On interpolation of two measures of non-compactness associated to Banach operator ideals

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THEOREM (Grothendieck, 1955)

A Banach space X has AP if and only if for every Banach space Y, the subspace $\mathcal{F}(Y, X)$ of finite-rank operators is $\|\cdot\|$-dense in the space $\mathcal{K}(Y, X)$ of compact operators from Y to X.

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A basic tool used by Grothendieck to study AP in a Banach space is the next characterization of a relatively compact set:

- A subset D of a Banach space X is relatively compact if and only if $D \subset \{\sum_{n=1}^{\infty} a_n x_n;\ (a_n) \in B_{\ell_1}\}$ for some sequence $(x_n) \in c_0(X)$.

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- Let $1 \leq p \leq \infty$ and let p' satisfy $\frac{1}{p} + \frac{1}{p'}$. A subset D of a Banach space X is said to be relatively p-compact if

$$D \subset p\text{-co}(x_n) := \left\{ \sum_{n=1}^{\infty} a_n x_n;\ (a_n) \in B_{\ell_{p'}} \right\} \text{ for some sequence } (x_n) \in \ell_p(X),$$

where the following conventions are understood:

$$(a_n) \in B_{c_0}, \text{ if } p = 1; \quad \text{and} \quad (x_n) \in c_0(X), \text{ when } p = \infty.$$

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According to this, relatively compact sets may be referred to as relatively ∞-compact sets.

- If $1 \leq p < q \leq \infty$, each relatively p-compact set is relatively q-compact.

The definition of relatively p-compact set leads in a natural way to the notion of p-compact operator (in the sense of Karn and Sinha):

- An operator $T \in \mathcal{L}(X, Y)$ is called p-compact operator if $T(B_X)$ is a relatively p-compact set in Y.

$\mathcal{K}_p$ will denote the class of all p-compact operators between Banach spaces.

This kind of p-compactness for operators is different from the concept of p-compact operator from the late of the seventies due to Fourie and Swart and, independently, to Pietsch.

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Relationship of $\mathcal{K}_p$ with other classes of operators:

THEOREM (Karn and Sinha, 2002)

For $T \in \mathcal{L}(X, Y)$, it holds that

- If T is p-compact, then T^* is p-summing.
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• Given $1 \leq p < \infty$, $T \in \mathcal{L}(X, Y)$ is called:
p-summing operator if there is $c \geq 0$ s.t. for each finite set $\{x_1, \ldots, x_m\}$ of X,
$\left(\sum_{i=1}^m \|Tx_i\|_Y^p\right)^{1/p} \leq c \sup\{\left(\sum_{i=1}^m |\langle x^*, x_i\rangle|^p\right)^{1/p}; x^* \in B_{X^*}\}$
quasi p-nuclear operator if there exists $(x_n^*) \in \ell_p(X^*)$ s.t.
$\|Tx\|_Y \leq \left(\sum_{n=1}^\infty |\langle x_n^*, x\rangle|^p\right)^{1/p}$, for any $x \in X$.

- If $T \in \mathcal{K}_p(X, Y)$, $T(B_X)$ is a relatively p-compact set in Y. Let

$$k_p(T) := \inf\{\|(y_n)\|_p ; (y_n) \in \ell_p(Y), T(B_X) \subset p\text{-co}(y_n)\}.$$

- If $T \in \Pi_p(X, Y)$, there is $c \geq 0$ s.t. for each finite set $\{x_1, \ldots, x_m\}$ of X,

$$\Big(\sum_{i=1}^{m} \|Tx_i\|_Y^p\Big)^{1/p} \leq c \sup\{\Big(\sum_{i=1}^{m} |\langle x^*, x_i\rangle|^p\Big)^{1/p} ; x^* \in B_{X^*}\}. \qquad (*)$$

Let $\pi_p(T)$ be the least c for which inequality $(*)$ always holds.

