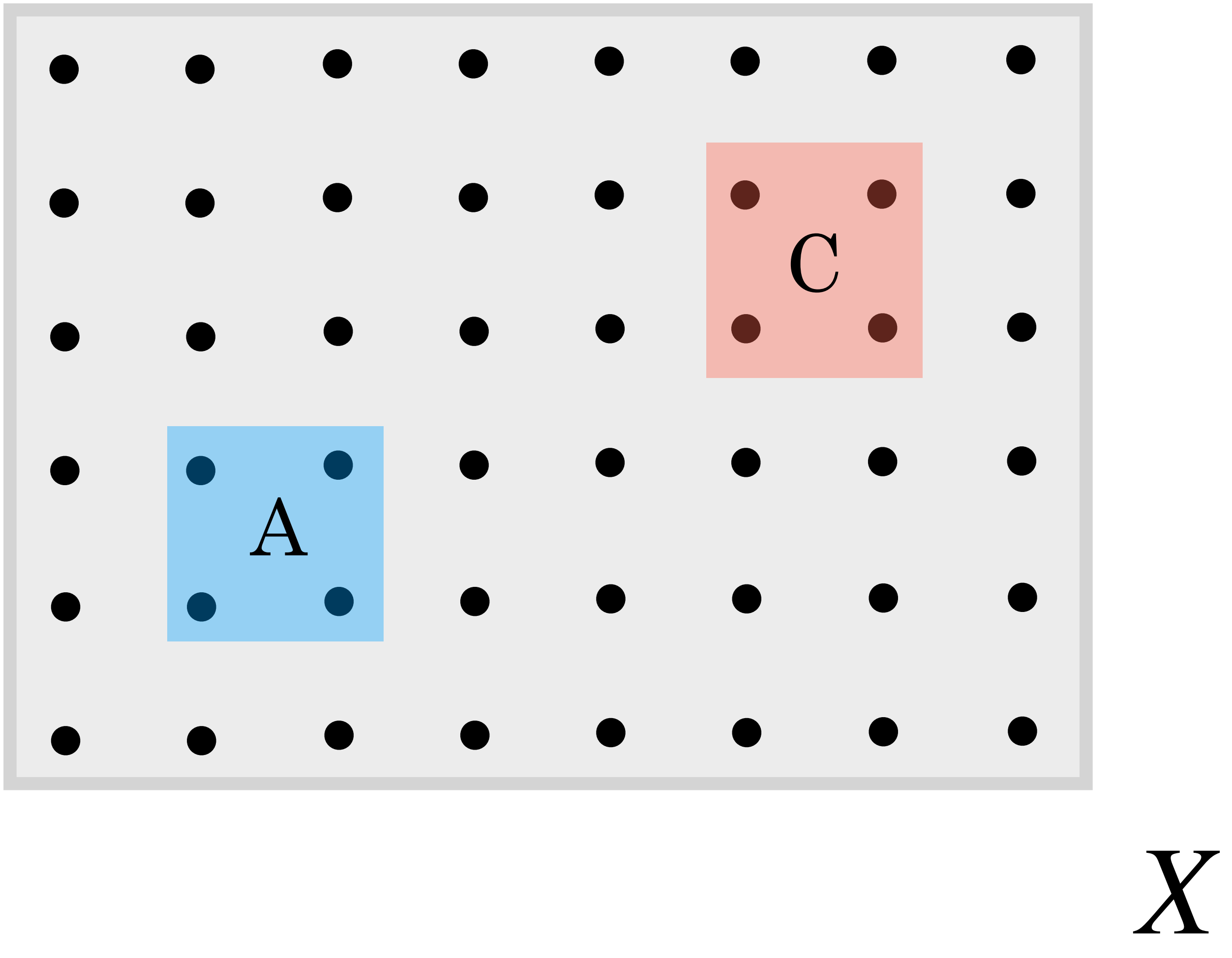
$$\le \sqrt{2 \cdot D(\sigma_{AC} \,||\, \sigma_A \otimes \sigma_C)}$$ Pinskner ineq



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In other words

$$\frac{1}{2}\mathrm{Corr}_\sigma(A : C)^2 \leq \frac{1}{2}\|\sigma_{AC} - \sigma_A \otimes \sigma_C\|_1^2 \leq I_\sigma(A : C)$$

Hence, Decay of MI $\Rightarrow$ Decay of operator correlation function

There are states with *small* operator correlation and *large* mutual information in quantum data hiding.

