Nuclear embeddings in function spaces – some recent results

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joint work with Leszek Skrzypczak (Adam Mickiewicz University Poznań) and Hans-Gerd Leopold (Friedrich Schiller University Jena, Germany)

DEE's question

The making-of ... how it all started (for us) ...

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- Nothing.
- So far.
- But now . . . ??? . . .

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but:

- X_1, X_2 ... preferably of Besov or Triebel-Lizorkin type
- no compact (let alone nuclear) embedding in general
- need to refine our setting, e.g. to spaces on (quasi-)bounded domains, weighted spaces, ...
- here: spaces on bounded domains, weighted function spaces

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- id_Ω nuclear ✓

⇢ results are known essentially, serve for matter of comparison, but essentially as inspiration for later results and settings

The setting, II

$X_1(\mathbb{R}^n, w), X_2(\mathbb{R}^n)$ function spaces, w some weight function

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The setting, II

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here:

- X_1, X_2 … Besov or Triebel-Lizorkin spaces
- w … (special) Muckenhoupt weights

The talk is based on our joint papers:

D.D. Haroske, H.-G. Leopold, and L. Skrzypczak.
Nuclear embeddings in general vector-valued sequence spaces with an application to Sobolev embeddings of function spaces on quasi-bounded domains.
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Weighted function spaces

Muckenhoupt $\mathcal{A}_p$ weights

$w \in L_1^{\mathrm{loc}}$, positive

- $w \in \mathcal{A}_p,\ 1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$:

$$\left(\frac{1}{|B|}\int_B w(x)\,\mathrm{d}x\right)^{1/p}\left(\frac{1}{|B|}\int_B w(x)^{-p'/p}\,\mathrm{d}x\right)^{1/p'} \le A$$

- $w \in \mathcal{A}_1$: $\quad Mw(x) \le Aw(x)$ for a.e. $x \in \mathbb{R}^n$
- $\mathcal{A}_\infty = \bigcup_{p>1} \mathcal{A}_p$

$$Mf(x) = \sup_{B(x,r)\in\mathcal{B}} \frac{1}{|B(x,r)|}\int_{B(x,r)} |f(y)|\,\mathrm{d}y, \quad x \in \mathbb{R}^n$$

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Notation $\quad w(\Omega) = \int_\Omega w(x)\,\mathrm{d}x$

Muckenhoupt weights

Examples

- polynomial weight: $\alpha, \beta > -n$

$$w_{\alpha,\beta}(x) = \begin{cases} |x|^\alpha & \text{if } |x| \le 1 \\ |x|^\beta & \text{if } |x| > 1 \end{cases}$$

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  $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)$, $\boldsymbol{\beta} = (\beta_1, \beta_2)$, $\alpha_1, \beta_1 > -n$, $\alpha_2, \beta_2 \in \mathbb{R}$

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  purely logarithmic weight: $\gamma_1, \gamma_2 \in \mathbb{R}$

$$w_{\gamma}^{\log}(x) = \begin{cases} (1-\log|x|)^{\gamma_1}, & \text{if} \quad |x| \le 1 \\ (1+\log|x|)^{\gamma_2}, & \text{if} \quad |x| > 1 \end{cases}$$

Smoothness Spaces

Besov and Triebel-Lizorkin spaces on $\mathbb{R}^n$

$0 < p, q \leq \infty,\ s \in \mathbb{R},\ (\varphi_j)_j$ smooth dyadic partition of unity

Besov space: $\left\| f | B^s_{p,q} \right\| = \left\| \left(2^{js} \left\| \mathcal{F}^{-1}(\varphi_j \mathcal{F} f) | L_p \right\| \right)_j | \ell_q \right\|$

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Rem.

- (quasi-)Banach spaces, $\mathcal{S} \hookrightarrow A^s_{p,q} \hookrightarrow \mathcal{S}'$
- (classical) Besov spaces for $p, q \geq 1$ and $s > 0$ via differences
- $p = q = \infty$: $B^s_{\infty,\infty} = \mathcal{C}^s$, $s > 0$ Hölder-Zygmund spaces
- $1 < p < \infty$, $s \in \mathbb{R}$: $F^s_{p,2} = H^s_p$, i.e., $F^k_{p,2} = W^k_p$, $k \in \mathbb{N}_0$

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Convention: $A^s_{p,q}$ with $A \in \{B, F\}$, $p < \infty$ when $A = F$

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Weighted spaces of type $A^s_{p,q}(\mathbb{R}^n, w)$

$$\|f|L_p(\mathbb{R}^n, w)\| = \left(\int_{\mathbb{R}^n} |f(x)|^p w(x)\,\mathrm{d}x\right)^{1/p}, \quad 0 < p < \infty$$

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Rem.