- If $T \in \mathcal{QN}_p(X, Y)$, there exists $(x_n^*) \in \ell_p(X^*)$ s.t.

$$\|Tx\|_Y \leq \Big(\sum_{n=1}^{\infty} |\langle x_n^*, x\rangle|^p\Big)^{1/p}, \text{ for any } x \in X. \qquad (**)$$

Let

$$\nu_p^Q(T) := \inf\{\|(x_n^*)\|_p ; (x_n^*) \in \ell_p(X^*) \text{ s.t. } (**) \text{ holds}\}.$$

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- $[\mathcal{L}, \|\cdot\|]$ and $[\mathcal{K}, \|\cdot\|]$ are Banach operator ideals.
  $[\mathcal{K}_p, k_p]$, $[\Pi_p, \pi_p]$ and $[\mathcal{QN}_p, \nu_p^Q]$ $(1 \leq p < \infty)$ are too.

We refer to classical books on operator theory by Diestel, Jarchow and Tonge, by Jarchow, and by Pietsch.

• An operator ideal $\mathcal{A}$ is defined as a method of ascribing to each pair of Banach spaces (X, Y) a linear subspace $\mathcal{A}(X, Y)$ of $\mathcal{L}(X, Y)$ such that

(I1) The operator $x^* \otimes y := \langle x^*, \cdot \rangle y \in \mathcal{A}(X, Y)$, for any $x^* \in X^*, y \in Y$;

(I2) If $S \in \mathcal{L}(U, X)$, $T \in \mathcal{A}(X, Y)$ and $R \in \mathcal{L}(Y, V)$, then $R \circ T \circ S \in \mathcal{A}(U, V)$.

If, in addition, there is a non-negative function $\alpha : \mathcal{A} \to \mathbb{R}$ in such a way that:

(N1) $\alpha(x^* \otimes y) = \|x^*\| \cdot \|y\|$, for all $x^* \in X^*, y \in Y$;

(N2) $\alpha(R \circ T \circ S) \leq \|R\| \cdot \alpha(T) \cdot \|S\|$, whenever U and V are Banach spaces and $S \in \mathcal{L}(U, X)$, $T \in \mathcal{A}(X, Y)$ and $R \in \mathcal{L}(Y, V)$;

(N3) For every pair of Banach spaces (X, Y), $(\mathcal{A}(X, Y), \alpha)$ is a Banach space;

then $[\mathcal{A}, \alpha]$ is called a Banach operator ideal.

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Besides approximation, the research of different properties (such as duality or factorization) in connection with p-compact sets and p-compact operators, as well as certain extensions of this form of compactness, has attracted the interest of a good number of authors recently (see papers by Ain, Delgado, Kim, Lee, Lillements, Oja, Piñeiro, Serrano, Zheng, among others).

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A more general approach using the notions of surjective $\mathcal{A}$-compactness and injective $\mathcal{A}$-compactness determined by an operator ideal $\mathcal{A}$, defined respectively by Carl and Stephani and by Stephani, allows the study of some of these questions under a wider framework. See, for example,

- J.M. Delgado, C. Piñeiro, *An approximation property with respect to an operator ideal.* Studia Math. **214** (2013), 67–75.
- J.M. Delgado, C. Piñeiro, *Duality of measures of non-$\mathcal{A}$-compactness, Studia Math.* **229** (2015), 95–112.
- S. Lasalle, P. Turco, *The Banach ideal of $\mathcal{A}$-compact operators and related approximation properties.* J. Funct. Anal. **265** (2013), 2452–2464.
- S. Lasalle, P. Turco, *On null sequences for Banach operator ideals, trace duality and approximation properties.* Math. Nachr. **290** (2017), 2308–2321.

For instance, Delgado and Piñeiro introduced an approximation property with respect to an operator ideal $\mathcal{A}$ that involves the notion of $\mathcal{A}$-compact set:

• Let $\mathcal{A}$ be an operator ideal. A Banach space X has the approximation property with respect to $\mathcal{A}$ ($AP_{\mathcal{A}}$) if I_X can be approximated by finite-rank operators uniformly on every $\mathcal{A}$-compact set of X.
Equivalently, X has $AP_{\mathcal{A}}$ if for every Banach space Y, $\mathcal{F}(Y, X)$ is $\|\cdot\|$-dense in the space $\mathcal{K}^{\mathcal{A}}(Y, X)$ of surjectively $\mathcal{A}$-compact operators from Y to X.

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and such that

• $AP_{\mathcal{A}} \equiv AP$, if $\mathcal{A}$ contains the ideal $\mathcal{K}$ of all compact operators.