(Hayden, leung, Shor, Winter, 2004)

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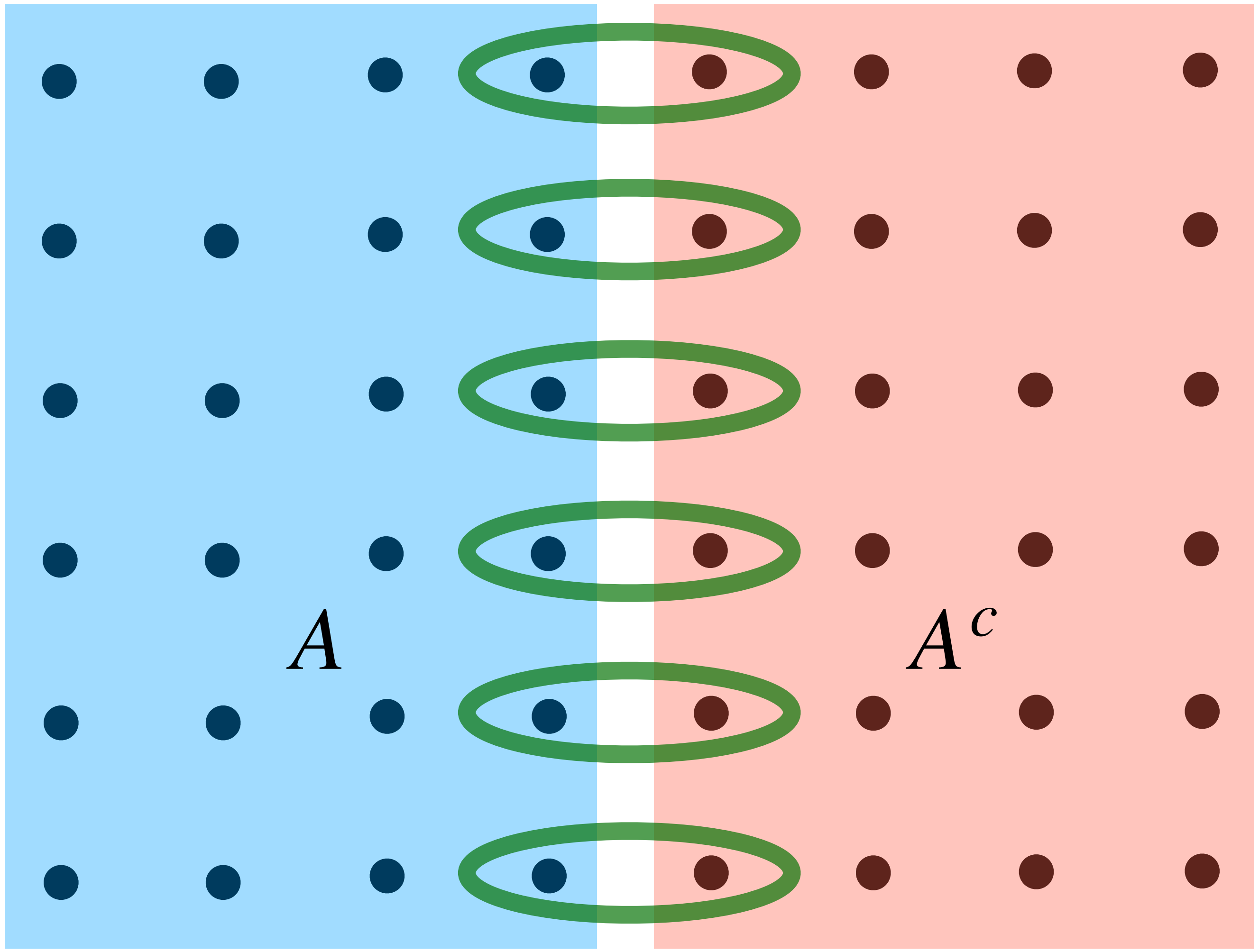
# Area Law




