Semigroups of composition operators on Hardy spaces of Dirichlet series

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Outline

1. Dirichlet series and Banach spaces of Dirichlet series
2. Composition operators in $\mathcal{H}^2$
   - Semigroups of analytic functions in $\mathbb{C}_+$
   - Semigroups of composition operators
3. Infinitesimal generators
4. Continuous semigroups in $\mathcal{H}^\infty$

Dirichlet series

Notation: given $\theta \in \mathbb{R}$, we set $\mathbb{C}_\theta := \{s \in \mathbb{C} : \mathrm{Re}(s) > \theta\}$ and $\mathbb{C}_+ = \mathbb{C}_0$.

We denote by $\mathcal{D}$ the space of convergent Dirichlet series, namely, the series

$$\varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s},$$

which are convergent in some half-plane $\mathbb{C}_\theta$.

Remarks:

- In this context, instead of radii of convergence, as is the case of Taylor series, we have abscissae of convergence. Hence, Dirichlet series converge in half-planes.
- In the half-plane of convergence, Dirichlet series define analytic functions.

To each Dirichlet series $\varphi(s) = \sum_{n=1}^{\infty} a_n n^{-s}$, one can associate the following abscissae:

$$\sigma_c(\varphi) = \inf\{\mathrm{Re}\, s : \sum_{n=1}^{\infty} a_n n^{-s} \text{ is convergent}\};$$

$$\sigma_u(\varphi) = \inf\{\sigma : \sum_{n=1}^{\infty} a_n n^{-s} \text{ is uniformly convergent on } \mathbb{C}_\sigma\};$$

$$\sigma_b(\varphi) = \inf\{\sigma : \sum_{n=1}^{\infty} a_n n^{-s} \text{ is bounded on } \mathbb{C}_\sigma\};$$

$$\sigma_a(\varphi) = \inf\{\mathrm{Re}\, s : \sum_{n=1}^{\infty} a_n n^{-s} \text{ is absolutely convergent}\}.$$

Theorem (Bohr)

There exists the following relation between the abcissae above

$$\sigma_c(\varphi) \leq \sigma_u(\varphi) = \sigma_b(\varphi) \leq \sigma_a(\varphi) \leq \sigma_c(\varphi) + 1.$$

The Hardy-Dirichlet space $\mathcal{H}^2$

We define the space $\mathcal{H}^2$ as the collection of Dirichlet series

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for which

$$\sum_{n=1}^{\infty} |a_n|^2 < \infty.$$

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- Banach space of analytic functions in $\mathbb{C}_{1/2}$ (Cauchy-Schwarz).

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Remarks:

- Banach space of analytic functions in $\mathbb{C}_{1/2}$ (Cauchy-Schwarz).
- Hilbert space structure endowed with the scalar product

$$\langle f, g \rangle = \sum_{n=1}^{\infty} a_n \overline{b}_n,$$

where $f(s) = \sum_{n=1}^{\infty} a_n n^{-s}$ and $g(s) = \sum_{n=1}^{\infty} b_n n^{-s}$ belong to $\mathcal{H}^2$.

The Hardy-Dirichlet space $\mathcal{H}^\infty$

- Let $\varepsilon \geq 0$. The space of bounded Dirichlet series $\mathcal{H}^\infty(\mathbb{C}_\varepsilon)$ consists of all analytic functions bounded in $\mathbb{C}_\varepsilon$ such that they can be written as a Dirichlet series in a certain half-plane. This is,

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- If we endow the space $\mathcal{H}^\infty(\mathbb{C}_\varepsilon)$ with the norm given by

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- For $\varepsilon = 0$, we simply write $\mathcal{H}^\infty$ to denote $\mathcal{H}^\infty(\mathbb{C}_+)$.