- $L_\infty(\mathbb{R}^n, w) = L_\infty(\mathbb{R}^n)$, $w \in \mathcal{A}_\infty$ $\dashrightarrow$ assume $p < \infty$ for weighted spaces
- $B^s_{p,\min(p,q)}(\mathbb{R}^n, w) \hookrightarrow F^s_{p,q}(\mathbb{R}^n, w) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^n, w)$, $w \in \mathcal{A}_\infty$
- $F^s_{p,2}(\mathbb{R}^n, w) = H^s_p(\mathbb{R}^n, w)$, $1 < p < \infty$, $s \in \mathbb{R}$, $w \in \mathcal{A}_p$; in particular, $F^k_{p,2}(\mathbb{R}^n, w) = W^k_p(\mathbb{R}^n, w)$, $k \in \mathbb{N}_0$ (classical) Sobolev spaces
- $w \in \mathcal{A}_\infty$ Bui, Paluszyński, Taibleson, Rychkov, Bownik, Roudenko ...

Wavelet decomposition

$Q_{\nu,m}$ cubes, centre/corner at $2^{-\nu}m$, side length $2^{-\nu}$, $\nu \in \mathbb{N}_0$, $m \in \mathbb{Z}^n$
ϕ, $\psi_i \in C^{N_1}(\mathbb{R}^n)$ with $\operatorname{supp}\phi, \operatorname{supp}\psi_i \subset [-N_2, N_2]^n$, $i = 1, \dots, 2^n - 1$,

$$\phi_{\nu,m}(x) = 2^{\nu n/2}\phi(2^\nu x - m), \qquad \psi_{i,\nu,m}(x) = 2^{\nu n/2}\,\psi_i(2^\nu x - m)$$

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Proposition 1

Let $0 < p < \infty$, $0 < q \leq \infty$, $s \in \mathbb{R}$ with $N_1 > |s|$, $\sigma = s + \frac{n}{2} - \frac{n}{p}$. Then $f \in \mathcal{S}'(\mathbb{R}^n)$ belongs to $B^s_{p,q}(\mathbb{R}^n, w)$, if, and only if,

$$\|f|B^s_{p,q}(\mathbb{R}^n, w)\|^* = \| \{\langle f, \phi_{0,m}\rangle\}_m |\ell_p(w)\| + \sum_{i=1}^{2^n-1} \|\{\langle f, \psi_{i,\nu,m}\rangle\}_{\nu,m}|b^\sigma_{p,q}(w)\|$$

is finite. Furthermore, $\|f|B^s_{p,q}(\mathbb{R}^n, w)\|^* \sim \|f|B^s_{p,q}(\mathbb{R}^n, w)\|$.

Rem. $$\|\gamma|\ell_p(w)\| \sim \Big(\sum_{m\in\mathbb{Z}^n} |\gamma_m|^p\ w(Q_{0,m})\Big)^{1/p}$$

$$\Big\|\lambda|b^s_{pq}(w)\Big\| \sim \Big\|\Big\{2^{\nu s}\Big(\sum_{m\in\mathbb{Z}^n} |\lambda_{\nu,m}|^p\ 2^{\nu n}\ w(Q_{\nu,m})\Big)^{1/p}\Big\}_{\nu\in\mathbb{N}_0}|\ell_q\Big\|$$

Wavelet decomposition

Strategy of the proofs – Reduction to sequence spaces

- we may restrict ourselves to B-spaces whenever the fine index q is not involved, as $B^s_{p,\min(p,q)}(\mathbb{R}^n, w) \hookrightarrow F^s_{p,q}(\mathbb{R}^n, w) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^n, w)$

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$$
\begin{array}{ccc}
B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_1) & \underset{T^{-1}}{\overset{T}{\rightleftharpoons}} & b^{\sigma_1}_{p_1,q_1}(w_1) \\
\mathrm{Id} \downarrow & & \downarrow \mathrm{id} \\
B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_2) & \underset{S^{-1}}{\overset{S}{\leftrightharpoons}} & b^{\sigma_2}_{p_2,q_2}(w_2)
\end{array}
$$

for suitable isomorphisms S, T (wavelets)

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B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_1) & \underset{T^{-1}}{\overset{T}{\rightleftarrows}} & b^{\sigma_1}_{p_1,q_1}(w_1) & \underset{A^{-1}}{\overset{A}{\rightleftarrows}} & b^{\sigma_1}_{p_1,q_1}(w_1/w_2) \\
\mathrm{Id}\downarrow & & \downarrow\mathrm{id} & & \downarrow\mathrm{id} \\
B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_2) & \underset{S^{-1}}{\overset{S}{\leftrightarrows}} & b^{\sigma_2}_{p_2,q_2}(w_2) & \underset{A}{\overset{A^{-1}}{\leftrightarrows}} & b^{\sigma_2}_{p_2,q_2}
\end{array}$$

  for suitable isomorphisms S, T (wavelets) and A (weight)
- it is sufficient to consider weighted source spaces only

