Balanced Fourier Truncations in Group von Neumann Algebras

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joint work with José M. Conde-Alonso[2] & Javier Parcet[1]

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Outline

Motivation

Problem

Given $2 < q < p$, $L_q \hookrightarrow L_p$ as metric spaces?

[Mankiewicz, 1972] No (differentiation argument)

[Naor, Schechtman, 2016] Metric invariant: metric X_p inequality

- L_p is a metric X_p space,
- L_q is NOT a metric X_p space

$\implies L_q \not\hookrightarrow L_p$ as metric spaces

[Naor, 2016] Sharp metric X_p inequalities via Fourier analysis

X_p inequality for Rademacher chaos

Let $n \in \mathbb{N}$, $k \in [n]$ and $p \in [2, \infty)$. For any $f \in L_p(\{-1,1\}^n)$ with zero mean, it holds

$$\frac{1}{\binom{n}{k}} \sum_{\substack{S \subseteq [n] \\ |S|=k}} \|\mathrm{E}_{[n]\setminus S} f\|^p_{L_p(\{-1,1\}^n)} \lesssim_p \frac{k}{n} \sum_{j=1}^{n} \|\partial_j f\|_p^p + \left(\frac{k}{n}\right)^{p/2} \|f\|_p^p.$$

- $\|f\|_{L_p(\{-1,1\}^n)} = (\mathbb{E}_\varepsilon |f|^p)^{1/p} = \left(\frac{1}{2^n} \sum_{\varepsilon \in \{-1,1\}^n} |f(\varepsilon)|^p\right)^{1/p}$.
- $\partial_j f(\varepsilon) = f(\varepsilon) - f(\varepsilon_1, \ldots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_n)$.
- $\mathrm{E}_{[n]\setminus S} f(\varepsilon) = \frac{1}{2^n} \sum_{\delta \in \{-1,1\}^n} f(\varepsilon_S + \delta_{[n]\setminus S})$

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Fourier analysis on the group $\{-1,1\}^n$

- Walsh functions: $W_A(\varepsilon) = \prod_{j \in A} \varepsilon_j$
- $\widehat{f}(A) = \mathbb{E}_\varepsilon (f \cdot W_A)$ and $f(\varepsilon) = \sum_{A \subseteq [n]} \widehat{f}(A)\, W_A(\varepsilon)$.
- Fourier truncations $\mathrm{E}_{[n]\setminus S} f(\varepsilon) = \sum_{A \subseteq S} \widehat{f}(A)\, W_A(\varepsilon)$.

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If f is linear, $f(\varepsilon) = \sum_{1 \leqslant j \leqslant n} \widehat{f}(\{j\})\, W_{\{j\}}(\varepsilon) = \sum_{1 \leqslant j \leqslant n} \widehat{f}(\{j\})\, \varepsilon_j$,

then it holds

$$\frac{1}{\binom{n}{k}} \sum_{\substack{S \subseteq [n] \\ |S|=k}} \| \sum_{j \in S} \widehat{f}(\{j\})\, \varepsilon_j \|_p^p$$

$$\lesssim_p \frac{k}{n} \sum_{j=1}^n \|\widehat{f}(\{j\}) \varepsilon_j\|_p^p + \left(\frac{k}{n}\right)^{p/2} \| \sum_{j=1}^n \widehat{f}(\{j\})\, \varepsilon_j \|_p^p.$$

Two questions

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[Johnson, Maurey, Schechtman, Tzafriri, 1979]
TRUE when $x_j \in \mathbb{R}$ ($\mathbb{C}$).

Question 1

Does Linear X_p hold for operator-valued sequences $\{x_j\}_{j=1}^n$?

[Junge, Xu, 2008] TRUE if $x_j \in S_p$ and even when $x_j \in L_p(\mathcal{M})$.

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Question 2

Does X_p inequality for chaos hold when replacing ε_j by some 'variables' over a different group?

