On a problem of Lions on real interpolation spaces. The quasi-Banach case

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"XX Encuentro de Análisis Real y Complejo"
Cartagena, 26 – 28 May 2022

27 May 2022

Plan of the talk

The talk is based on joint work with

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- Preliminaries: Quasi-norms and the real interpolation method
- Solution to Lions's problem for quasi-Banach couples
- Application to spaces of operators defined by approximation numbers

Quasi-normed and r-normed spaces

- A quasi-norm resp. an r-norm $(0 < r \leq 1)$ on a linear space satisfies the norm axioms, but the triangle inequality is replaced by the

  quasi-triangle inequality: $\|x+y\| \leq C\,(\|x\|+\|y\|)$ for some $C \geq 1$

  r-triangle inequality: $\|x+y\|^r \leq \|x\|^r + \|y\|^r$

- norm $\iff$ quasi-norm with $C = 1$ $\iff$ r-norm with $r = 1$

- A quasi-Banach resp. r-Banach space is a linear space which is complete with respect to a quasi-norm resp. r-norm.

- Every r-norm is an s-norm for all $0 < s < r$, and also a quasi-norm. (In fact, the quasi-triangle constant is then $C = 2^{1/r-1}$.)

- Aoki-Rolewicz-Theorem. Every quasi-norm with constant $C > 1$ is equivalent to an r-norm, where r is defined by $C = 2^{1/r-1}$.

- Example: $(\ell_r, \|.\|_r)$ is an r-Banach space, $0 < r < 1$.

Real interpolation spaces

- A quasi-Banach couple (A_0, A_1) is formed by two quasi-Banach spaces, both embedded into a common topological Hausdorff space.

- Peetre's K-functional with respect to a quasi-Banach couple (A_0, A_1) is defined for $t > 0$ and $a \in A_0 + A_1$ by

$$K(t, a) := \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}.$$

- The quasi-norms of $A_0 + A_1$ and $A_0 \cap A_1$ are given by

$$\|a\|_{A_0+A_1} := K(1, a) \quad , \quad \|a\|_{A_0\cap A_1} := \max\{\|a\|_{A_0}, \|a\|_{A_1}\}.$$

- Let $0 < \theta < 1$, $0 < p \leq \infty$. The real interpolation space $(A_0, A_1)_{\theta,p}$ consists of all $a \in A_0 + A_1$ with finite quasi-norm

$$\|a\|_{\theta,p} := \begin{cases} \left(\int_0^\infty \left(t^{-\theta}K(t, a)\right)^p \frac{dt}{t}\right)^{1/p} & \text{if } 0 < p < \infty \\ \sup\limits_{t>0} t^{-\theta}K(t, a) & \text{if } p = \infty. \end{cases}$$

Equivalent quasi-norms on $(A_0, A_1)_{\theta,p}$

- For every $0 < r < 1$ we have the equivalence

$$K(t,a) \sim K_r(t,a) := \inf \left(\|a_0\|_{A_0}^r + t^r \|a_1\|_{A_1}^r \right)^{1/r} ,$$

  where the inf is taken over all decompositions $a = a_0 + a_1, a_j \in A_j$.

- Discretizing the integral and setting $j_m(a) := 2^{-m\theta} K_r(2^m, a)$ we get

$$\|a\|_{\theta,p} \sim \|a\|_{\theta,p;r} := \begin{cases} \left(\sum_{m\in\mathbb{Z}} j_m(a)^p \right)^{1/p} & \text{if } 0 < p < \infty \\ \sup_{m\in\mathbb{Z}} j_m(a) & \text{if } p = \infty \, . \end{cases}$$

- If A_0 and A_1 are both r-Banach spaces and $0 < r < p$, then $\|a\|_{\theta,p;r}$ is an r-norm on $(A_0, A_1)_{\theta,p}$, and $K_r(1,a)$ is an r-norm on $A_0 + A_1$.

Gagliardo couples

Let (A_0, A_1) be a (quasi-)Banach couple.

