Strong $BV-$extension and $W^{1,1}-$extension domains

(XX Encuentros de Análisis Real y Complejo)

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$W^{1,p}$ and BV spaces

Let $\Omega \subset \mathbb{R}^n$ be a domain for some $n \geq 2$. We always work with $\mathbb{R}$-valued functions. $u : \Omega \to \mathbb{R}$, $u : \mathbb{R}^n \to \mathbb{R}$.

Definition

1. For every $1 \leq p \leq \infty$, we define the **Sobolev space** $W^{1,p}(\Omega)$ to be

$$W^{1,p}(\Omega) = \{u \in L^p(\Omega) : \nabla u \in L^p(\Omega; \mathbb{R}^n)\},$$

   with norm $\|u\|_{W^{1,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \|\nabla u\|_{L^p(\Omega)}$.

2. The space of **functions of bounded variation** is

$$BV(\Omega) = \{u \in L^1(\Omega) : \|Du\|(\Omega) < \infty\}$$

   where $\|Du\|(\Omega) = \sup\left\{\int_\Omega u \,\mathrm{div}(v)\, dx : v \in C_c^\infty(\Omega; \mathbb{R}^n), |v| \leq 1\right\}$ denotes the total variation of u on Ω. We endow this space with the norm $\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + \|Du\|(\Omega)$.

- $\mathbf{W^{1,1}(\Omega) \subset BV(\Omega)}$.
- Examples of BV but not $W^{1,1}$: **Heaviside and Cantor functions**.

$W^{1,p}-$ and $BV-$extension domains

We say that $\Omega \subset \mathbb{R}^n$ is a $\mathbf{W^{1,p}}-$**extension domain** or $\mathbf{BV}-$**extension domain** if there exists a (not necessarily linear) operator

$$T\colon W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^n) \ \ or \ \ T\colon BV(\Omega) \to BV(\mathbb{R}^n)$$

and a constant $C > 0$ so that $Tu|_\Omega = u$ and

$$\|Tu\|_{W^{1,p}(\mathbb{R}^n)} \leq C\|u\|_{W^{1,p}(\Omega)} \ \ or \ \ \|Tu\|_{BV(R^n)} \leq C\|u\|_{BV(\Omega)}$$

for every $u \in W^{1,p}(\Omega)$ or $u \in BV(\Omega)$ respectively.

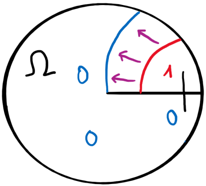
Theorem

1. *(**Calderón** and **Stein**, 1961, 1970): Lipschitz domains are $W^{1,p}-$extension domains for all $1 \leq p \leq \infty$.*
2. *(**Jones**, 1981): Uniform domains are $W^{1,p}-$extension domains for all $1 \leq p \leq \infty$.*

Despite many partial results in the last 30 years, a complete characterization of Sobolev extension domains is still missing!

- **Ω is $\mathbf{W^{1,1}}$ – extension domain $\Rightarrow$ Ω is $\mathbf{BV}$ – extension domain.**

The converse is not true! The slit disc is a counterexample.



$\Omega \equiv$ Slit disc

Let $u \in W^{1,1}(\Omega)$,

$$\begin{cases} u \equiv 1 & \text{in red part} \\ u \equiv 0 & \text{in blue part} \end{cases}$$

Any $Eu \in W^{1,1}(\mathbb{R}^2)$ extension is not abs. cont on a.e. line parallel to the coordinate axes.

Theorem

$f \in W^{1,p}(\Omega)$ if and only if $f \in L^p(\Omega)$ and f has a representative that is absolutely continuous on $\mathcal{H}^{n-1}$–almost every line within Ω parallel to the coordinate axes and whose partial derivatives belong to $L^p(\Omega)$.

About $W^{1,1}-$ and $BV-$extension domains

QUESTION: What prevents a $BV-$extension domain to be a $W^{1,1}$ extension domain?

About $W^{1,1}-$ and $BV-$extension domains

Let us think about $\mathbb{R}^2$.

Definition

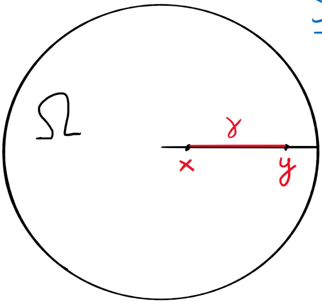
A domain $\Omega \subset \mathbb{R}^n$ is **quasiconvex** if $\exists C > 0$ so that for all $x, y \in \Omega$ there is a rectifiable curve $\gamma \subset \Omega$ so that $\ell(\gamma) \leq C|x-y|$.