Function spaces of type $A^s_{p,q}(\Omega)$

Let

- $\Omega \subset \mathbb{R}^n$ bounded Lipschitz domain,
- $0 < p \leq \infty$ ($p < \infty$ when $A = F$), $0 < q \leq \infty$, $s \in \mathbb{R}$,

and $A^s_{p,q}(\Omega)$ be defined by restriction:

$$A^s_{p,q}(\Omega) := \left\{ f \in \mathcal{D}'(\Omega) : \ \exists\, g \in A^s_{p,q}(\mathbb{R}^n) : \ f = g|_\Omega \right\}$$

$$\text{with} \quad \left\| f \, | A^s_{p,q}(\Omega) \right\| = \inf_{f = g|_\Omega} \left\| g \, | A^s_{p,q}(\mathbb{R}^n) \right\|$$

Rem. $\Omega \subset \mathbb{R}^n$ *bounded Lipschitz domain*: Ω bounded domain, $\partial\Omega \subset \bigcup_{k=1}^N B_k$ open balls, $\partial\Omega \cap B_k$ part of a Lipschitz function

Compact embeddings of function spaces

The settings I & II revisited

$\Omega \subset \mathbb{R}^n$ bounded Lipschitz domain, $w \in \mathcal{A}_\infty$

$$\mathrm{id}_\Omega : A^{s_1}_{p_1,q_1}(\Omega) \longrightarrow A^{s_2}_{p_2,q_2}(\Omega)$$

$$\mathrm{id}_w : A^{s_1}_{p_1,q_1}(\mathbb{R}^n, w) \longrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^n)$$

$s_1 \geq s_2,\ 0 < p_1, p_2 \leq \infty$ ($p_i < \infty$ when $A = F$), $0 < q_1, q_2 \leq \infty$

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Notation

- $\delta = \left(s_1 - \frac{n}{p_1}\right) - \left(s_2 - \frac{n}{p_2}\right)$ differential dimension
- $\frac{1}{p^*} = \left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+, \quad \frac{1}{q^*} = \left(\frac{1}{q_2} - \frac{1}{q_1}\right)_+$

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Rem. $w \in \mathcal{A}_1$: never compact; concentrate on example weights

Spaces on bounded domains

Compactness

$\Omega \subset \mathbb{R}^n$ bounded Lipschitz domain, $A^s_{p,q}(\Omega)$ defined by restriction

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Proposition 2

Let $s_i \in \mathbb{R}$, $0 < p_i, q_i \le \infty$ ($p_i < \infty$ if $A{=}F$), $i = 1, 2$. Then

$$\mathrm{id}_\Omega : A^{s_1}_{p_1,q_1}(\Omega) \to A^{s_2}_{p_2,q_2}(\Omega) \quad \text{compact} \iff \frac{s_1 - s_2}{n} > \left(\frac{1}{p_1} - \frac{1}{p_2}\right)_+ .$$

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Rem.

- Edmunds/Triebel ('89, '92)
- entropy numbers $\quad e_k(\mathrm{id}_\Omega) \sim k^{-\frac{s_1-s_2}{n}}$, $k \in \mathbb{N}$
- id_Ω compact $\iff \delta > \frac{n}{p^*}$

Compact embeddings

Polynomial weight $w = w_{\alpha,\beta}$

$$\mathrm{id}_{\alpha,\beta} : A^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha,\beta}) \to A^{s_2}_{p_2,q_2}(\mathbb{R}^n), \quad \alpha,\beta > -n, \quad w_{\alpha,\beta}(x) = \begin{cases} |x|^{\alpha} & \text{if } |x| \leq 1 \\ |x|^{\beta} & \text{if } |x| > 1 \end{cases}$$

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Corollary 3

$\mathrm{id}_{\alpha,\beta} : A^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha,\beta}) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^n)$ compact if, and only if,

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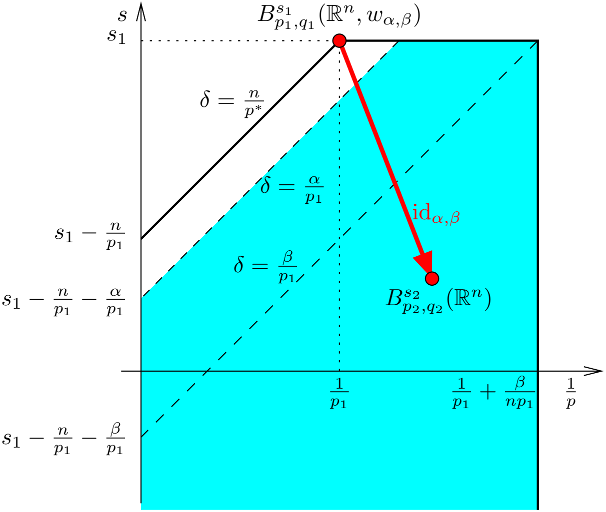
Rem.