• $AP_{\mathcal{A}} \equiv$ Reinov approximation property of order p (AP_p), when $1/2 \leq p < 1$, if $\mathcal{A}$ is the ideal of all operators mapping bounded sets to Bourgain-Reinov q-compact sets, $q = p/(1-p)$.

• $AP_{\mathcal{A}} \equiv$ Karn-Sinha p-AP, if $\mathcal{A}$ is the ideal $\mathcal{K}_p$ of all p-compact operators.

This more general approach based on the notion of surjective $\mathcal{A}$-compactness, as well as on the concept of injective $\mathcal{A}$-compactness, is also used by Delgado and Piñeiro to provide a quantitative version of the aforementioned result:

$$T \in \mathcal{QN}_p \quad \text{iff} \quad T^* \in \mathcal{K}_p.$$

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Theorem (Delgado and Piñeiro, 2015)

Under certain conditions on the operator ideal $\mathcal{A}$, there is $C > 0$ s.t. for every $T \in (\mathcal{A}^d)^{inj}(X,Y)$,

$$\frac{1}{C}\chi_{\mathcal{A}}(T^*) \leq n_{\mathcal{A}^d}(T) \leq C\chi_{\mathcal{A}}(T^*).$$

As a consequence, $n_{\Pi_p}(T) = \chi_{\Pi_p^d}(T^*)$ for $T \in \Pi_p(X,Y)$.

Due to $\mathcal{QN}_p$ = injectively Π_p-compact operators, $T \in \mathcal{QN}_p$ iff $n_{\Pi_p}(T) = 0$. Analogously, $\mathcal{K}_p$ = surjectively Π_p^d-compact operators, and so $T^* \in \mathcal{K}_p$ iff $\chi_{\Pi_p^d}(T^*) = 0$. Therefore,

$$T \in \mathcal{QN}_p \quad \text{iff} \quad T^* \in \mathcal{K}_p.$$

- Given an operator ideal $\mathcal{A}$, $\mathcal{A}^d$ stands for the dual ideal of $\mathcal{A}$, i.e.
$$\mathcal{A}^d(X,Y) = \{T \in \mathcal{L}(X,Y);\ T^* \in \mathcal{A}(Y^*,X^*)\}.$$

If $[\mathcal{A},\alpha]$ is a Banach operator ideal, $[\mathcal{A}^d,\alpha^d]$ is also a Banach operator ideal, with $\alpha^d(T) := \alpha(T^*)$, for $T \in \mathcal{A}^d(X,Y)$.

An operator ideal $\mathcal{A}$ is said to be surjective whenever $\mathcal{A} = \mathcal{A}^{sur}$, where $\mathcal{A}^{sur}$ is the surjective hull ideal, whose components are
$$\mathcal{A}^{sur}(X,Y) := \{T \in \mathcal{L}(X,Y);\ T(B_X) \subset S(B_Z), S \in \mathcal{A}(Z,Y)\}.$$

Analogously, $\mathcal{A}$ is called injective when $\mathcal{A} = \mathcal{A}^{inj}$, where $\mathcal{A}^{inj}$ is the injective hull ideal, whose components are
$$\mathcal{A}^{inj}(X,Y) := \{T \in \mathcal{L}(X,Y);\ \|Tx\|_Y \le \|Sx\|_Z \text{ for } x \in X,\ S \in \mathcal{A}(X,Z)\}.$$

If $[\mathcal{A},\alpha]$ is a Banach operator ideal, $[\mathcal{A}^{sur},\alpha^{sur}]$ and $[\mathcal{A}^{inj},\alpha^{inj}]$ become Banach operator ideals, where
$\alpha^{sur}(T) := \inf\{\alpha(S);\ T(B_X) \subset S(B_Z), S \in \mathcal{A}(Z,Y)\} = \alpha(T \circ Q_X)$,
$\alpha^{inj}(T) := \inf\{\alpha(S);\ \|Tx\|_Y \le \|Sx\|_Z \text{ for } x \in X,\ S \in \mathcal{A}(X,Z)\} = \alpha(J_Y \circ T)$.
Here $Q_X : \ell_1(B_X) \to X$ is the metric surjection $Q_X(\lambda_x)_{x\in B_X} := \sum_{x\in B_X} \lambda_x x$, and $J_X : X \to \ell_\infty(B_{X^*})$ is the metric injection $J_X x := (\langle x^*, x\rangle)_{x^*\in B_{X^*}}$.