Replacing Walsh functions $W_A(\varepsilon) = \prod_{j\in A} \varepsilon_j$ by … 'characters'?

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Brief review: Fourier Analysis on groups

Let Γ be a compact abelian topological group.

Dual group of Γ

$\widehat{\Gamma} =: G$ is the discrete abelian group of **characters** $\chi_g : \Gamma \to S^1$,

$$\chi_g(ww') = \chi_g(w)\chi_g(w'), \quad \overline{\chi_g(w)} = \chi_g(w^{-1})$$

Example

$$(\ \mathbb{T} \ni x \mapsto e^{2\pi i \langle x,n \rangle}\ ) \ \longleftrightarrow \ n \in \mathbb{Z}$$

$\{e^{2\pi i \langle \cdot, n \rangle}\}_{n \in \mathbb{Z}}$ orthonormal basis of $L_2(\mathbb{T})$

Noncommutative integration

Let $\Gamma = \{-1, 1\}^n$ so $G = \{W_A\}_{A \subset [n]} \simeq \mathbb{Z}_2^n$

- $W_A(\cdot) = e^{\pi i \langle \cdot, e_A \rangle}$ ONB of $L_2(\Gamma)$,
- $e^{\pi i \langle \cdot, g \rangle} \simeq \lambda(g) : e^{\pi i \langle \cdot, h \rangle} \mapsto e^{\pi i \langle \cdot, g \rangle} e^{\pi i \langle \cdot, h \rangle} = e^{\pi i \langle \cdot, g+h \rangle}$

Set G noncommutative? Problem: we do not have characters.

Let G be a discrete group.

- $\ell_2(G) = \overline{\text{span}}\{\delta_h \ : \ h \in G\}$,
- *left regular representation* operators

$$\lambda(g)\delta_h = \delta_{gh} \quad \text{for any } g, h \in G.$$

Group von Neumann algebra for G

$$\mathcal{L}G = \overline{\text{span}\{\lambda(g) \ : \ g \in G\}}^{w^*}$$

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- $\mathcal{L}G \simeq L_\infty(\Gamma)$ whenever G commutative
  Example: $\mathcal{L}\mathbb{Z}^n \simeq L_\infty(\mathbb{T}^n)$

Trace on $\mathcal{L}G$

$$\tau(\lambda(g)) = \langle \lambda(g)\delta_e, \delta_e \rangle_{\ell_2(G)} = \langle \delta_g, \delta_e \rangle_{\ell_2(G)}.$$

- $L_p(\mathcal{L}G) = \overline{\mathcal{L}G}^{\|\cdot\|_p}$ where $\|f\|_p = \tau(|f|^p)^{1/p}$
- $(1 \leqslant p < \infty)$ $(L_p(\mathcal{L}G))^* = L_{p'}(\mathcal{L}G)$ where $1/p + 1/p' = 1$.
- Hölder, Minkowski, duality formulas, interpolation, etc.

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Differential structures

Let G be a discrete group.

Left cocycle $(\mathcal{H}, \alpha, \beta)$

- $\mathcal{H}$ real Hilbert space with basis $\{e_j\}_{j\geqslant 1}$,
- $\beta : G \to \mathcal{H}$,
- orthogonal action $\alpha : G \to \mathcal{O}(\mathcal{H})$ satisfying

$$\beta(gh) = \alpha_g(\beta(h)) + \beta(g).$$

Length function $\psi : G \longrightarrow [0, \infty)$

Cocycles $\Longleftrightarrow$ Length functions

$$(\mathcal{H}, \beta, \alpha) \Leftrightarrow \psi(g) = \langle \beta(g), \beta(g) \rangle_{\mathcal{H}}$$

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Differential operators on $\mathcal{L}G$