Theorem

Let $\Omega \subset \mathbb{R}^2$ be a bounded and simply connected domain.

1. *(**Koskela, Miranda, Shanmugalingam**, 2010) Ω is a $BV-$extension domain if and only if Ω^c is quasiconvex.*
2. *(**Koskela, Rajala, Zhang**, preprint) Ω is a $W^{1,1}-$extension domain if and only if there is $C > 0$ so that for every $x, y \in \Omega^c$ there exists a curve $\gamma \subset \Omega^c$ connecting x and y with*

$$\ell(\gamma) \leq C|x-y|, \text{ and } \mathcal{H}^1(\gamma \cap \partial\Omega) = 0.$$



Split disc

$$\Omega = \mathbb{D} \setminus \left([0,1]\times\{0\}\right)$$

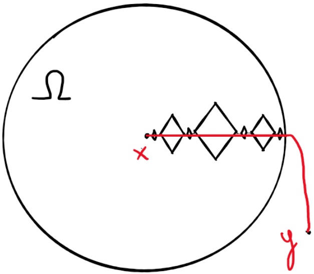
→ Ω^c is quasiconvex $\Rightarrow$

$\Rightarrow \Omega$ is BV- ext. dom.

→ Ω is not $W^{1,1}$- ext. dom.

$\forall \gamma \subseteq \Omega^c$ connecting x and y , $\mathcal{H}^1(\gamma \cap \partial\Omega) > 0$

Examples



Let $C \subseteq [0,1]$ be a Cantor set with $\mathcal{H}^1(C)=0$

$$N := \{(x_1,x_2) : x_1 \in [0,1],\ |x_2| \le \mathrm{dist}(x_1, C\times\{0\})\}$$

$$\Omega = \mathbb{D} \setminus N$$

$\Rightarrow \Omega$ is BV- and $W^{1,1}$-extension domain.

Planar Jordan domains

Theorem (Koskela, Rajala, Zhang, preprint)

Let $\Omega \subset \mathbb{R}^2$ be a Jordan domain. The following are equivalent:

1. *Ω is a $W^{1,1}-$extension domain.*
2. *Ω is a $BV-$extension domain.*
3. *$\mathbb{R}^2 \setminus \overline{\Omega}$ is quasiconvex.*

Still thinking...
what are the differences between $BV-$ and $W^{1,1}-$extension domains in $\mathbb{R}^n$?

Naive idea

Assume $\Omega \subset \mathbb{R}^n$ is a $BV-$extension domain and let us try to see if Ω could be a $W^{1,1}-$extension domain as well.

- Take $u \in W^{1,1}(\Omega)$.
- Since $W^{1,1}(\Omega) \subset BV(\Omega)$ there exists an extension $Tu \in BV(\mathbb{R}^n)$ so that $\|Tu\|_{BV(\mathbb{R}^n)} \leq C\|u\|_{BV(\Omega)} \leq C\|u\|_{W^{1,1}(\Omega)}$.
- We can "smooth" Tu on $\mathbb{R}^n \setminus \overline{\Omega}$ to get $\widetilde{u} \in W^{1,1}(\mathbb{R}^n \setminus \partial\Omega)$ and so that $\|\widetilde{u}\|_{W^{1,1}(\mathbb{R}^n \setminus \partial\Omega)} \leq C\|u\|_{W^{1,1}(\Omega)}$.
- In an ideal world we would like to assume that $\|D\tilde{u}\|(\partial\Omega) = 0$. In such a case $\tilde{u} \in W^{1,1}(\mathbb{R}^n)$ and we would have proved that Ω is a $W^{1,1}-$extension operator.

Conclusion: We want to avoid $\|D\tilde{u}\|(\partial\Omega) > 0$. This is achieved if we work with the following stronger definition of $BV-$extension domain.

Definition

A domain $\Omega \subset \mathbb{R}^n$ is a **strong $\mathbf{BV}$–extension domain** if $\exists C > 0$ so that $\forall u \in BV(\Omega)$, $\exists Tu \in BV(\mathbb{R}^n)$ with

1. $Tu|_\Omega = u$
2. $\|Tu\|_{BV(\mathbb{R}^n)} \leq C\|u\|_{BV(\Omega)}$, and
3. $\|D(Tu)\|(\partial\Omega) = 0$.