$X$

$$H_X = H_A + H_{A^c} + H_{\partial A} \qquad \sigma^X = e^{-\beta H_X}/\mathrm{Tr}(e^{-\beta H_X})$$

$$I_\sigma(A : A^c) \le 2\beta \, \|H_\partial\|_\infty \lesssim \beta \, |\partial A|$$ (Wolf, Verstraete, Hastings, Cirac, 2008)

$$I_\sigma(A : A^c) \lesssim \beta^{2/3} \, |\partial A|$$ (Kuwahara, Alhambra, Anshu, 2020)

Also for variants of the mutual information replacing relative entropy with Rényi divergences. (Scalet, Alhambra, Styliaris, Cirac, 2021)

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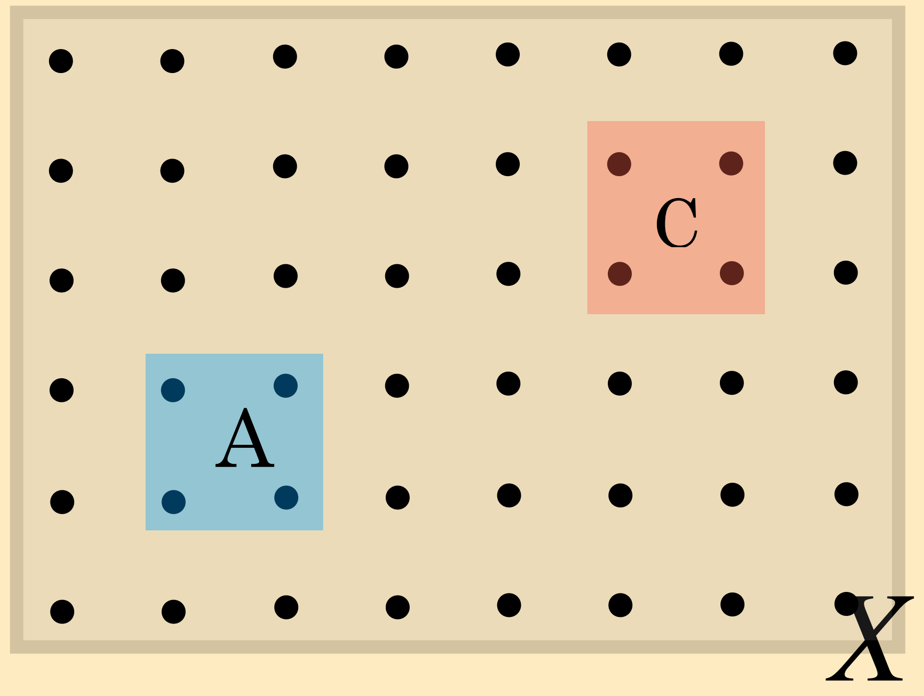


**Theorem (Kuwahara, Kato, Brandao, 2020)** There is $\beta^*$ (depending on $\mathbb{Z}^D$) such that for every $0 < \beta < \beta^*$ there exist $K(\beta), \alpha(\beta) > 0$ satisfying that for every finite $X \subset \mathbb{Z}^D$ and every $A, C \subset X$

$$I_{\sigma^X}(A:C) \leq K(\beta) \min\{|\partial A|, |\partial C|\}\, e^{-\alpha(\beta)\,\mathrm{dist}(A,C)}$$

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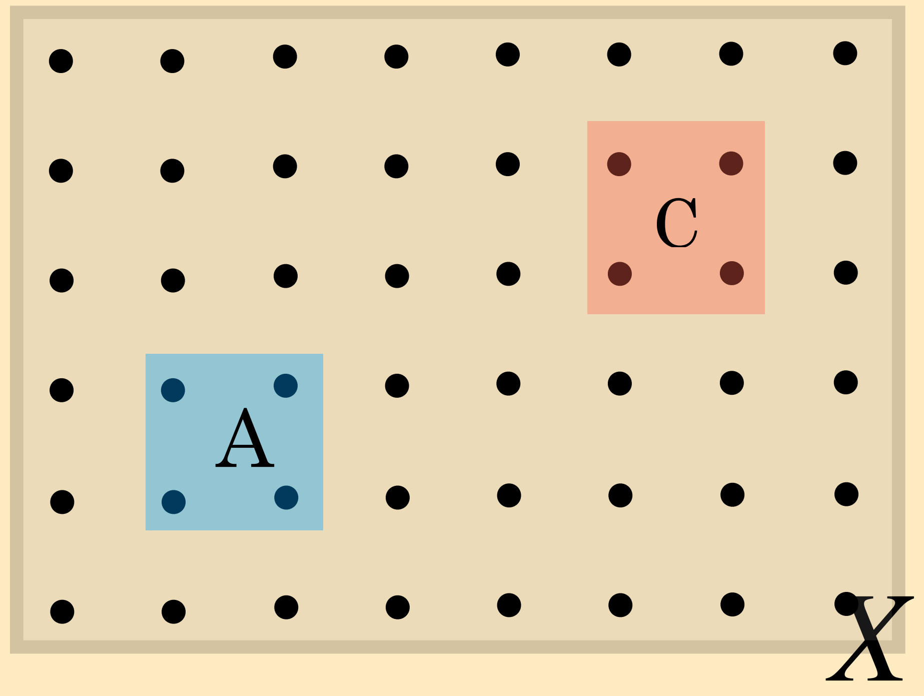