- The Gagliardo completion $A_j^\sim$ of A_j consists of all $a \in A_0 + A_1$ s.t.

$$\|a\|_{A_0^\sim} := \sup_{t>0} K(t,a) = \lim_{t\to\infty} K(t,a) < \infty \quad \text{resp.}$$

$$\|a\|_{A_1^\sim} := \sup_{t>0} \frac{K(t,a)}{t} = \lim_{t\to 0} \frac{K(t,a)}{t} < \infty\,.$$

- In other words: $A_0^\sim = (A_0, A_1)_{0,\infty}$ and $A_1^\sim = (A_0, A_1)_{1,\infty}$
- (A_0, A_1) is called a Gagliardo couple, if $A_0^\sim = A_0$ and $A_1^\sim = A_1$.
- This is a rather mild condition, it is satisfied in many concrete cases.
- Example: If $0 < p_0 \neq p_1 \leq \infty$, then (L_{p_0}, L_{p_1}) is a Gagliardo couple.

Lions's problem

Lions's problem

When does a given family of interpolation spaces effectively depend on its parameters, i.e. when are all these spaces different from each other?

- For Banach couples and the real method with parameters $0 < \theta < 1$ and $1 \leq p \leq \infty$ the solution is as follows:

Theorem (Janson, Nilson, Peetre, Zafran 1984)

Let $0 < \theta, \eta < 1$ *and* $1 \leq p, q \leq \infty$. *If* (A_0, A_1) *is a Banach couple such that* $\boxed{(*) \quad A_0 \cap A_1 \text{ is NOT closed in } A_0 + A_1,}$ *then*

$$(A_0, A_1)_{\theta,p} \neq (A_0, A_1)_{\eta,q} \quad \text{unless} \quad (\theta, p) = (\eta, q)$$

- Note that condition $(*)$ is necessary!
  Otherwise, due to the J-description of the K-method, $(A_0, A_1)_{\theta,p} = A_0 \cap A_1$ for all parameters.

Related results

- J. D. Stafney (Pac. J. Math. 1970)

  Similar results for the complex method.

- J. Almira and P. Fernández-Martínez (J. Math. Anal. Appl. 2021)

  considered the real method for ordered quasi-Banach couples

Subspaces of $(A_0, A_1)_{\theta,p}$ - the Banach case

Let (A_0, A_1) be a Banach couple, $0 < \theta < 1$ and $1 \leq p \leq \infty$.

Proposition (Mireille Levy, paper 1979 and PhD 1980)

$(A_0, A_1)_{\theta,p}$ closed in $A_0 + A_1$ $\Longrightarrow$ $A_0 \cap A_1$ closed in $A_0 + A_1$

Due to the J-description of $(A_0, A_1)_{\theta,p}$, the implication $\Longleftarrow$ is trivial.

In the proof duality arguments are used.

Theorem (M. Levy)

Let $1 \leq p < \infty$. If $(A_0, A_1)_{\theta,p}$ is NOT closed in $A_0 + A_1$, then it contains a complemented subspace isomorphic to ℓ_p.

Idea of proof: For every $0 < \varepsilon < 1$ one can find recursively a sequence $(x_n) \subset (A_0, A_1)_{\theta,p}$ that is $(1+\varepsilon)$-equivalent to the unit vector basis in ℓ_p. Essential for this construction is the previous proposition.

Subspaces of $(A_0, A_1)_{\theta,p}$ - the quasi-Banach case

Let (A_0, A_1) be a quasi-Banach couple, $0 < \theta < 1$ and $0 < p \leq \infty$.

Proposition (Cobos-Cwikel-K. 2022)

$(A_0, A_1)_{\theta,p}$ *closed in* $A_0 + A_1 \quad \Longrightarrow \quad A_0^{\sim} \cap A_1^{\sim}$ *closed in* $A_0 + A_1$

Levy used duality arguments, which are no longer available in the quasi-Banach case. Instead our proof is based on computations with the K-functional, combined with an iterative procedure.

Theorem (Cobos-Cwikel-K. 2022)

Let (A_0, A_1) *be a Gagliardo couple and* $0 < p < \infty$. *If* $(A_0, A_1)_{\theta,p}$ *is NOT closed in* $A_0 + A_1$, *then it contains a subspace isomorphic to* ℓ_p.

Proof: Similar construction as in Levy's paper. But due to the lack of duality, we cannot show that the subspace is complemented.