B. Carl, I. Stephani, *On $\mathcal{A}$-compact operators, generalized entropy numbers and entropy ideals.* Math. Nachr. **119** (1984), 77–95.

• Let $\mathcal{A}$ be an operator ideal. Let X be a Banach space. A subset D of X is called $\mathcal{A}$-compact when $D \subset \{\sum_{n=1}^{\infty} a_n x_n;\ (a_n) \in B_{\ell_1}\}$, for some $(x_n) \subset X$ which is $\mathcal{A}$-convergent to zero (a sequence $(x_n) \subset X$ is $\mathcal{A}$-convergent to zero if there exist $S \in \mathcal{A}(Z, X)$ so that, given any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ s.t. $x_n \in \varepsilon \cdot S(B_Z)$ for $n > n_0$).

Equivalently, D is $\mathcal{A}$-compact if there are a Banach space Z and an operator $S \in \mathcal{A}(Z, X)$ s.t. $D \subset S(K)$ for some compact $K \subset Z$.

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• $T \in \mathcal{L}(X, Y)$ is surjectively $\mathcal{A}$-compact if $T(B_X)$ is an $\mathcal{A}$-compact set in Y. Let $\mathcal{K}^{\mathcal{A}}$ be the class of all surjectively $\mathcal{A}$-compact operators.

B. Carl, I. Stephani, *On $\mathcal{A}$-compact operators, generalized entropy numbers and entropy ideals.* Math. Nachr. **119** (1984), 77–95.

- Let $\mathcal{A}$ be an operator ideal. Let X be a Banach space. A subset D of X is called $\mathcal{A}$-compact when $D \subset \{\sum_{n=1}^{\infty} a_n x_n;\ (a_n) \in B_{\ell_1}\}$, for some $(x_n) \subset X$ which is $\mathcal{A}$-convergent to zero (a sequence $(x_n) \subset X$ is $\mathcal{A}$-convergent to zero if there exist $S \in \mathcal{A}(Z, X)$ so that, given any $\varepsilon > 0$, there is $n_0 \in \mathbb{N}$ s.t. $x_n \in \varepsilon \cdot S(B_Z)$ for $n > n_0$).

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- $\mathcal{K}^{\mathcal{A}}$ is a surjective operator ideal and $\mathcal{K}^{\mathcal{A}} = \mathcal{A}^{sur} \circ \mathcal{K}$.

(if $\mathcal{U}$ and $\mathcal{V}$ are operator ideals, $T \in \mathcal{L}(X, Y)$ belongs to the product ideal $\mathcal{V} \circ \mathcal{U}$ if there are a Banach space G and operators $T_1 \in \mathcal{U}(X, G)$ and $T_2 \in \mathcal{V}(G, Y)$ s.t $T = T_2 \circ T_1$).

In particular, $\mathcal{K}^{\mathcal{L}} = \mathcal{K}$, $\mathcal{K}^{\mathcal{K}} = \mathcal{K}$, $\mathcal{K}^{\mathcal{A}} = \mathcal{K}^{\mathcal{A}^{sur}}$ and $\mathcal{K}^{\mathcal{A}} \subset \mathcal{A}^{sur}$.

- $\mathcal{K}^{\Pi_p^d} = \mathcal{K}_p$, since $\mathcal{K}^{\Pi_p^d} = \Pi_p^d \circ \mathcal{K}$ and also $\mathcal{K}_p = \Pi_p^d \circ \mathcal{K}$.

A natural question is to estimate in some sense the distance between an operator in the surjective hull of $\mathcal{A}$ and $\mathcal{K}^{\mathcal{A}}$. The next characterization of a surjectively $\mathcal{A}$-compact operator was established by Carl and Stephani:

THEOREM (Carl and Stephani, 1984)

- Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. An operator T from X to Y is surjectively $\mathcal{A}$-compact iff for every $\varepsilon > 0$, there are finitely many elements $y_1, \ldots, y_n \in Y$, a Banach space Z and $S \in \mathcal{A}(Z, Y)$, with $\alpha(S) \leq \varepsilon$, s.t.

$$T(B_X) \subset \bigcup_{k=1}^{n} \{y_k + S(B_Z)\}.$$

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and it motivated the definition of the following function considered by Delgado and Piñeiro:

- Given $T \in \mathcal{A}^{sur}(X, Y)$,

$$\chi_{\mathcal{A}}(T) := \inf\left\{\varepsilon > 0;\ T(B_X) \subset \bigcup_{k=1}^{n} \{y_k + S(B_Z)\}\right\},$$

where the infimum is taken over all possible sets of finitely many elements $y_1, \ldots, y_n \in Y$, Banach spaces Z and operators $S \in \mathcal{A}(Z, Y)$ with $\alpha(S) \leq \varepsilon$. Note that $T \in \mathcal{A}^{sur}(X, Y)$ ensures that the infimum is taken on a nonempty set of positive numbers. Indeed, $\chi_{\mathcal{A}}(T) \leq \alpha^{sur}(T)$.

We can say that $\chi_{\mathcal{A}}$ is a measure of surjective non-$\mathcal{A}$-compactness in the sense that

- $T \in \mathcal{A}^{sur}(X, Y)$ is surjectively $\mathcal{A}$-compact if and only if $\chi_{\mathcal{A}}(T) = 0$.

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We note that

- $\chi_{\mathcal{L}} =$ (ball) measure of non-compactness of an operator

$$\gamma(T) := \inf\Big\{\sigma > 0;\ T(B_X) \subset \bigcup_{k=1}^{n} \{y_k + \sigma B_Y\},\ y_k \in Y,\ n \in \mathbb{N}\Big\}.$$

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- $\chi_{\mathcal{A}}$ is a different notion from the (outer) measure $\gamma_{\mathcal{A}}$ defined by Astala:

$$\gamma_{\mathcal{A}}(T) := \inf\Big\{\varepsilon > 0;\ T(B_X) \subset \varepsilon B_Y + S(B_Z),\ \text{for some Banach space } Z \text{ and operator } S \in \mathcal{A}(Z, Y)\Big\}.$$

In fact, choosing the ideal $[\Pi_p, \pi_p]$ and taking T as the embedding map from ℓ_1 to c_0, it follows that $\gamma_{\Pi_p}(T) = 0$, because T is a 1-integral operator and so it is a p-summing operator. Nevertheless, it was proved by Delgado and Piñeiro that $\chi_{\Pi_p}(T) > 0$.

I. Stephani, *Injectively $\mathcal{A}$-compact operators, generalized inner entropy numbers and Gelfand numbers.* Math. Nachr. **133** (1987), 247–272.

• Let $\mathcal{A}$ be an operator ideal. Let X, Y be Banach spaces. $T \in \mathcal{L}(X, Y)$ is an injectively $\mathcal{A}$-compact operator if there are a Banach space Z and an operator $S \in \mathcal{A}(Z, X)$ s.t. $T(S^{-1}(B_Z))$ is a relatively compact subset in Y.

Equivalently, $T \in \mathcal{L}(X, Y)$ is injectively $\mathcal{A}$-compact if there exist a Banach space Z, a sequence $(z_n^*) \in c_0(Z^*)$ and an operator $S \in \mathcal{A}^{inj}(X, Z)$ s.t. $\|Tx\|_Y \leq \sup_{n\in\mathbb{N}} |\langle z_n^*, Sx\rangle|$ for any $x \in X$.

Let $\mathcal{H}^{\mathcal{A}}$ be the class of all injectively $\mathcal{A}$-compact operators.

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Let $\mathcal{H}^{\mathcal{A}}$ be the class of all injectively $\mathcal{A}$-compact operators.

• $\mathcal{H}^{\mathcal{A}}$ is an injective operator ideal and $\mathcal{H}^{\mathcal{A}} = \mathcal{K} \circ \mathcal{A}^{inj}$.

In particular, $\mathcal{H}^{\mathcal{L}} = \mathcal{K}$, $\mathcal{H}^{\mathcal{K}} = \mathcal{K}$, $\mathcal{H}^{\mathcal{A}} = \mathcal{H}^{\mathcal{A}^{inj}}$ and $\mathcal{H}^{\mathcal{A}} \subset \mathcal{A}^{inj}$.

• $\mathcal{H}^{\Pi_p} = \mathcal{QN}_p$, since $\mathcal{H}^{\Pi_p} = \mathcal{K} \circ \Pi_p$ and as well $\mathcal{QN}_p = \mathcal{K} \circ \Pi_p$.