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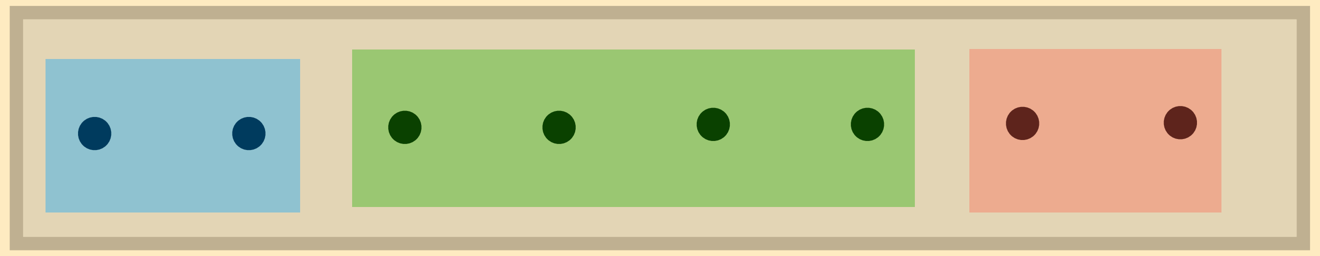
**Theorem (Bluhm, Capel, A.P.H., 2022)** In **1D** and if the interactions are translation-invariant, then the above property actually holds for every $\beta > 0$.



$I = ABC$

# Upper bounds on the relative entropy

Umegaki relative entropy

$$D(\rho\|\sigma) = \mathrm{Tr}[\,\rho\,(\log\rho \;-\; \log\sigma)]$$ $\rho, \sigma$ invertible states

$D(\rho\|\sigma)$

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BS relative entropy (Belavkin, Staszewski,1982)

$$\widehat{D}(\rho\|\sigma) = \mathrm{Tr}[\,\rho\,\log\big(\rho^{1/2}\sigma^{-1}\rho^{1/2}\big)]$$

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$$\widehat{D}_{\alpha}(\rho\|\sigma) = \frac{1}{\alpha-1}\log\mathrm{Tr}[\sigma^{1/2}\,\big(\sigma^{-1/2}\,\rho\,\sigma^{-1/2}\big)^{\alpha}\sigma^{1/2}]$$

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- $\widehat{D}_{\alpha}(\rho\|\sigma) \;\le\; \widehat{D}_{\gamma}(\rho\|\sigma)$ if $\alpha < \gamma$
- $\lim_{\alpha\to 1^+} \widehat{D}_{\alpha}(\rho\|\sigma) = \widehat{D}\,(\rho\|\sigma)$

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$$\begin{aligned}
\widehat{D}_\alpha(\rho\|\sigma) &= \frac{1}{\alpha-1}\log\mathrm{Tr}\left[\rho\left(\rho^{1/2}\sigma^{-1}\rho^{1/2}\right)^{\alpha-1}\right] \\
&\le \frac{1}{\alpha-1}\log\left\|\left(\rho^{1/2}\sigma^{-1}\rho^{1/2}\right)^{\alpha-1}\right\|_\infty \\
&\le \log\|\rho^{1/2}\sigma^{-1}\rho^{1/2}\|_\infty \\
&= \log\left(\|\rho^{1/2}\sigma^{-1}\rho^{1/2} - \mathbf{1} + \mathbf{1}\|_\infty\right) \\
&\le \log\left(\|\rho^{1/2}\sigma^{-1}\rho^{1/2} - \mathbf{1}\|_\infty + 1\right) \\
&\le \|\rho^{1/2}\sigma^{-1}\rho^{1/2} - \mathbf{1}\|_\infty \\
&\le \|\sigma^{-1}\rho - \mathbf{1}\|_\infty
\end{aligned}$$