Lions's problem in the quasi-Banach case

Theorem (Cobos-Cwikel-K. 2022)

Let (A_0, A_1) be a quasi-Banach Gagliardo couple such that $A_0 \cap A_1$ is NOT closed in $A_0 + A_1$, and let $0 < \theta, \eta < 1$ and $0 < p, q \leq \infty$. Then

$$(A_0, A_1)_{\theta,p} \neq (A_0, A_1)_{\eta,q} \quad \textit{unless} \quad (\theta, p) = (\eta, q).$$

- This solves Lions's problem in the quasi-Banach setting.
- We need a mild extra assumption: (A_0, A_1) is a Gagliardo couple
- Extended range of the parameters: $p < 1$ and/or $q < 1$ is possible
- Interesting dichotomy: The spaces $(A_0, A_1)_{\theta,p}$

  either all coincide (if $A_0 \cap A_1$ is closed in $A_0 + A_1$)

  or are all different (if $A_0 \cap A_1$ is not closed in $A_0 + A_1$)

Sketch of the proof

We proceed by contradiction.

- Case 1. First assume that for some $0 < \theta < 1$ and $0 < p < q < \infty$

$$X := (A_0, A_1)_{\theta,p} = (A_0, A_1)_{\theta,q} \quad \text{with equivalence of quasi-norms}\,.$$

  Then one can construct – as in the proof of the subspace-theorem, taking the parameter ε small enough – a sequence $(x_n) \subset X$ that is equivalent to the unit vector basis in both ℓ_p and ℓ_q, a contradiction.

- Case 2. Assume now that for some $0 < \theta \neq \eta < 1$ and $0 < p, q \leq \infty$

$$(A_0, A_1)_{\theta,p} = (A_0, A_1)_{\eta,q}\,.$$

  By reiteration it follows that the spaces $(A_0, A_1)_{\lambda,r}$ with $\lambda = \frac{\theta+\eta}{2}$ do not depend on r, for $0 < r \leq \infty$. Thus we are back in Case 1, and the proof is finished.

An application

- We want to give an application concerning spaces of operators defined by the behaviour of their approximation numbers.

- But first we need some preparations.

Approximation numbers

- The n-th approximation number of a (bounded linear) operator $T \in \mathcal{L}(X, Y)$ between two Banach spaces X and Y is defined by

$$a_n(T) := \inf\{\|T - A\| : A \in \mathcal{L}(X, Y), \operatorname{rank} A < n\}.$$

- $\lim_{n\to\infty} a_n(T) = 0 \quad \Longrightarrow \quad T$ compact
  $\Longleftarrow$ fails by Enflo's counter-example

- The rate of decay of $a_n(T)$ as $n \to \infty$ can be viewed as a measure of the 'degree' of compactness of T.

- For compact operators between Hilbert spaces and all $n \in \mathbb{N}$ one has

$$a_n(T) = s_n(T) = \sqrt{\lambda_n(T^*T)} = n\text{-th singular number}.$$

Operator classes defined by approximation numbers

- For $0 < p < \infty$ and $0 < q \leq \infty$ we consider the class

$$\mathcal{A}_{p,q}(X,Y) := \left\{ T \in \mathcal{L}(X,Y) : \big((a_n(T) \big)_{n\in\mathbb{N}} \in \ell_{p,q} \right\},$$

  where $\ell_{p,q}$ are the Lorentz sequence spaces.

- $\mathcal{A}_{p,q}(X,Y)$ is a quasi-Banach space w.r.t. the quasi-norm

$$\|T\|_{p,q} := \begin{cases} \left(\sum_{n\in\mathbb{N}} \left(n^{1/p-1/q} a_n(T) \right)^q \right)^{1/q} & \text{if } q < \infty \\ \sup_{n\in\mathbb{N}} n^{1/p} a_n(T) & \text{if } q = \infty . \end{cases}$$

  In general, the quasi-norm $\|.\|_{p,q}$ is not equivalent to a norm .

- If H and G are Hilbert spaces, then

$$\mathcal{A}_{p,q}(H,G) = \mathcal{S}_{p,q}(H,G) = \text{Schatten classes} .$$

A problem on the scale $\{\mathcal{A}_p(X,Y)\}_{p>0}$

$\ell_{p,p} = \ell_p \quad \curvearrowright \quad$ short notation: $\mathcal{A}_p(X,Y) := \mathcal{A}_{p,p}(X,Y)$

Problem (Albrecht Pietsch, Sept. 2021)

Show that, for arbitrary infinite-dimensional Banach spaces X and Y,

the scale $\{\mathcal{A}_p(X,Y)\}_{p>0}$ *is strictly increasing.*

This motivated us to consider Lions's problem for quasi-Banach couples.