- Derivatives $\partial_j \lambda(g) = 2\pi i \, \langle \beta(g), e_j \rangle \, \lambda(g)$
- Laplacian $\Delta\lambda(g) = \psi(g) \, \lambda(g)$
- Riesz transforms

$$R_j \lambda(g) = \partial_j \Delta^{-1/2} \lambda(g) = 2\pi i \; \psi(g)^{-1/2} \, \langle \beta(g), e_j \rangle \, \lambda(g)$$

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$$R_j \lambda(g) = \partial_j \Delta^{-1/2} \lambda(g) = 2\pi i \; \psi(g)^{-1/2} \, \langle \beta(g), e_j \rangle \, \lambda(g)$$

Differential operators on $\mathcal{L}G$

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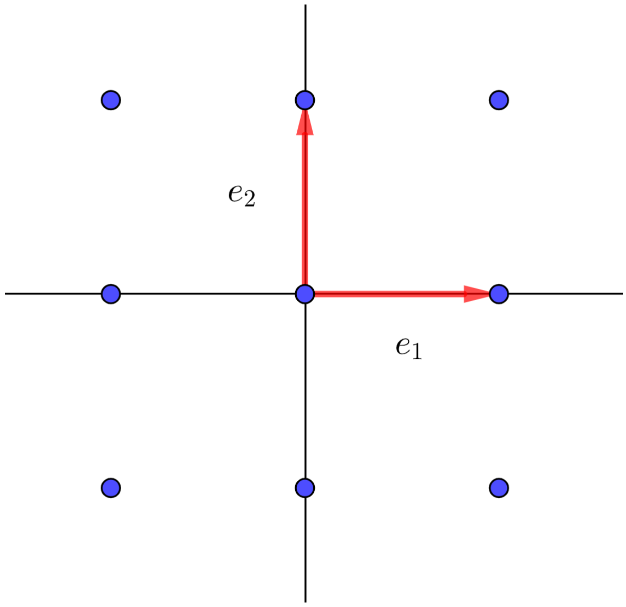
Example: $G = \mathbb{Z}^n$, $\mathcal{L}\mathbb{Z}^n \simeq L_\infty(\mathbb{T}^n)$

- $\partial_j \, e^{2\pi i \langle x,g \rangle} = 2\pi i \; g_j \; e^{2\pi i \langle x,g \rangle} = 2\pi i \; \langle g, e_j \rangle_{\mathbb{R}^n} \; e^{2\pi i \langle x,g \rangle}$
- $\Delta e^{2\pi i \langle x,g \rangle} = -4\pi^2 (g_1^2 + \ldots + g_n^2) \; e^{2\pi i \langle x,g \rangle}$



$$
\begin{aligned}
\mathcal{H} &= \mathbb{R}^n \\
\beta(g) &= g \\
\alpha_g &= \mathrm{id} \\
\psi(g) &= \|\beta(g)\|_{\mathbb{R}^n}^2 = \|g\|_2^2 \\
&= g_1^2 + \ldots + g_n^2
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Example: $G = \mathbb{Z}_2^n$, $\mathcal{L}\mathbb{Z}_2^n \simeq L_\infty(\{-1,1\}^n)$

- $$\begin{aligned} \partial_j \; W_A(\varepsilon) &= W_A(\varepsilon) - W_A(\varepsilon - 2\varepsilon_j e_j) \\ &= 2 \; \delta_{j\in A} W_A(\varepsilon) = 2\langle e_A, e_j \rangle \; W_A(\varepsilon) \end{aligned}$$
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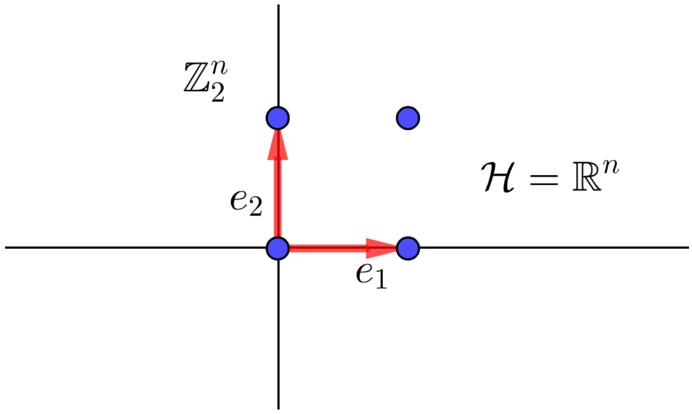