1st main theorem (G-B, Rajala, preprint)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then the following are equivalent:

1. Ω is a $W^{1,1}$–extension domain.
2. Ω is a strong BV–extension domain.

BV extension domains and extension of sets of finite perimeter

A Lebesgue measurable subset $E \subset \mathbb{R}^n$ has **finite perimeter** in Ω if $\chi_E \in BV(\Omega)$. We set $P(E,\Omega) = \|D\chi_E\|(\Omega) = \mathcal{H}^{n-1}(\partial^M E \cap \Omega)$.
The **measure theoretic boundary** is the set

$$\partial^M E = \left\{ x \in \mathbb{R}^n : \limsup_{r \searrow 0} \frac{|E \cap B(x,r)|}{|B(x,r)|} > 0 \text{ and } \limsup_{r \searrow 0} \frac{|(\mathbb{R}^n \setminus E) \cap B(x,r)|}{|B(x,r)|} > 0 \right\}$$

Theorem (Burago, Mazya, 1967 + Koskela, Miranda and Shanmugalingam 2010)

If Ω is a bounded domain in $\mathbb{R}^n$ then Ω is a $BV-$extension domain if and only if any set $E \subset \Omega$ of finite perimeter in Ω admits an extension $\widetilde{E} \subset \mathbb{R}^n$ satisfying

1. $\widetilde{E} \cap \Omega = E$,
2. $P(\widetilde{E}, \mathbb{R}^n) \leq C P(E,\Omega)$ *where $C > 0$ is some absolute constant.*

Recall the definition of strong $BV-$extension domain.

Definition

A domain $\Omega \subset \mathbb{R}^n$ is a **strong $\mathbf{BV}-$extension domain** if $\exists C > 0$ so that $\forall u \in BV(\Omega)$, $\exists Tu \in BV(\mathbb{R}^n)$ with

1. $Tu|_\Omega = u$
2. $\|Tu\|_{BV(\mathbb{R}^n)} \le C\|u\|_{BV(\Omega)}$, and
3. $\|D(Tu)\|(\partial\Omega) = 0$.

Translate it to the language of extension of sets of finite perimeter.

Definition

A domain $\Omega \subset \mathbb{R}^n$ has the **strong extension property for sets of finite perimeter** if $\exists C > 0$ so that $\forall E \subset \Omega$ of finite perimeter in Ω, $\exists \widetilde{E} \subset \mathbb{R}^n$ such that

1. $\widetilde{E} \cap \Omega = E$,
2. $P(\widetilde{E}, \mathbb{R}^n) \le CP(E, \Omega)$, and
3. $\mathcal{H}^{n-1}(\partial^M \widetilde{E} \cap \partial\Omega) = 0$.

2^{nd} main result

2nd main theorem (G-B, Rajala, preprint)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Then the following are equivalent:

1. Ω is a $W^{1,1}-$extension domain.
2. Ω is a strong $BV-$extension domain.
3. Ω has the strong extension property for sets of finite perimeter.

Corollary

Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded $W^{1,1}-$extension domain. Let Ω_i, for $i \in I$, be the connected components of $\mathbb{R}^n \setminus \overline{\Omega}$. Then

$$\mathbf{H} = \partial\mathbf{\Omega} \setminus \bigcup_{\mathbf{i}\in\mathbf{I}} \overline{\mathbf{\Omega_i}} \text{ is purely } (\mathbf{n}-\mathbf{1})-\text{unrectifiable,}$$

i.e., for every Lipschitz map $f\colon \mathbb{R}^{n-1} \to \mathbb{R}^n \Rightarrow \mathcal{H}^{n-1}(H \cap f(\mathbb{R}^{n-1})) = 0$.

Necessary geometric condition

We have found a **necessary geometric condition** in the boundary for $W^{1,1}-$extension domains.

Corollary

Suppose that $\Omega \subset \mathbb{R}^n$ is a bounded $W^{1,1}-$extension domain. Let Ω_i, for $i \in I$, be the connected components of $\mathbb{R}^n \setminus \overline{\Omega}$. Then

$$\mathbf{H} = \partial\mathbf{\Omega} \setminus \bigcup_{\mathbf{i}\in\mathbf{I}} \overline{\mathbf{\Omega_i}} \text{ is purely } (\mathbf{n}-\mathbf{1}) - \text{unrectifiable,}$$

i.e., for every Lipschitz map $f\colon \mathbb{R}^{n-1} \to \mathbb{R}^n \Rightarrow \mathcal{H}^{n-1}(H \cap f(\mathbb{R}^{n-1})) = 0.$

Question: Is this condition sufficient for a bounded $BV-$extension domain in order to be a $W^{1,1}-$extension domain?