It is natural to wonder about the distance between an operator in the injective hull of $\mathcal{A}$ and $\mathcal{H}^{\mathcal{A}}$. The next characterization of an injectively $\mathcal{A}$-compact operator is due to Stephani:

THEOREM (Stephani, 1987)

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. An operator T from X to Y is injectively $\mathcal{A}$-compact if and only if for every $\varepsilon > 0$, there are finitely many $x_1^*, \ldots, x_n^* \in X^*$, a Banach space Z and $S \in \mathcal{A}(X, Z)$, with $\alpha(S) \leq \varepsilon$, s.t.

$$\|Tx\|_Y \leq \sup_{1 \leq k \leq n} |\langle x_k^*, x \rangle| + \|Sx\|_Z, \quad x \in X.$$

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$$\|Tx\|_Y \leq \sup_{1\leq k\leq n} |\langle x_k^*, x\rangle| + \|Sx\|_Z, \quad x \in X.$$

It motivated the definition of the following function given by Delgado and Piñeiro:

- For $T \in \mathcal{A}^{inj}(X, Y)$,

$$n_{\mathcal{A}}(T) := \inf\Big\{\varepsilon > 0\,;\ \|Tx\|_Y \leq \sup_{1\leq k\leq n} |\langle x_k^*, x\rangle| + \|Sx\|_Z,\ x \in X\Big\},$$

where the infimum is taken over all choices of finitely many $x_1^*, \ldots, x_n^* \in X^*$, Banach spaces Z and operators $S \in \mathcal{A}(X, Z)$ with $\alpha(S) \leq \varepsilon$.
The condition $T \in \mathcal{A}^{inj}(X, Y)$ ensures that this infimum is taken over a nonempty set of positive numbers. Indeed, $n_{\mathcal{A}}(T) \leq \alpha^{inj}(T)$.

Therefore, $n_{\mathcal{A}}$ can be considered as a measure of injective non-$\mathcal{A}$-compactness in the sense that

- $T \in \mathcal{A}^{inj}(X, Y)$ is injectively $\mathcal{A}$-compact if and only if $n_{\mathcal{A}}(T) = 0$.

Therefore, $n_{\mathcal{A}}$ can be considered as a measure of injective non-$\mathcal{A}$-compactness in the sense that

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We observe that

- $n_{\mathcal{L}} =$ seminorm $\|\cdot\|_m$ studied by Lebow and Schechter

$$\|T\|_m := \inf\Big\{\sigma > 0; \text{ there is a subspace } M \text{ of } X \text{ with } \operatorname{codim}(M) < \infty \\ \text{such that } \|Tx\|_Y \leq \sigma\|x\|_X, \text{ for any } x \in M\Big\}.$$

Thus, $\chi_{\mathcal{L}}(T)/2 \leq n_{\mathcal{L}}(T) \leq 2\chi_{\mathcal{L}}(T)$.

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Thus, $\chi_{\mathcal{L}}(T)/2 \leq n_{\mathcal{L}}(T) \leq 2\chi_{\mathcal{L}}(T)$.

- $n_{\mathcal{A}}$ is a different concept from the (inner) measure $\beta_{\mathcal{A}}$ introduced by Tylli:

$$\beta_{\mathcal{A}}(T) := \inf\Big\{\varepsilon > 0;\ \text{there are a Banach space } Z \text{ and } S \in \mathcal{A}(X, Z) \text{ such that } \|Tx\|_Y \leq \varepsilon\|x\|_X + \|Sx\|_Z,\ \text{for any } x \in X\Big\}.$$

In fact, consider $[\Pi_p, \pi_p]$ and let T be the inclusion map from ℓ_1 into ℓ_2. Since T is p-summing, it follows that $\beta_{\Pi_p}(T) = 0$. However, $n_{\Pi_p}(T) = \chi_{\Pi_p^d}(T^*)$, with T^* being the identity operator from ℓ_2 into ℓ_∞, which is not a p-compact and so $\chi_{\Pi_p^d}(T^*) > 0$.

We are interested in studying the measures $\chi_{\mathcal{A}}$ and $n_{\mathcal{A}}$ when considering a general Banach operator ideal $\mathcal{A}$. In this talk, we focus on their behaviour under interpolation. As far as we know, there is no previous result in this sense.