- general criterion: Kühn/Leopold/Sickel/Skrzypczak (2006)
- case of the *admissible weights* $w^\beta(x) = \langle x\rangle^\beta \sim w_{0,\beta}(x)$

$$\mathrm{id}^\beta : A^{s_1}_{p_1,q_1}(\mathbb{R}^n, w^\beta) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^n) \quad \text{compact} \quad \Longleftrightarrow \quad \frac{\beta}{p_1} > \frac{n}{p^*} \quad \text{and} \quad \delta > \frac{n}{p^*}$$

- H./Skrzypczak ('08): entropy and approximation numbers


s
s_1
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$\delta = \frac{n}{p^*}$
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$s_1 - \frac{n}{p_1}$
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Compact embeddings

Almost polynomial weight $w = w_{(\boldsymbol{\alpha},\boldsymbol{\beta})}$

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Let $\alpha_1, \beta_1 > -n$, $\alpha_2, \beta_2 \in \mathbb{R}$. Then $\mathrm{id}_{(\alpha,\beta)}$ is compact if, and only if,

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Rem.

- *purely logarithmic weight*: $\gamma_1, \gamma_2 \in \mathbb{R}$, $w^{\log}_{\gamma}(x) = \begin{cases} (1-\log|x|)^{\gamma_1}, & \text{if } |x| \le 1 \\ (1+\log|x|)^{\gamma_2}, & \text{if } |x| > 1 \end{cases}$

  $\mathrm{id}_{\log} : A^{s_1}_{p_1,q_1}(\mathbb{R}^n, w^{\log}_{\gamma}) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^n)$ compact $\iff p_1 \le p_2, \ \delta > 0, \ \gamma_2 > 0$

- H./Skrzypczak (2011), results on entropy numbers and F-case known

The concept and recent results

Nuclear maps

X, Y Banach spaces, X' dual space of X, $T \in \mathcal{L}(X,Y)$

- T *nuclear*, i.e. $T \in \mathcal{N}(X,Y)$, if there exist elements $a_j \in X'$, $y_j \in Y$, $j \in \mathbb{N}$, such that

$$\sum_{j=1}^{\infty} \|a_j\|_{X'} \|y_j\|_Y < \infty \quad \text{and} \quad Tx = \sum_{j=1}^{\infty} a_j(x) y_j, \quad x \in X$$

- *nuclear norm* $\quad \nu(T) = \inf \left\{ \sum_{j=1}^{\infty} \|a_j\|_{X'} \|y_j\|_Y : \ T = \sum_{j=1}^{\infty} a_j(\cdot) y_j \right\}$
- $\mathcal{N}(X,Y)$ with $\nu(\cdot)$ Banach space
- $T \in \mathcal{N}(X,Y) \ \curvearrowright \ T$ compact
- $T \in \mathcal{N}(X,Y)$, $S \in \mathcal{L}(X_0, X)$, $R \in \mathcal{L}(Y, Y_0) \ \curvearrowright \ RTS \in \mathcal{N}(X_0, Y_0)$,

$$\nu(RTS) \le \|R\| \nu(T) \|S\|$$

Rem. Grothendieck (1955), Pietsch (1980, 1987)

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The concept and recent results

Tong's result (1969)

Let $r_1, r_2 \in [1,\infty]$, and $t(r_1, r_2) \in [1,\infty]$ be given by

$$\frac{1}{t(r_1,r_2)} = \begin{cases} 1, & \text{if } 1 \leq r_2 \leq r_1 \leq \infty, \\ 1 - \frac{1}{r_1} + \frac{1}{r_2}, & \text{if } 1 \leq r_1 \leq r_2 \leq \infty. \end{cases}$$

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Proposition 5

Let $1 \le r_1, r_2 \le \infty$, $\tau = (\tau_j)_j$, $D_\tau : x = (x_j)_j \mapsto (\tau_j x_j)_j$ diagonal operator

- D_τ nuclear $\iff$ $\tau \in \ell_{\mathrm{t}(r_1,r_2)}$, with $\ell_{\mathrm{t}(r_1,r_2)} = c_0$ if $\mathrm{t}(r_1,r_2) = \infty$. Moreover,
$$\nu(D_\tau : \ell_{r_1} \to \ell_{r_2}) = \|\tau|\ell_{\mathrm{t}(r_1,r_2)}\|.$$
- Let $D_\tau^m : \ell_{r_1}^m \to \ell_{r_2}^m$ the diagonal operator $D_\tau^m : (x_j)_{j=1}^m \mapsto (\tau_j x_j)_{j=1}^m$, $m \in \mathbb{N}$. Then
$$\nu(D_\tau^m : \ell_{r_1}^m \to \ell_{r_2}^m) = \left\|(\tau_j)_{j=1}^m|\ell_{\mathrm{t}(r_1,r_2)}^m\right\|.$$

