Linear combinations of iterates of Blaschke products and Peano curves

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Lacunary series

A power series $\sum_{k=1}^{\infty} a_k z^{n_k}$ is said to be lacunar if there exists $\lambda > 1$ such that

$$\frac{n_{k+1}}{n_k} \geq \lambda \text{ for } k = 1, 2, \cdots$$

In many contexts lacunary series behave as partial sums of independent random variables and many stochastic results have the corresponding translation for these series.

Kintchine-Kolmogorov's Theorem

Suppose (X_n) is a sequence of real centered independent random variables with finite variance.
Then

$$\sum X_n \text{ converges a.s. } \iff \sum E(X_n^2) < \infty.$$

In the case of lacunary series, we have the corresponding result:

If $\sum_{k=1}^{\infty} a_k z^{n_k}$ is a lacunary series, then $\sum_{k=1}^{\infty} a_k \xi^{n_k}$ converges a.e. $\xi \in \partial\mathbb{D}$ if and only if $\sum_{n=1}^{\infty} |a_k|^2 < \infty$.

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Central Limit Theorem for lacunary series

Let's consider the trigonometric series $\sum_{k=1}^{\infty} a_k \cos(n_k x)$, where the sequence of frequencies is lacunar.

Let's put $A_N^2 = \sum_{k=1}^{N} a_k^2$ and assume that $A_N \to \infty$ and that $a_N^2 = o\left(A_N^2\right)$ as $N \to \infty$.

Then, for any $y \in \mathbb{R}$,

$$\frac{1}{2\pi} m\left(\left\{x \in [0, 2\pi] \,:\, \frac{\sum_{k=1}^{N} a_k \cos(n_k x)}{\sqrt{\frac{1}{2} A_N^2}} \leq y\right\}\right) \to \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-t^2/2}\, dt$$

Law of the iterated logarithm for lacunary series (M. Weiss)

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Assume that $A_N^2 = \sum_{k=1}^{N} a_k^2$ tends to infinity and that $a_N^2 = o\left(A_N (\log\log A_N)^{1/2}\right)$ as $N \to \infty$.
Then

$$\limsup_{N\to\infty} \frac{\sum_{k=1}^{N} a_k \cos(n_k x)}{\sqrt{2A_N^2 \log\log A_N}} = 1 \text{ a.e. } x \in [0, 2\pi]$$

Objective

We want to study the behaviour of linear combinations of the iterates of finite Blaschke products which vanish at the origin, that is iterates of

$$f(z) = z \prod_{k=1}^{N} \frac{z - z_k}{1 - \overline{z_k} z}.$$

Iterates of Blaschke products

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$$f(z) = z \prod_{k=1}^{N} \frac{z - z_k}{1 - \overline{z_k} z}.$$

If f^n denotes the n-th iterate of f and (a_k) is a sequence of complex numbers,

- What can we say about the partial sums of $\sum_{k=1}^{\infty} a_k f^k(\zeta)$ for $\zeta \in \partial\mathbb{D}$?
- How about the radial behaviour of the analytic function $\sum_{k=1}^{\infty} a_k f^k(z)$ in $\mathbb{D}$?

Iterates of Blaschke products

Theorem. Nicolau 2022

Let f a finite Blaschke product such that $f(0) = 0$ which is not a rotation, and let (a_n) a sequence of complex numbers. Then, the following propositions are equivalent.

1. The sequence (a_n) satisfies $\sum_{k=1}^{\infty} |a_k|^2 < \infty$.
2. The series $\sum_{k=1}^{\infty} a_k f^k(\zeta)$ converges a.e. $\zeta \in \partial\mathbb{D}$.
3. The set $\left\{\zeta \in \partial\mathbb{D} \,:\, \sup_N \left|\sum_{k=1}^{N} a_k f^k(\zeta)\right| < +\infty\right\}$ has positive Lebesgue measure.
4. The function defined on $\mathbb{D}$ by $F(z) = \sum_{k=1}^{\infty} a_k f^k(z)$ belongs to VMOA.

What happens if $\sum_{k=1}^{\infty} |a_k|^2 = \infty$ and $(a_k) \in \ell^{\infty}$?

In this case, the function $b(z) = \sum_{k=1}^{\infty} a_k f^k(z)$ defined by the series belongs to the Bloch space, that is,

$$\sup_{z\in\mathbb{D}}(1 - |z|^2)|b'(z)| < \infty$$

and has radial limit almost nowhere.

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and has radial limit almost nowhere.