We have obtained interpolation formulas for $\chi_{\mathcal{A}}$ and $n_{\mathcal{A}}$ in the cases in which one of the Banach couples reduces to a single Banach space.

From these estimates for the measures $\chi_{\mathcal{A}}$ and $n_{\mathcal{A}}$, we deduce results on interpolation of surjectively $\mathcal{A}$-compact operators and injectively $\mathcal{A}$-compact operators, when $\mathcal{A}$ is an arbitrary Banach operator ideal.

As a consequence, we establish in particular interpolation results on p-compact operators (including the case of compact operators) and on quasi p-nuclear operators, respectively, by applying our interpolation formulas to $\mathcal{A} = \Pi_p^d$ and $\mathcal{A} = \Pi_p$, respectively.

• Let $\bar{A} = (A_0, A_1)$ be a Banach couple, i.e., A_0 and A_1 are Banach spaces continuously embedded in some Hausdorff topological vector space. Then, $\Sigma(\bar{A}) := A_0 + A_1$ and $\Delta(\bar{A}) := A_0 \cap A_1$ are Banach spaces endowed with $K(1, \cdot)$ and $J(1, \cdot)$, respectively, where

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$$K(t, a) = K(t, a; A_0, A_1) := \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_i \in A_i\}, a \in \Sigma(\bar{A}).$$

$$J(t, a) = J(t, a; A_0, A_1) := \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}, \ a \in \Delta(\bar{A}).$$

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• Important examples of intermediate spaces:

- The real interpolation space $\bar{A}_{\theta,q} = (A_0, A_1)_{\theta,q}$.
- The complex interpolation space $\bar{A}_{[\theta]} = (A_0, A_1)_{[\theta]}$.
- A_i°, i.e., the closure of $\Delta(\bar{A})$ in A_i equipped with the norm of A_i $(i = 0, 1)$.
- $A_i^\sim$, i.e., the space formed by all those $a \in \Sigma(\bar{A})$ for which there exists a sequence $(a_n)_{n\in\mathbb{N}} \subset A_i$ s.t.

$$\sup_{n\in\mathbb{N}} \|a_n\|_{A_i} < \infty \quad \text{and} \quad \lim_{n\to\infty} \|a - a_n\|_{\Sigma(\bar{A})} = 0. \tag{1}$$

The norm in $A_i^\sim$ is $\|a\|_{A_i^\sim} = \inf\left\{ \sup_{n\in\mathbb{N}} \|a_n\|_{A_i} ; (a_n)_n \text{ satisfies } (1)\right\}$.
$A_i^\sim$ is called the Gagliardo completion of A_i $(i = 0, 1)$ in $\Sigma(\bar{A})$.

- All of these examples are in fact interpolation spaces.

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An intermediate space A with respect to $\bar{A} = (A_0, A_1)$ is said to be an interpolation space if for any operator $T\colon \bar{A} \to \bar{A}$ (i.e., T is a bounded linear operator from $\Sigma(\bar{A})$ into $\Sigma(\bar{A})$ whose restriction to each A_i defines a bounded operator from A_i into A_i for $i = 0, 1$), it holds that $T \in \mathcal{L}(A, A)$. In that case, there is a constant $C = C(A, \bar{A})$ s.t.

$$\|T\|_{A,A} \leq C\|T\|_{\bar{A},\bar{A}}, \quad \text{for all } T\colon \bar{A} \to \bar{A}, \tag{2}$$

where $\|T\|_{\bar{A},\bar{A}} := \max\{\|T\|_{A_0,A_0}, \|T\|_{A_1,A_1}\}$.

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An intermediate space A is called a rank-one interpolation space if (2) is fulfilled for all $T\colon \bar{A} \to \bar{A}$ of the form $T = f \otimes a$, with $a \in \Delta(\bar{A})$ and $f \in \Sigma(\bar{A})^*$.

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An example of an intermediate space A with respect to (L_1, L_∞) which is not an interpolation space can be found in

S.G. Krein, J.I. Petunin, E.M. Semenov, *Interpolation of Linear Operators.* Amer. Math. Soc., Providence R. I, 1982.

Nevertheless, such a space A is a rank-one interpolation space because is lying between Lorentz and Marcinkiewicz spaces with the same fundamental function.