The concept and recent results

Tong's result, continued

Examples

- $D_\tau = \mathrm{id} \iff \tau \equiv 1 \notin \{c_0, \ell_p, p < \infty\} \curvearrowright$ no nuclearity in ℓ_r
- $m \in \mathbb{N} \curvearrowright \nu(\mathrm{id} : \ell_{r_1}^m \to \ell_{r_2}^m) = \begin{cases} m & \text{if} \quad 1 \leq r_2 \leq r_1 \leq \infty, \\ m^{1-\frac{1}{r_1}+\frac{1}{r_2}} & \text{if} \quad 1 \leq r_1 \leq r_2 \leq \infty. \end{cases}$

  In particular, $\nu(\mathrm{id} : \ell_r^m \to \ell_r^m) = m$, and $\nu(\mathrm{id} : \ell_1^m \to \ell_\infty^m) = 1.$

Rem. Tong (1969)

The concept and recent results

Tong's result, continued

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Rem. Tong (1969)

Observation

- $$\frac{1}{\mathrm{t}(r_1, r_2)} = 1 - \left(\frac{1}{r_1} - \frac{1}{r_2}\right)_+ \ge \frac{1}{r^*} = \left(\frac{1}{r_2} - \frac{1}{r_1}\right)_+$$
- $\mathrm{t}(r_1, r_2) = r^* \in \{1, \infty\} \iff \{r_1, r_2\} = \{1, \infty\}$

The concept and recent results

Tong's result, continued

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  In particular, $\nu(\mathrm{id} : \ell_r^m \to \ell_r^m) = m$, and $\nu(\mathrm{id} : \ell_1^m \to \ell_\infty^m) = 1$.

Rem. Tong (1969)

Observation

- $\frac{1}{\mathrm{t}(r_1, r_2)} = 1 - \left(\frac{1}{r_1} - \frac{1}{r_2}\right)_+ \geq \frac{1}{r^*} = \left(\frac{1}{r_2} - \frac{1}{r_1}\right)_+$
- $\mathrm{t}(r_1, r_2) = r^* \in \{1, \infty\} \iff \{r_1, r_2\} = \{1, \infty\}$

The concept and recent results

Tong's result, continued

Examples

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Rem. later: *compactness* $\dashrightarrow$ *nuclearity* corresponds to $r^* \dashrightarrow \mathrm{t}(r_1, r_2)$

The concept and recent results

Recent results: the case of bounded domains

Proposition 6

Let $\Omega \subset \mathbb{R}^n$ be a bounded Lipschitz domain, $1 \le p_i, q_i \le \infty$, $s_i \in \mathbb{R}$.
Then $\mathrm{id}_\Omega : A^{s_1}_{p_1,q_1}(\Omega) \to A^{s_2}_{p_2,q_2}(\Omega)$ is nuclear if, and only if,

$$s_1 - s_2 > n - n\left(\frac{1}{p_2} - \frac{1}{p_1}\right)_+ .$$

Rem. Pietsch/Triebel (1968), Pietsch (1971), Edmunds/Lang (2014), Edmunds/Gurka/Lang (2016), Triebel (2017), H./Skrzypczak (2020)

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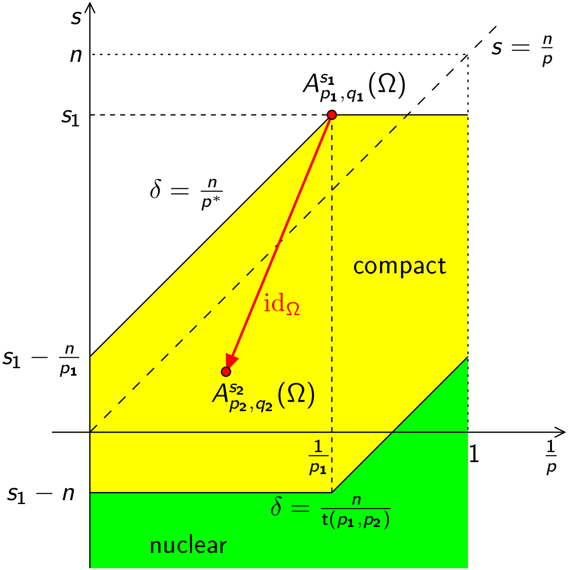

s
n
$s = \frac{n}{p}$
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s_1
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id_Ω
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1
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Nuclear embeddings of weighted spaces