In this situation is interesting to remember Rohde's theorem,

Theorem (Makarov, Rohde)

- A Bloch function f is radially bounded on a set of Hausdorff dimension one.
- If f belongs to $\mathcal{B}_0$ and has radial limit almost nowhere, then for any point $w \in \mathbb{C}$ there exists a set $E \subset \partial\mathbb{D}$ with $\dim_H(E) = 1$, such that for any $\zeta \in E$,

$$\lim_{r \to 1} f(r\zeta) = w.$$

Statement of our results

Theorem (D., Nicolau)

Let f be a finite Blaschke product with $f(0) = 0$ which is not a rotation and suppose that (a_n) is a sequence such that

$$\sum_{k=1}^{\infty} |a_k| = \infty \text{ and } a_k \to 0 \text{ as } k \to \infty.$$

Then, for any point $w \in \mathbb{C}$ there exists a set $E_w \subset \partial\mathbb{D}$ of positive dimension such that if $\zeta \in E_w$, $\displaystyle\sum_{k=1}^{\infty} a_k f^k(\zeta)$ converges and

$$\sum_{k=1}^{\infty} a_k f^k(\zeta) = w.$$

Obviously, the interesting case is when $\sum_{k=1}^{\infty} |a_k|^2 < \infty$, because by Nicolau's theorem, the series $\displaystyle\sum_{k=1}^{\infty} a_k f^k(\zeta)$ converges a.e.

Statement of our results

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In order to state it, we have the following result that can be understood as a version of Abel's Theorem in our context.

Theorem (D., Nicolau)

Let f be a finite Blaschke product with $f(0) = 0$ which is not a rotation. Let (a_n) be a sequence and suppose that for some $\zeta \in \partial\mathbb{D}$ we have that $\sum_{k=1}^{\infty} a_k f^k(\zeta)$ converges. Then the non-tangencial limit $\lim_{\substack{z \to \zeta \\ \not<}} \sum_{k=1}^{\infty} a_k f^k(z)$ exists and it is equal to $\sum_{k=1}^{\infty} a_k f^k(\zeta)$. The converse is true.

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Let f be a finite Blaschke product with $f(0) = 0$ which is not a rotation and suppose that (a_n) is a sequence such that

$$\sum_{k=1}^{\infty} |a_k| = \infty \text{ and } a_k \to 0 \text{ as } k \to \infty.$$

Then, for any point $w \in \mathbb{C}$ there exists a set $E_w \subset \partial\mathbb{D}$ of positive dimension such that if $\zeta \in E_w$, the function $F(z) = \sum_{k=1}^{\infty} a_k f^k(z)$ has nontangential limit w at ζ

Statement of our results

A possible way to improve last result is the following one:
Given an analytic function $g : \mathbb{D} \to \mathbb{C}$ and a point $\zeta \in \partial\mathbb{D}$, the radial cluster set of g at the point ζ is defined to be

$$\mathrm{Cl}(g,\zeta) = \bigcap_{r<1} \overline{\{g(s\zeta)\ :\ s \geq r\}}.$$

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$$\sum_{k=1}^{\infty} |a_k| = \infty \text{ and } a_k \to 0 \text{ as } k \to \infty,$$

and consider the function $F(z) = \sum_{k=1}^{\infty} a_k f^k(z)$.
Then, for any closed connected set $K \subset \mathbb{C}_\infty$ there exists a set of positive dimension E_K such that if $\zeta \in E_K$, then $\mathrm{Cl}(F, \zeta) = K$.

How about the case $\sum_{k=1}^{\infty} |a_k| < \infty$?

It is clear that in this case the function defined by

$$F(z) = \sum_{k=1}^{\infty} a_k f^k(z)$$

is continuous in $\overline{\mathbb{D}}$.

In order to state our result, let's observe this elementary Calculus fact.

Suppose we have a sequence of positive numbers (x_n) such that $\sum_{n=1}^{\infty} x_n = S < \infty$. Suppose that the convergence of this series is slow in the sense that for any n,

$$\frac{x_n}{\sum_{j>n} x_j} < 1.$$

Adding and substracting at choice every term x_n, what is the set of all possible values we can obtain? That is, what is the set

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Answer: This set is $[-S, S]$.

Statement of our results

Keeping in mind last fact, we have the following result.

Theorem (D., Nicolau)

Let f be a finite Blaschke product with $f(0) = 0$ which is not a rotation. Let (a_n) be a sequence such that $\sum_{n=1}^{\infty} |a_n| < \infty$ and

$$\lim_{n\to\infty} \frac{|a_n|}{\sum_{j>n} |a_j|} = 0.$$

Then the image of $\partial\mathbb{D}$ under the function $F = \sum_{n\geq 1} a_n f^n$ is a Peano curve, that is, $F(\partial\mathbb{D})$ contains a disk.