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$$\frac{1}{2}\mathrm{Corr}_{\rho}(A:C)^2 \leq I_{\rho}(A:C) \leq \|\,\rho_A^{-1}\otimes\rho_C^{-1}\,\rho_{AC} - Id\,\|_{\infty}$$

# Mutual information

**Definition.** Let $\mathscr{H}_{AC} = \mathscr{H}_A \otimes \mathscr{H}_C$ be a bipartite quantum system and let $\sigma_{AC}$ be a full-rank state. The *mutual information* of $\sigma_{AC}$ between $A$ and $C$ is defined as

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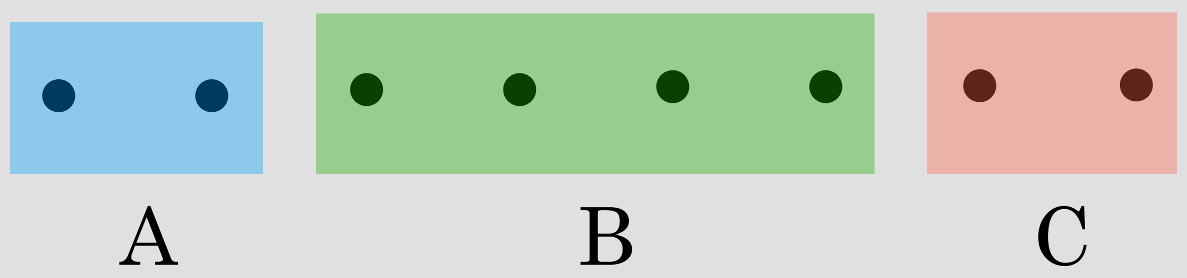


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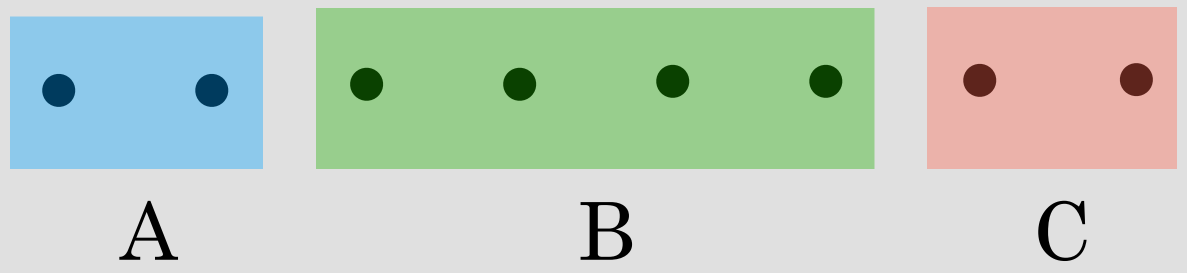


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**Theorem (Bluhm, Capel, P.H., 2021)** The following assertions are equivalent for a fixed temperature $\beta > 0$:

(i) Exponential Uniform Clustering

(ii) Exponential Decay of Mutual Information

(iii) Exponential Decay Mixing Condition

1. Setting

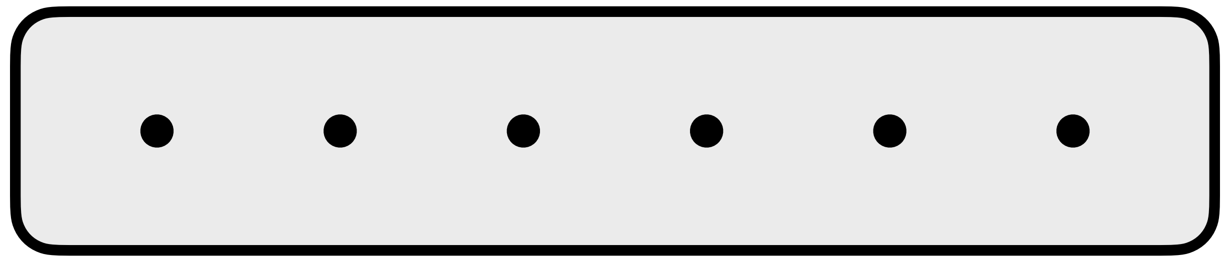
2. Operator correlation function

3. Mutual Information

**3.1 Application: mixing time in 1D thermalization**

4. Approximate recoverability

# Mixing time in 1D thermalization



System size: $n$ qudits

$\rho$

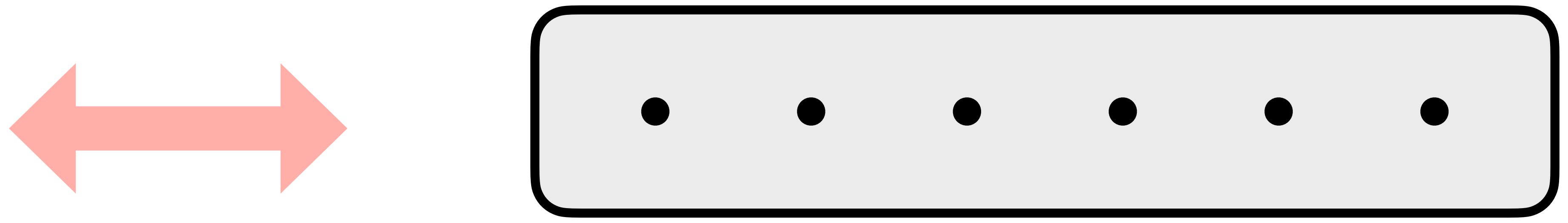
$\mathscr{H}_n \equiv \mathbb{C}^d \otimes \ldots \otimes \mathbb{C}^d$

$\mathfrak{A}_n = \mathscr{L}(\mathscr{H}_n)$ observables

$H_n$ local Hamiltonian

# Mixing time in 1D thermalization

Thermal bath at fixed temperature $\beta > 0$



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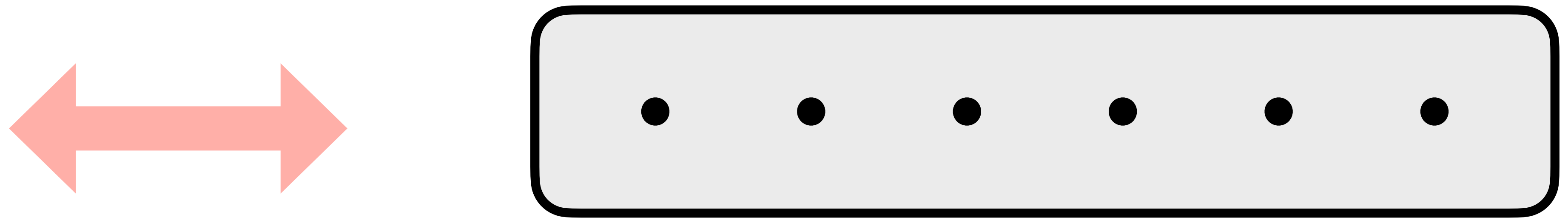

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**Definition (Mixing time)** $t_{mix}(\varepsilon) = \inf\{t \geq 0 : \ \forall \rho \ , \|e^{t\mathscr{D}_n}(\rho) - \sigma_\beta\| < \varepsilon \}$

**Theorem 3.1.** *Let $\Lambda = [\![1, n]\!]$. For any $\beta > 0$, we denote by $\sigma \equiv \sigma^\beta$ the Gibbs state of a finite-range, translation-invariant, commuting Hamiltonian at inverse temperature $\beta > 0$. Consider $\mathcal{L}^D_{\Lambda *}$ the Davies generator of a quantum Markov semigroup $\{e^{t\mathcal{L}^D_{\Lambda *}}\}_{t \geq 0}$ with unique fixed point $\sigma$. Then, there exists $\alpha_n = \Omega(\ln(n)^{-1})$ such that, for all $\rho \in \mathcal{D}(\mathcal{H}_\Lambda)$ and all $t \geq 0$,*

$$D(\rho_t \| \sigma) \leq e^{-\alpha_n t} D(\rho \| \sigma) , \quad (25)$$

*where $\rho_t := e^{t\mathcal{L}^D_{\Lambda *}}(\rho)$. Moreover, $\alpha_n = e^{-\mathcal{O}(\beta)}$ as a function of $\beta$.*