Remark

If $\dim X < \infty$ *and/or* $\dim Y < \infty$, *then*

$$\mathcal{A}_{p,q}(X,Y) = \mathcal{L}(X,Y) \quad \text{for all } 0 < p < \infty \text{ and } 0 < q \le \infty .$$

Proof. Every $T \in \mathcal{L}(X,Y)$ has finite rank $\curvearrowright$ $a_n(T) = 0 \;\; \forall\, n > \operatorname{rank} T$

Real interpolation of the classes $\mathcal{A}_p(X, Y)$

Theorem (Hermann König 1978)

Let $0 < p_0 < p_1 < \infty$ *.*

(i) The K-functional of the couple $(\mathcal{A}_{p_0}(X, Y), \mathcal{A}_{p_1}(X, Y))$ *satisfies*

$$K(t, T) \sim \Big(\sum_{n \le \lfloor t^r \rfloor} a_n(T)^{p_0} \Big)^{1/p_0} + \Big(\sum_{n > \lfloor t^r \rfloor} a_n(T)^{p_1} \Big)^{1/p_1}, t > 0,$$

where $1/r = 1/p_0 - 1/p_1$.

(ii) Let $0 < \theta < 1$ *and* $0 < q \le \infty$ *. Then*

$$\big(\mathcal{A}_{p_0}(X, Y), \mathcal{A}_{p_1}(X, Y)\big)_{\theta,q} = \mathcal{A}_{p,q}(X, Y) \quad , \quad \frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}.$$

By reiteration, even $\big(\mathcal{A}_{p_0,q_0}(X, Y), \mathcal{A}_{p_1,q_1}(X, Y)\big)_{\theta,q} = \mathcal{A}_{p,q}(X, Y)$ holds.

Further properties

Let $0 < p_0 < p_1 < \infty$. König's interpolation formula implies

Lemma (Cobos, Cwikel, K., 2022)

$(\mathcal{A}_{p_0}(X,Y), \mathcal{A}_{p_1}(X,Y))$ *is a quasi-Banach Gagliardo couple.*

Since $\mathcal{A}_{p_0} \subset \mathcal{A}_{p_1}$, we have $\qquad \mathcal{A}_{p_0} = \mathcal{A}_{p_0} \cap \mathcal{A}_{p_1}\,,\ \mathcal{A}_{p_1} = \mathcal{A}_{p_0} + \mathcal{A}_{p_1}$.

Lemma (Cobos-Cwikel-K. 2022)

If $1/p_0 - 1/p_1 > 1$ *then* $\mathcal{A}_{p_0}(X,Y)$ *is not closed in* $\mathcal{A}_{p_1}(X,Y)$.

Proof: By Dvoretzky's theorem one can construct a sequence of finite-rank operators $T_n \in \mathcal{L}(X,Y)$, such that $\lim\limits_{n\to\infty} \frac{\|T_n\|_{p_0}}{\|T_n\|_{p_1}} = \infty$, hence the quasi-norms $\|.\|_{p_0}$ and $\|.\|_{p_1}$ are not equivalent on $\mathcal{A}_{p_0}$, and therefore $\mathcal{A}_{p_0}(X,Y)$ cannot be closed in $\mathcal{A}_{p_1}(X,Y)$.

Solutions to Pietsch's problem

Combining these two lemmata gives the following more general result.

Theorem (Cobos-Cwikel-K. 2022)

Let X and Y be arbitrary infinite-dimensional Banach spaces. Then

$$\mathcal{A}_{p_0,q_0}(X,Y) \neq \mathcal{A}_{p_1,q_1}(X,Y) \quad \textit{unless} \quad (p_0,q_0) = (p_1,q_1)\,.$$

In particular, the scale $\{\mathcal{A}_p(X,Y)\}_{p>0}$ is strictly increasing.

In fact, the spaces $\mathcal{A}_{p,q}(X,Y)$ are lexicographically ordered, similarly to the ordering of the Lorentz sequence spaces $\ell_{p,q}$, i.e.

$$\mathcal{A}_{p_0,q_0}(X,Y) \hookrightarrow \mathcal{A}_{p_1,q_1}(X,Y)\ , \text{ if } \begin{cases} p_0 < p_1 & \text{or} \\ p_0 = p_1 \text{ and } q_0 < q_1\,. \end{cases}$$

Moreover, all these embeddings are strict.

Dear Fernando,

Congratulations once again to your 65th birthday

and all my best wishes for many years to come!

Thank you for your attention!