Composition operators on $\mathcal{H}^2$

Given an analytic function $\Phi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$, we define the associated composition operator C_Φ on $\mathcal{H}^2$ as

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Definition

Let $\Phi : \mathbb{C}_+ \to \mathbb{C}_+$ be an analytic function. We say that Φ belongs to the Gordon-Hedenmalm class $\mathcal{G}$ if

- *There exists $c_\Phi \in \mathbb{N} \cup \{0\}$ and φ a Dirichlet series such that*

$$\Phi(s) = c_\Phi s + \varphi(s), \quad s \in \mathbb{C}_+.$$

- *If $c_\Phi = 0$, then $\Phi(\mathbb{C}_+) \subset \mathbb{C}_{1/2}$.*

The value c_Φ is known as the characteristic of the function Φ.

Boundedness of composition operators

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- *If* $c_\Phi = 0$*, then* $\Phi(\mathbb{C}_+) \subset \mathbb{C}_{1/2}$.

The value c_Φ *is known as the characteristic of the function* Φ.

Theorem (Gordon-Hedenmalm, '99)

An analytic function $\Phi : \mathbb{C}_{1/2} \to \mathbb{C}_{1/2}$ *defines a bounded composition operator* $\mathcal{C}_\Phi : \mathcal{H}^2 \to \mathcal{H}^2$ *if and only if* Φ *has a holomorphic extension to* $\mathbb{C}_+$ *that belongs to the class* $\mathcal{G}$.

Remark: even though the elements in $\mathcal{H}^2$ define analytic functions in $\mathbb{C}_{1/2}$, the symbol Φ must have an analytic extension to $\mathbb{C}_+$.

Definition

A family $\{\Phi_t\}_{t\geq 0}$ *of analytic functions* $\Phi_t : \mathbb{C}_+ \to \mathbb{C}_+$ *is a semigroup if it verifies:*

(i) $\Phi_0(s) = s$.

(ii) *For every* $t, u \geq 0$, $\Phi_t \circ \Phi_u(s) = \Phi_{t+u}(s)$.

If, in addition, it satisfies

(iii) $\Phi_t \to \Phi_0$

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Let X be a Banach space and $\{T_t\}_{t\geq 0}$ such that $T_t : X \to X$ bounded. We say that $\{T_t\}_{t\geq 0}$ is a semigroup if:

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(ii) *For every* $t, u \geq 0$, $T_t \circ T_u = T_{t+u}$.

If, in addition, it satisfies that

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In our context:

- Continuous semigroups of analytic functions in the class $\mathcal{G}$.
- Semigroups of composition operators in $X = \mathcal{H}^2$.

- Clearly, given a semigroup $\{\Phi_t\}_t$ such that $\Phi_t \in \mathcal{G}$ for every t, $T_t = C_{\Phi_t}$ defines a semigroup of composition operators.

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- Consider the continuous semigroup $\Phi_t(s) = s + at$, $\mathrm{Re}\,(a) \geq 0$, in the class $\mathcal{G}$. The semigroup of composition operators $\{T_t\}_t$, $T_t = C_{\Phi_t}$ is strongly continuous.

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Question

Is it true in general that given a continuous semigroups of analytic functions $\{\Phi_t\}$ in $\mathcal{G}$ induce strongly continuous semigroups of composition operators $\{T_t\}_t$, $T_t = C_{\Phi_t}$? Does the converse hold?

Theorem

Let $\{\Phi_t\}_{t\geq 0}$ be a semigroup of analytic functions, such that $\Phi_t \in \mathcal{G}$ for every $t > 0$ and denote by T_t the composition operator $T_t(f) = f \circ \Phi_t$. Then, the following assertions are equivalent:

a. $\{T_t\}_{t\geq 0}$ *is a strongly continuous semigroup in* $\mathcal{H}^2$.
b. $\{\Phi_t\}_{t\geq 0}$ *is a continuous semigroup.*
c. $\Phi_t(s) \to s$, *as* t *goes to* 0, *uniformly in* $\mathbb{C}_\varepsilon$, *for every* $\varepsilon > 0$.