Comments

In general, results like ours, requires arguments based on a "Scaping balls lemma".

Suppose (W_t) is a planar brownian motion with $W_0 = 0$ and consider a circle C centered at the origin with radius r. If J is an arc of this circle, then we know that the probability that W_t scapes from C across J is $|J|/2\pi r$.

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For instance, a possible way to prove Rohde's Theorem is the following.

Theorem (Rohde)

If f belongs to $\mathcal{B}_0$ and has radial limit almost nowhere, then for any point $w \in \mathbb{C}$ there exists a set $E \subset \partial\mathbb{D}$ with $\dim_H(E) = 1$, such that for any $\zeta \in E$,

$$\lim_{r \to 1} f(r\zeta) = w.$$

Comments

- Given the function f it is possible to associate it a complex discrete martingale (W_n) defined on $\partial\mathbb{D}$ with increments tending to 0 which governs the radial behaviour of f, that is,

$$|f(r\zeta) - W_n(\zeta)| \text{ is small if } r \sim 1 - 2^{-n}.$$

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- This implies, in particular that this martingale diverges a.e., consequently, if C is a circle or radius R centered at the origin, the stopping time $\tau(\zeta) = \inf\{n : |W_n(\zeta)| > R\}$ is finite a.e. $\zeta \in \partial\mathbb{D}$.

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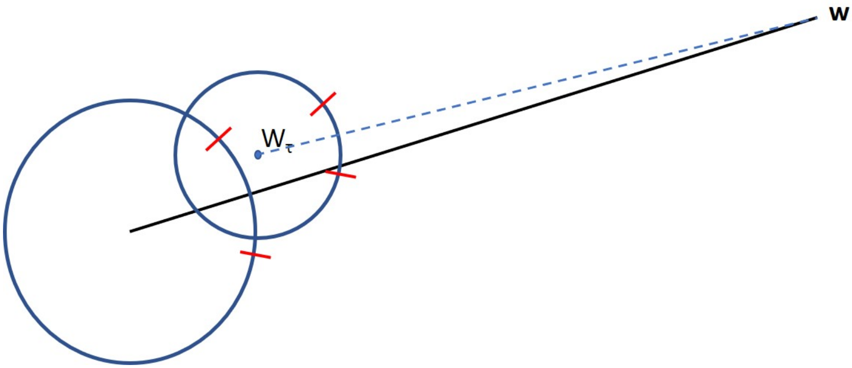
- The fact that f is analytic, allows us to obtain something extra: when the martingale scapes from C, the values $W_{\tau(\zeta)}(\zeta)$ are, essentially, uniformly distributed along C (a sort of isotropy) and if Γ is an arc in C, the set of points $\zeta \in \partial\mathbb{D}$ for which the martingale scapes across Γ has measure essentially $|\Gamma|/2\pi R$.

Comments



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Our proofs are inspired in the ideas of M. Weiss but in absence of lacunarity, we explote a nice interplay between the dynamical properties of f as a selfmapping of the unit disk and the dynamics of f^n at the boundary.

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Theorem

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Fix $w \in \mathbb{C}$. The idea of the proof of this result is to find a sort of "scaping balls lemma" a little bit technical and suitable decomposition of the sum $\sum_{k=1}^{\infty} a_k f^k$ into blocks.

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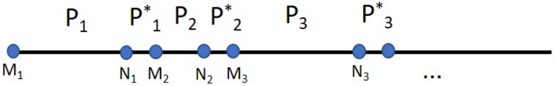
That is, we will find a sequence of indices
$1 = M_1 < N_1 < M_2 < N_2 < M_3 < \cdots$ and we will define blocks

$$P_j = \sum_{n=M_j}^{N_j} a_n f^n, \qquad P_j^* = \sum_{n=N_j+1}^{M_{j+1}-1} a_n f^n$$



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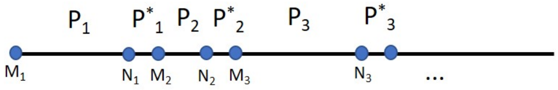

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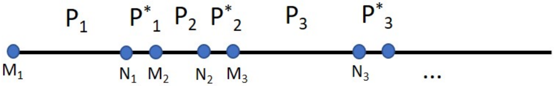

such that

- The blocks P^* will have fix length (maybe large) and have the mission to prepare the blocks P for the application of the "scaping balls lemma".
- The blocks P have no controled length and they have the mission to make that the sequence $\sum_{k=1}^{\infty} a_k f^k$ converges to w at some point $\zeta \in \partial\mathbb{D}$.

THANK YOU