(Barden, Capel, Gao, Lucia, Pérez García, Rouzé 2022)

$t_{mix}(\varepsilon) \leq C \log(n) \left(\log(1/\varepsilon) + \log(n)\right)$

*Assumption 1* (mixing condition). *Let $\Lambda \subset\subset \mathbb{Z}$ be a finite chain, and let $C, D \subset \Lambda$ be the union of non-overlapping finite-sized segments of $\Lambda$. Let $\sigma_\Lambda$ be the Gibbs state of a commuting Hamiltonian. The following inequality holds for certain positive constants $K_1, K_2$ independent of $\Lambda, C, D$:*

$$\left\| \sigma_C^{-1/2} \otimes \sigma_D^{-1/2} \sigma_{CD} \sigma_C^{-1/2} \otimes \sigma_D^{-1/2} - \mathbb{1}_{CD} \right\|_\infty \leq K_1 e^{-K_2 d(C,D)},$$

*where $d(C, D)$ is the distance between $C$ and $D$, i.e., the minimum distance between two segments of $C$ and $D$.*

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$$I_\sigma(A : C \,|\, B) \ := \ S(\sigma_{AB}) + S(\sigma_{BC}) - S(\sigma_{ABC}) - S(\sigma_B) \qquad \text{where} \qquad S(\rho) = -\operatorname{Tr}(\rho \log \rho)$$

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Strong subadditivity: (Lieb and Ruskai 73)

$$I_\sigma(A:C|B) \geq 0$$

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$\Leftrightarrow$ There is a CPTP map $\mathscr{R}_{B\to BC}$ such that

$$\sigma_{ABC} = \mathscr{R}_{B\to BC}(\sigma_{AB})$$

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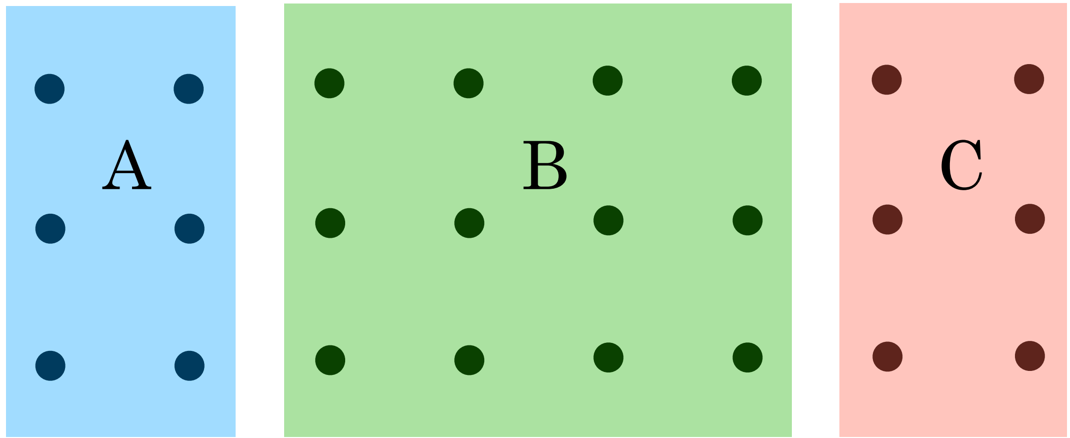
Let $X = ABC$ and $\sigma = \sigma^X$ Gibbs state. In arbitrary dimensions and high temperature $(\beta < \beta^*)$, we have exponential decay:




$$I_\sigma(A:C|B) \leq K(\beta)\min\{|\partial A|, |\partial C|\}\, e^{-\alpha(\beta)\,\mathrm{dist}(A,C)}$$

(Kuwahara, Kato, Brandao, 2020)