3^{rd} main result

• For the case of bounded $BV-$extension domains $\mathbf{\Omega} \subset \mathbb{R}^{\mathbf{2}}$ the answer is **YES**.

3^{rd} main theorem (G-B, Rajala, preprint)

Let $\Omega \subset \mathbb{R}^2$ be a bounded $BV-$extension domain. Then Ω is a $W^{1,1}-$extension domain if and only if the set

$$H = \partial\Omega \setminus \bigcup_{i \in I} \overline{\Omega_i}$$

is purely $1-$unrectifiable, where $\{\Omega_i\}_{i \in I}$ are the connected components of $\mathbb{R}^2 \setminus \overline{\Omega}$.

• For the case of bounded $BV-$extension domains $\Omega \subset \mathbb{R}^n$ with $\mathbf{n \geq 3}$ the answer is **NO**.

With our 3^{rd} main result one can "easily" recover some previous results.

Theorem (Koskela, Rajala, Zhang, preprint)

$\Omega \subset \mathbb{R}^2$ bounded simply connected is a $W^{1,1}-$extension domain if and only if for every $x, y \in \Omega^c$ there exists a curve $\gamma \subset \Omega^c$ connecting x and y with

$$\ell(\gamma) \leq C|x-y|, \text{ and } \mathcal{H}^1(\gamma \cap \partial\Omega) = 0.$$

Theorem (Koskela, Rajala, Zhang, preprint)

$\Omega \subset \mathbb{R}^2$ Jordan domain is a $W^{1,1}-$extension domain if and only if it is a $BV-$extension domain. Indeed for this case one has

$$H = \partial\Omega \setminus \bigcup_{i \in I} \overline{\Omega_i} = \emptyset.$$

Other consequences: Bi-Lipschitz invariance

Definition

Given two domains $\Omega, \Omega' \subset \mathbb{R}^n$ we say they are **bi-Lipschitz equivalent** if there exists a homeomorphism $f : \Omega \to \Omega'$ and a constant $L > 0$ so that

$$L^{-1}|x-y| \leq |f(x)-f(y)| \leq L|x-y|$$

for all $x, y \in \Omega$.

QUESTION: If $\Omega, \Omega' \subset \mathbb{R}^n$ are two bi-Lipschitz equivalent domains and Ω is a $W^{1,p}-$extension domain, is then Ω' a $W^{1,p}-$extension domain?

Observation

Suppose given $f : \Omega \to \Omega'$ bi-Lipschitz there exists a bi-Lipschitz homeomorphic extension $F : \mathbb{R}^n \to \mathbb{R}^n$. In such a case the problem is trivial. However in general no such extension exists!

Other consequences: Bi-Lipschitz invariance

QUESTION: If $\Omega, \Omega' \subset \mathbb{R}^n$ are two bi-Lipschitz equivalent domains and Ω is a $W^{1,p}-$extension domain, is then Ω' a $W^{1,p}-$extension domain?

1. Case $1 < p \leq \infty$: YES. **Hajłasz, Koskela and Tuominen** (2008).
2. Case $p = 1$:
   1. If $\Omega \subset \mathbb{R}^2$ bounded simply connected. YES. **Koskela, Miranda, Shanmugalingam** (2010).
   2. Otherwise unknown.
3. BV case.
   1. If $\Omega \subset \mathbb{R}^2$ bounded simply connected. YES. **Koskela, Miranda, Shanmugalingam** (2010).
   2. Otherwise unknown.

Other consequences: Bi-Lipschitz invariance

We were able to solve the case of **bounded domains $\boldsymbol{\Omega} \subset \mathbb{R}^{\mathbf{2}}$** (not necessarily simply connected).

Theorem (G-B, Rajala, Zhu, 2021)

Let $\Omega, \Omega' \subset \mathbb{R}^2$ be bi-Lipschitz equivalent bounded domains. Then

1. *Ω is a $BV-$extension domain if and only if Ω' is a $BV-$extension domain.*
2. *Ω is a $W^{1,1}-$extension domain if and only if Ω' is a $W^{1,1}-$extension domain.*

References

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4. **P. Koskela, T. Rajala and Y. Zhang**, *Planar* $W^{1,1}-$*extension domains*, preprint 2017.

THANKS FOR YOUR ATTENTION !