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- a) *$\{T_t\}_{t\geq 0}$ is a strongly continuous semigroup in $\mathcal{H}^2$.*
- b) *$\{\Phi_t\}_{t\geq 0}$ is a continuous semigroup.*
- c) *$\Phi_t(s) \to s$, as t goes to 0, uniformly in $\mathbb{C}_\varepsilon$, for every $\varepsilon > 0$.*

The proof of $b) \Leftrightarrow c)$ depends strongly on the properties of $\mathcal{G}$. This implication fails for general continuous semigroups of analytic functions in $\mathbb{C}_+$. Indeed, consider

$$\Phi_t(s) = \left(\frac{t}{2} + s^{1/2}\right)^2, \quad s \in \mathbb{C}_+$$

where we are taking the principal branch of the square root. Clearly,

- $\Phi_0(s) = s, \qquad \Phi_{t+u}(s) = \Phi_t(\Phi_u(s))$.
- $\Phi_t(s) - s = \frac{t^2}{4} + ts^{1/2} \to 0$ as $t \to 0$ uniformly on compact subsets of $\mathbb{C}_+$. However, $\Phi_t(s) \not\to s$ uniformly in half-planes $\mathbb{C}_\epsilon$, $\epsilon > 0$.

Ideas of the proof

We need an auxiliary Lemma.

Proposition

Let $\{\Phi_t\}_{t\geq 0}$ be a continuous semigroup of analytic functions in the class $\mathcal{G}$. Set $c_t = c_{\Phi_t}$. Then, the characteristics $\{c_t\}_{t\geq 0}$ of the symbols $\{\Phi_t\}_t$ is constantly equal to 1.

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Key ideas:

- $t \mapsto c_t$ measurable and $c_t c_u = c_{t+u}$, $t, u \geq 0 \Rightarrow c_t \in \{0, 1\}$.
- Elements of a continuous semigroup are injective.
- Dirichlet series are never injective.

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- Dirichlet series are never injective.

Conclusion: given a continuous semigroups $\{\Phi_t\}_t$ of elements in $\mathcal{G}$,

$$\Phi_t(s) = s + \varphi_t(s), \quad \varphi_t \in \mathcal{D}.$$

$a) \Rightarrow c)$ ($\{T_t\}$ strongly continuous $\Longrightarrow$ $\{\Phi_t\}$ continuous)

Long proof (Bayart theorem, Baire's category Theorem). The semigroup structure is essential. In general, given $\{T_n\}_{n\in\mathbb{N}}$, $T_n f = f \circ \Phi_n$, convergent to the identity operator in the SOT $\not\Longrightarrow$ local uniform convergence to the identity of $\{\Phi_n\}_{n\in\mathbb{N}}$.

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Example

By Kronecker's Theorem, we can take a sequence $\{x_n\}_n \in \mathbb{R}$ such that $|x_n| \to \infty$ when $n\to\infty$ and $m^{-ix_n} \to 1$ as $n\to\infty$ for all $m\in\mathbb{N}$.

Let $\Phi_n(s) = s + ix_n$, $s\in\mathbb{C}_+$. Define $T_n f = f\circ\Phi_n$.
Then, if $f(s) = \sum_{m\geq 1} a_m m^{-s} \in \mathcal{H}^2$, using the DCT we obtain

$$\lim_{n\to\infty} \|f - f\circ\Phi_n\|^2_{\mathcal{H}^2} = \lim_{n\to\infty} \sum_{m=1}^{\infty} |a_m|^2 |1 - m^{-ix_n}|^2 = 0.$$

However, by the definition of $\{x_n\}_n$, $\Phi_n(s) \not\to s$ as $n\to\infty$.

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- The pointwise evaluation functional δ_s is bounded for $s \in \mathbb{C}_{1/2}$.

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The conclusion follows from a standard compactness argument involving the ideas above.