The polynomial case $w_{\alpha,\beta}$

$$\mathrm{id}_{\alpha,\beta} : A^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_{\alpha,\beta}) \to A^{s_2}_{p_2,q_2}(\mathbb{R}^n), \quad \alpha,\beta > -n, \quad w_{\alpha,\beta}(x) = \begin{cases} |x|^\alpha & \text{if } |x| \leq 1 \\ |x|^\beta & \text{if } |x| > 1 \end{cases}$$

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Let $1 \le p_1 < \infty$, $1 \le p_2 \le \infty$ ($p_2 < \infty$ if $A = F$), $1 \le q_i \le \infty$, $s_i \in \mathbb{R}$, $i = 1, 2$. Then $\mathrm{id}_{\alpha,\beta}$ is nuclear if, and only if,

$$\frac{\beta}{p_1} > \frac{n}{\mathrm{t}(p_1,p_2)} \qquad \text{and} \qquad \delta > \max\left(\frac{n}{\mathrm{t}(p_1,p_2)}, \frac{\alpha}{p_1}\right).$$

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Rem.

- recall: $\mathrm{id}_{\alpha,\beta}$ compact $\iff$ $\frac{\beta}{p_1} > \frac{n}{p^*}$ and $\delta > \max\left(\frac{n}{p^*}, \frac{\alpha}{p_1}\right)$
- as before: compactness vs. nuclearity: replace p^* by $\mathrm{t}(p_1,p_2)$
- H./Skrzypczak (2020), proof by wavelet decomposition & Tong

Strategy of the proofs – revisited

Wavelet decomposition and reduction to sequence spaces

- we may restrict ourselves to *B-spaces* whenever the fine index q is not involved, as $B^s_{p,\min(p,q)}(\mathbb{R}^n, w) \hookrightarrow F^s_{p,q}(\mathbb{R}^n, w) \hookrightarrow B^s_{p,\max(p,q)}(\mathbb{R}^n, w)$

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- we may reduce our function spaces argument to sequence spaces

$$\begin{array}{ccc} B^{s_1}_{p_1,q_1}(\mathbb{R}^n, w_1) & \underset{T^{-1}}{\overset{T}{\rightleftarrows}} & b^{\sigma_1}_{p_1,q_1}(w_1) \\ \mathrm{Id}\big\downarrow & & \big\downarrow\mathrm{id} \\ B^{s_2}_{p_2,q_2}(\mathbb{R}^n, w_2) & \underset{S^{-1}}{\overset{S}{\leftrightarrows}} & b^{\sigma_2}_{p_2,q_2}(w_2) \end{array}$$

for suitable isomorphisms S, T (wavelets)

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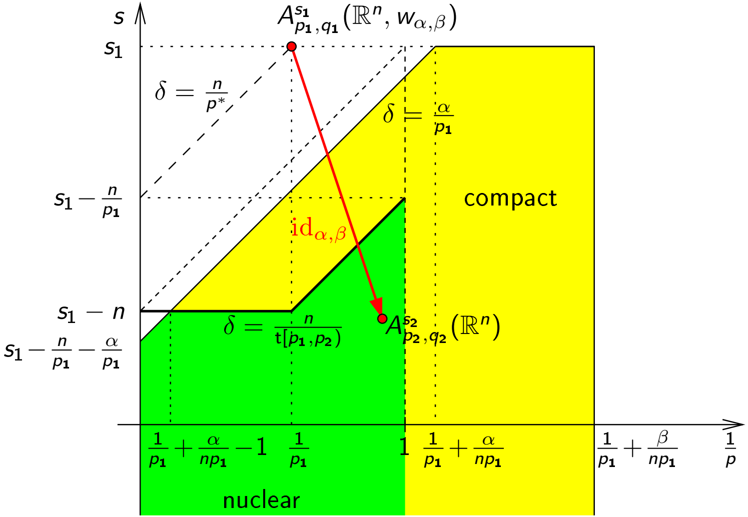
  for suitable isomorphisms S, T (wavelets) and A (weight)
- it is sufficient to consider weighted source spaces only
- we benefit from Tong's result on sequence spaces for nuclearity



the case $\frac{\beta}{np_1} > 1 > \frac{\alpha}{np_1} > 1 - \frac{1}{p_1} \geq 0$

Some consequences

Limiting cases, Admissible weights, Almost polynomial weights

- *limiting cases:* if $p^* = \mathrm{t}(p_1, p_2) \iff \{p_1, p_2\} = \{1, \infty\}$, then

$$\mathrm{id} \quad \text{compact} \iff \mathrm{id} \quad \text{nuclear},$$

where $\mathrm{id} = \mathrm{id}_\Omega$ or $\mathrm{id} = \mathrm{id}_{\alpha,\beta}$