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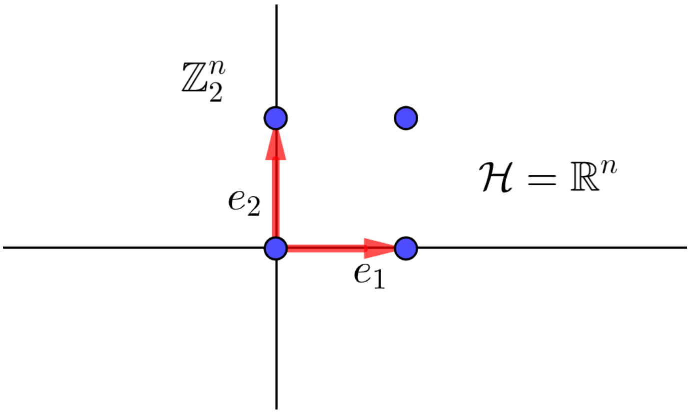

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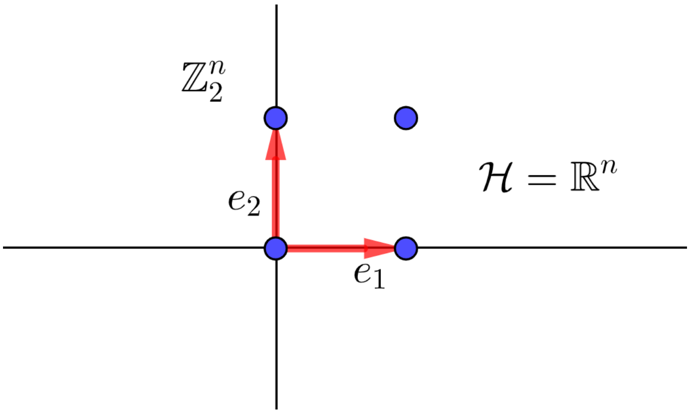

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X_p inequality for Rademacher chaos

Let $n \in \mathbb{N}$, $k \in [n]$ and $p \in [2, \infty)$. For any $f \in L_p(\{-1,1\}^n)$ with zero mean, it holds

$$\frac{1}{\binom{n}{k}} \sum_{\substack{S \subseteq [n] \\ |S|=k}} \|\mathrm{E}_{[n]\setminus S} f\|^p_{L_p(\{-1,1\}^n)} \lesssim_p \frac{k}{n} \sum_{j=1}^{n} \|\partial_j f\|_p^p + \left(\frac{k}{n}\right)^{p/2} \|f\|_p^p.$$

✓ $\|f\|_{L_p(\{-1,1\}^n)} = (\mathbb{E}_\varepsilon |f|^p)^{1/p} = \left(\frac{1}{2^n} \sum_{\varepsilon \in \{-1,1\}^n} |f(\varepsilon)|^p\right)^{1/p}$.

✓ $\partial_j f(\varepsilon) = f(\varepsilon) - f(\varepsilon_1, \ldots, \varepsilon_{j-1}, -\varepsilon_j, \varepsilon_{j+1}, \ldots, \varepsilon_n)$.