Infinitesimal generators

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Theorem (Berkson-Porta, '78)

Let $\{\Phi_t\}_t$ be a continuous semigroup of analytic functions in $\mathbb{C}_+$. Then, there exists

$$H(s) = \lim_{t \to 0^+} \frac{\Phi_t(s) - s}{t}, \qquad \textit{for all } s \in \mathbb{C}_+$$

and such limit is uniform on compact sets of $\mathbb{C}_+$. Moreover, given $s \in \mathbb{C}_+$, $\Phi_t(s)$ is the solution to the Cauchy problem

$$\begin{cases} y'(t) = H(y(t)) \\ y(0) = s \end{cases}$$

The holomorphic function H is called the *infinitesimal generator* of the semigroup $\{\Phi_t\}_t$.

- For continuous semigroups $\{\Phi_t\}_t$ in $\mathcal{G}$, we have that $H : \mathbb{C}_+ \to \overline{\mathbb{C}_+}$.

  Indeed, we know that for every $t > 0$, $\Phi_t(s) = s + \varphi_t(s)$, where $\varphi_t \in \mathcal{D}$ and $\varphi_t : \mathbb{C}_+ \to \mathbb{C}_+$. By the definition of H,

$$\mathrm{Re}(H(s)) = \lim_{t\to 0} \mathrm{Re}\left(\frac{\Phi_t(s) - s}{t}\right) = \lim_{t\to 0} \mathrm{Re}\left(\frac{\varphi_t(s)}{t}\right) \geq 0.$$

- Berkson-Porta: given $H : \mathbb{C}_+ \to \overline{\mathbb{C}_+}$ holomorphic, then H is the infinitesimal generator of a continuous semigroup $\{\Phi_t\}_t$ of analytic functions in $\mathbb{C}_+$ such that $\Phi_t(\infty) = \infty$.

Question

Can we describe the infinitesimal generators associated to continuous semigroups $\{\Phi_t\}_t$ in the class $\mathcal{G}$?

Theorem

Let $H : \mathbb{C}_+ \to \overline{\mathbb{C}}_+$ be analytic. Then, the following statements are equivalent:

a. *H is the infinitesimal generator of a continuous semigroup of elements in the class $\mathcal{G}$.*
b. *$H \in \mathcal{H}^\infty(\mathbb{C}_\varepsilon)$, for all $\varepsilon > 0$.*
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Comments:

- $b) \iff c)$: well-known result in Dirichlet series theory.
- $a) \implies c)$: not that surprising, locally uniform limit of Dirichlet series.
- $b) \implies a)$: most surprising. Use an adapted Cauchy-Picard type argument to see that the solution of the Cauchy Problem is still in the class $\mathcal{G}$.

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Answer: there are 'many' continuous semigroups in $\mathcal{G}$.

Example: for $H(s) = 1 + 2^{-s}$, we obtain the semigroup in $\mathcal{G}$ given by

$$\Phi_t(s) = s + t + \frac{1}{\ln 2}\mathrm{Log}(1 + 2^{-s}(1 - 2^{-t})), t \geq 0,\ s \in \mathbb{C}_+.$$

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Theorem (Bayart'02)

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Remark: every bounded composition operator in $\mathcal{H}^2$ is bounded in $\mathcal{H}^\infty$. The converse is false.

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Remark: every bounded composition operator in $\mathcal{H}^2$ is bounded in $\mathcal{H}^\infty$. The converse is false.

Theorem

Let $\{T_t\}_{t \geq 0}$ be a strongly continuous semigroup of composition operators in $\mathcal{H}^\infty$. Then, $T_t = \mathrm{Id}$ for every $t \geq 0$.

Idea: for a strongly continuous semigroup, the operator $f \mapsto Hf'$, $f \in \mathcal{H}^\infty$, must have dense range. If this is the case, there exists a point in $i\mathbb{R}$ such that **every** function in $\mathcal{H}^\infty$ has non-tangential limit there. Impossible.

Thank You!