- *admissible weights:* $w^\beta(x) = \langle x\rangle^\beta$, $\mathrm{id}^\beta : A^{s_1}_{p_1,q_1}(\mathbb{R}^n, w^\beta) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^n)$

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nuclearity of $\mathrm{id}_{(\alpha,\beta)}$ corresponds exactly to compactness criterion when p^* is replaced by $\mathrm{t}(p_1, p_2)$ and q^* by $\mathrm{t}(q_1, q_2)$

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Nuclear embeddings of weighted spaces

Consequence: The purely logarithmic case

$$\mathrm{id}_{\log} : A^{s_1}_{p_1,q_1}(\mathbb{R}^n, w^{\log}_{\gamma}) \hookrightarrow A^{s_2}_{p_2,q_2}(\mathbb{R}^n), \quad w^{\log}_{\gamma}(x) = \begin{cases} (1-\log|x|)^{\gamma_1}, & \text{if} \quad |x| \leq 1 \\ (1+\log|x|)^{\gamma_2}, & \text{if} \quad |x| > 1 \end{cases}$$

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Let $\gamma_1, \gamma_2 \in \mathbb{R}$, $1 \leq p_1 < \infty$, $1 \leq p_2 \leq \infty$ ($p_2 < \infty$ in the F-case), $1 \leq q_i \leq \infty$, $s_i \in \mathbb{R}$, $i = 1, 2$. Then

- $\mathrm{id}_{\log}$ is compact $\iff$ $\delta > 0$, $\gamma_2 > 0$, and $p_1 \leq p_2$.
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Rem. nuclearity of $\mathrm{id}_{\log}$ only in the limiting case, when compactness and nuclearity coincide, that is, when $p^* = \mathrm{t}(p_1, p_2)$ $\dashrightarrow$ never in F-case

Spaces on quasi-bounded domains

Quasi-bounded domains

Let $\Omega \subset \mathbb{R}^n$ be unbounded, $\Omega \neq \mathbb{R}^n$. Ω is called quasi-bounded, if

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Spaces on quasi-bounded domains

Quasi-bounded domains: examples

Examples

- $\Omega_\alpha = \{(x,y) \in \mathbb{R}^2 : \; x > 1, \; |y| < x^{-\alpha}\} \subset \mathbb{R}^2, \quad \alpha > 0$

$$\curvearrowright \quad b(\Omega_\alpha) = \begin{cases} 1 + \frac{1}{\alpha}, & 0 < \alpha < 1 \\ 2, & \alpha \geq 1 \end{cases}$$

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Rem. Leopold/Skrzypczak (2013, 2015): details, explanations, further examples ...

Spaces on quasi-bounded domains

Definition of Besov spaces, and wavelet decomposition

- assume now $1 \le p, q \le \infty$, $s \in \mathbb{R}$, $\Omega \subsetneq \mathbb{R}^n$
- $\widetilde{B}^s_{p,q}(\Omega) = \left\{ f \in \mathcal{D}'(\Omega) : \ \exists\, g \in B^s_{p,q}(\mathbb{R}^n) : \ f = g|_\Omega \ , \mathrm{supp}\, g \in \overline{\Omega} \right\}$ equipped with the quotient norm

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Wavelet decomposition

If $\Omega \subsetneq \mathbb{R}^n$ is quasi-bounded and uniformly E-porous, then there exists an orthonormal basis $\{\Phi^j_r\}_{j,r}$ in $L_2(\Omega)$, sufficiently smooth, such that

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{M_j} \lambda^j_r 2^{-j\frac{n}{2}} \Phi^j_r, \qquad f \in \overline{B}^s_{p,q}(\Omega) \longleftrightarrow \{\lambda^j_r\}_{j,r} \in \ell_q\left(2^{j(s-\frac{n}{p})} \ell_p^{M_j}\right)$$

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- $\widetilde{B}^s_{p,q}(\Omega) = \{f \in \mathcal{D}'(\Omega) : \exists\, g \in B^s_{p,q}(\mathbb{R}^n) : f = g|_\Omega, \operatorname{supp} g \in \overline{\Omega}\}$ equipped with the quotient norm

Define $\quad \overline{B}^s_{p,q}(\Omega) = \begin{cases} \widetilde{B}^s_{p,q}(\Omega), & s > 0, \\ B^s_{p,q}(\Omega), & s \leq 0. \end{cases}$