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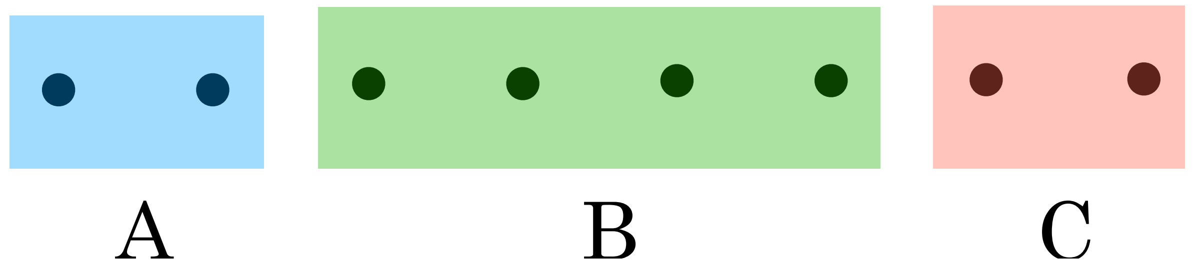
In **1D**, for every $\beta > 0$ there exists $K(\beta), \alpha(\beta) > 0$ such that for every finite interval $I = ABC$ and $\sigma = e^{-\beta H_I}/\mathrm{Tr}[e^{-\beta H_I}]$




$$I_\sigma(A : C|B) \;\leq\; K(\beta)\; e^{-\alpha(\beta)\sqrt{|B|}}$$

(Kato, Brandao, 2019)

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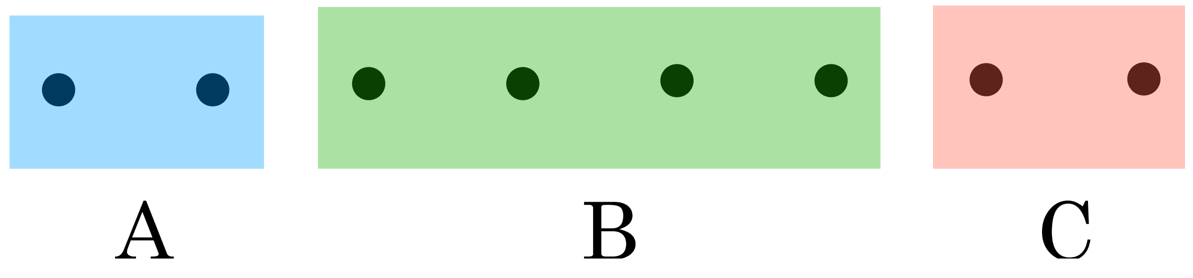


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$$\|\sigma_{ABC} - \sigma_{BC}^{1/2}\, \sigma_B^{-1/2}\, \sigma_{AB}\, \sigma_B^{-1/2}\, \sigma_{BC}^{1/2}\|_1 \; \leq \; \delta_\beta(|B|)$$

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**Theorem (Bluhm, Capel, P.H. 2021)** For each $\beta > 0$ there is a positive function $\ell \mapsto \delta_\beta(\ell)$ which decays faster than any exponential such that: for every finite interval $I = ABC$ the Gibbs state $\sigma_{ABC} = e^{-\beta H_{ABC}}/\operatorname{Tr}(e^{-\beta H_{ABC}})$ satisfies

$$\|\sigma_{ABC} - \sigma_{AB}\,\sigma_B^{-1}\,\sigma_{BC}\|_1 \leq \delta_\beta(|B|)$$

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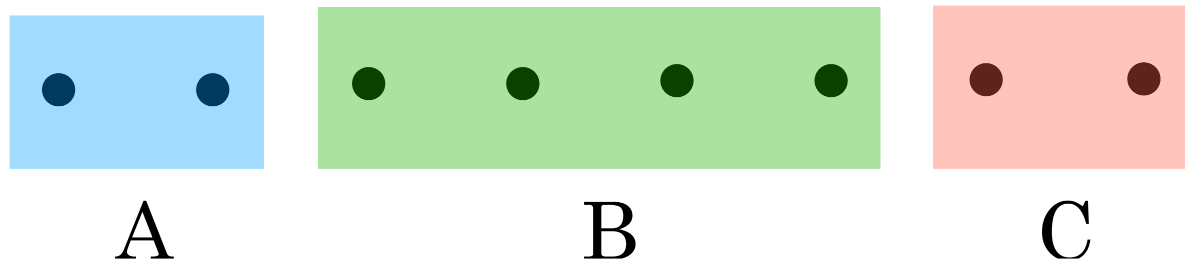


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# Muchas gracias

A. Bluhm, Á. Capel and A. Pérez-Hernández, Exponential decay of mutual information for Gibbs states of local Hamiltonians, Quantum, 6, 650, 2022,
DOI: 10.22331/q-2022-02-10-650. arXiv: 2104.04419