- $\mathrm{E}_{[n]\setminus S} f(\varepsilon) = \frac{1}{2^n} \sum\limits_{\delta \in \{-1,1\}^n} f(\varepsilon_S + \delta_{[n]\setminus S})$

Bounded truncations

Let G be a discrete group. We say that G is equipped with a family of

L_p-bounded truncations

if there exists a family $\{B_S\}_{S\subset[n]}$ such that the maps

$$\mathrm{E}_{[n]\setminus S} f = \sum_{g\in B_S} \widehat{f}(g)\ \lambda(g)$$

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Example: $G = \mathbb{Z}_2^n$

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Main result

Theorem [C-M, Conde-Alonso, Parcet, 2022]

Let $p \in [2, \infty)$. Let G be a discrete group with a cocycle $(\mathcal{H}, \beta, \alpha)$ such that there exists $\{B_S\}_{S \subset [n]}$ satisfying

- $\{\mathrm{E}_{[n] \setminus S}\}_{S \subset [n]}$ are completely bounded maps on $L_p(\mathcal{L}G)$,
- for some decomposition $\mathcal{H} = \bigoplus_{\ell=1}^{n} \mathcal{H}_\ell$ it holds

$$\beta(B_S) \subset \mathcal{H}_S = \bigoplus_{\ell \in S} \mathcal{H}_\ell.$$

Then, for any $k \in [n]$, it holds

$$\frac{1}{\binom{n}{k}} \sum_{\substack{S \subseteq [n] \\ |S| = k}} \|\mathrm{E}_{[n] \setminus S} f\|_p^p$$

$$\lesssim_p \frac{k}{n} \sum_{\ell=1}^{n} \Big\| \Big(\sum_{e_j \in \mathcal{H}_\ell} |\partial_j f|^2 + |\partial_j (f^*)|^2 \Big)^{1/2} \Big\|_p^p + \Big(\frac{k}{n} \Big)^{p/2} \|f\|_p^p.$$

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Example. $G = \mathbb{Z}^n$ with length $\psi(g) = \sum_{j=1}^{n} |g_j|$

- $\mathcal{H} = \mathbb{R}[\mathbb{Z}^n]/\operatorname{Ker}(\langle \cdot, \cdot \rangle) = \overline{\operatorname{span}}\{\delta_h : h \in \mathbb{Z}^n\}$ where

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If there exists $\{\tilde{\partial}_j\}_{j=1}^n$ such that

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$$\frac{1}{\binom{n}{k}} \sum_{\substack{S \subseteq [n] \\ |S| = k}} \|\mathrm{E}_{[n]\setminus S} f\|_p^p \lesssim_p \frac{k}{n} \sum_{j=1}^n \|\tilde{\partial}_j f\|_p^p + \|\tilde{\partial}_j (f^*)\|_p^p + \left(\frac{k}{n}\right)^{p/2} \|f\|_p^p.$$

Example. $G = \mathbb{F}_n$, $g = g_{i_1}^{\ell_1} \ldots g_{i_r}^{\ell_r}$

- Word length $\psi(g) = \sum_{j=1}^r |\ell_j|$
- Cocycle derivatives $\{\partial_g\}_{g \in \mathbb{F}_n \setminus \{e\}}$.
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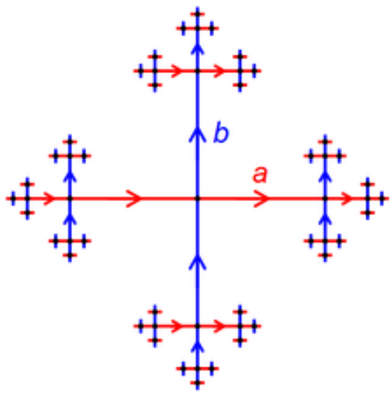

Figure 1: Wikipedia: Free group

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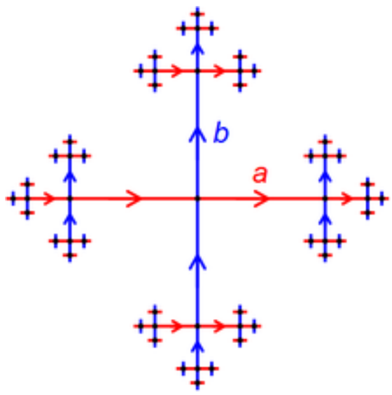

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Thanks for your attention.

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