Wavelet decomposition

If $\Omega \subsetneq \mathbb{R}^n$ is quasi-bounded and uniformly E-porous, then there exists an orthonormal basis $\{\Phi^j_r\}_{j,r}$ in $L_2(\Omega)$, sufficiently smooth, such that

$$f = \sum_{j=0}^{\infty} \sum_{r=1}^{M_j} \lambda^j_r 2^{-j\frac{n}{2}} \Phi^j_r, \qquad f \in \overline{B}^s_{p,q}(\Omega) \longleftrightarrow \{\lambda^j_r\}_{j,r} \in \ell_q\left(2^{j(s-\frac{n}{p})} \ell_p^{M_j}\right)$$

Spaces on quasi-bounded domains

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Rem. Triebel (2008) for uniformly E-porous domains, $b_j(\Omega) \sim M_j$, above examples Ω_α, $\alpha > 0$, and $\Omega_{1,\beta}$, $\beta > 0$, are E-porous

Spaces on quasi-bounded domains

Compactness of weighted vector-valued sequence spaces

using wavelet decomposition ⇢ reduction to appropriate sequence spaces $\ell_q\big(\beta_j \ell_p^{M_j}\big)$, $0 < p, q \leq \infty$, $(\beta_j)_{j\in\mathbb{N}_0}$ with $\beta_j > 0$, $(M_j)_{j\in\mathbb{N}_0} \subset \mathbb{N}$

$$\ell_q\left(\beta_j \ell_p^{M_j}\right) = \Big\{ x = (x_{j,k})_{j\in\mathbb{N}_0, k=1,\ldots,M_j} \ : \ x_{j,k} \in \mathbb{C},$$

$$\left\| x|\ell_q\left(\beta_j \ell_p^{M_j}\right)\right\| = \Big(\sum_{j=0}^{\infty} \beta_j^q \Big(\sum_{k=1}^{M_j} |x_{j,k}|^p\Big)^{\frac{q}{p}}\Big)^{\frac{1}{q}} < \infty \Big\}$$

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Proposition 9

$$\mathrm{id}_\beta : \ell_{q_1}\left(\beta_j \ell_{p_1}^{M_j}\right) \to \ell_{q_2}\left(\ell_{p_2}^{M_j}\right) \quad \text{compact} \iff \left(\beta_j^{-1} M_j^{\frac{1}{p^*}}\right)_{j\in\mathbb{N}_0} \in \ell_{q^*}$$

with ℓ_∞ replaced by c_0 when $q^* = \infty$

Rem. Leopold (2000)

Spaces on quasi-bounded domains

Nuclearity of weighted vector-valued sequence spaces

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Spaces on quasi-bounded domains

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Rem.

- H./Leopold/Skrzypczak (2022)
- $\beta_j = \tau_j^{-1}$, $M_j \equiv 1$ $\dashrightarrow$ Tong's result
- if $\{p_1, p_2\} = \{1, \infty\}$ and $\{q_1, q_2\} = \{1, \infty\}$, i.e., $\mathrm{t}(p_1, p_2) = p^*$ and $\mathrm{t}(q_1, q_2) = q^*$ $\curvearrowright$ id_β compact $\iff$ id_β nuclear

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Spaces on quasi-bounded domains

Compact embedding

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(i) If $b(\Omega) = \infty$, then the embedding

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is compact $\iff$ $p_1 \le p_2$ and $s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > 0$.

(ii) If $b(\Omega) < \infty$, then $\mathrm{id}_{\overline{\Omega}}$ is compact if $s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) > \frac{b(\Omega)}{p^*}$.

If $\mathrm{id}_{\overline{\Omega}}$ is compact, then

$$s_1 - s_2 - n\left(\frac{1}{p_1} - \frac{1}{p_2}\right) \begin{cases} > 0, & \text{if} \quad p^* = \infty, \\ \ge \frac{b(\Omega)}{p^*}, & \text{if} \quad p^* < \infty. \end{cases}$$

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Rem. Leopold/Skrzypczak (2013), in (i): $p^* = \infty$; in (ii): almost sharp

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Rem. H./Leopold/Skrzypczak (2022)

The end

'... but orderly to end where I begun ... ' (Hamlet, Shakespeare)

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D.D. Haroske, H.-G. Leopold, S.D. Moura, and L. Skrzypczak.
Compact and nuclear embeddings in function spaces of generalised smoothness.
work in progress

D.D. Haroske and L. Skrzypczak.
Nuclear embeddings in smoothness Morrey spaces.
Preprint

D.D. Haroske, L. Skrzypczak, and H. Triebel.
Nuclear Fourier transforms.
Preprint; arXiv:2205.03128.

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Happy belated birthday, Fernando!

Thank you for all the marvellous ideas and inspiring papers (so far) – we already wait for the next ones ...

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and, finally, . . .


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Apolda, Germany
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Fernando Cobos (Madrid, Spain)
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Eero Saksman (Helsinki, Finland)
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Sergey Tikhonov (Barcelona, Spain)
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