To establish our interpolation formulas for $\chi_{\mathcal{A}}$ and $n_{\mathcal{A}}$ we follow an approach inspired by

F. Cobos, M. Cwikel, P. Matos, *Best possible compactness results of Lions-Peetre type.* Proc. Edinburgh Math. Soc. **44** (2001), 153–172.

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In such formulas we consider interpolated operators acting between intermediate spaces. Let us note that surjective $\mathcal{A}$-compactness cannot be interpolated when considering an arbitrary ideal and general intermediate spaces, in the sense that

- Given a Banach couple $\bar{X} = (X_0, X_1)$, an intermediate space X, a Banach space Y and $T \in \mathcal{A}^{sur}(\Sigma(\bar{X}), Y)$, only one of the following assumptions

$T : X_0 \to Y$ is surject. $\mathcal{A}$-compact or $T : X_1 \to Y$ is surject. $\mathcal{A}$-compact,

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In fact, for $[\mathcal{L}, \|\cdot\|]$, $\bar{X} = (\ell_\infty(2^n), \ell_\infty)$, $X = c_0$, $Y = \ell_\infty$ and $T = \text{Identity}$,

$$T : \ell_\infty(2^n) \to \ell_\infty \text{ is compact (i.e., is surjectively } \mathcal{L}\text{-compact)},$$

but

$$T : c_0 \to \ell_\infty \text{ is not compact (i.e., is not surjectively } \mathcal{L}\text{-compact)}.$$

We consider two functions (variants of those studied by Dmitriev, by Maligranda and Mastyło and by Pustylnik) that describe the "position" of an intermediate space within the couple.

- Given a Banach couple $\bar{A} = (A_0, A_1)$ and an intermediate space A, define

$$\psi_A(t) = \psi_A(t; \bar{A}) := \sup\{K(t,a);\ a \in A, \|a\|_A = 1\},$$

$$\rho_A(t) = \rho_A(t; \bar{A}) := \inf\{J(t,a);\ a \in \Delta(\bar{A}), \|a\|_A = 1\}.$$

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- An intermediate space A is said to be of class $C_K(\theta; \bar{A})$ (resp., of class $C_J(\theta; \bar{A})$), where $0 < \theta < 1$, if there is $C > 0$ s.t. for all $t > 0$ and $a \in A$ (resp., $a \in \Delta(\bar{A})$),

$$K(t,a) \leq Ct^{\theta}\|a\|_A \quad (\text{resp., } \|a\|_A \leq Ct^{-\theta}J(t,a)).$$

Equivalently, A is of class $C_K(\theta; \bar{A})$ (resp., of class $C_J(\theta; \bar{A})$) if and only if

$$\Delta(\bar{A}) \hookrightarrow A \hookrightarrow \bar{A}_{\theta,\infty} \quad (\text{resp., } \bar{A}_{\theta,1} \hookrightarrow A \hookrightarrow \Sigma(\bar{A})).$$

We consider two functions (variants of those studied by Dmitriev, by Maligranda and Mastyło and by Pustylnik) that describe the "position" of an intermediate space within the couple.

- Given a Banach couple $\bar{A} = (A_0, A_1)$ and an intermediate space A, define

$$\psi_A(t) = \psi_A(t; \bar{A}) := \sup\{K(t,a);\ a \in A, \|a\|_A = 1\},$$

$$\rho_A(t) = \rho_A(t; \bar{A}) := \inf\{J(t,a);\ a \in \Delta(\bar{A}), \|a\|_A = 1\}.$$

- An intermediate space A is said to be of class $C_K(\theta; \bar{A})$ (resp., of class $C_J(\theta; \bar{A})$), where $0 < \theta < 1$, if there is $C > 0$ s.t. for all $t > 0$ and $a \in A$ (resp., $a \in \Delta(\bar{A})$),

$$K(t,a) \le C t^{\theta} \|a\|_A \quad (\text{resp., } \|a\|_A \le C t^{-\theta} J(t,a)).$$

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If A is of class $C_K(\theta; \bar{A})$ (resp., of class $C_J(\theta; \bar{A})$), then

$$\lim_{t\to 0} \psi_A(t) = \lim_{t\to\infty} \frac{\psi_A(t)}{t} = 0 \quad \left(\text{resp., } \lim_{t\to 0} \frac{t}{\rho_A(t)} = \lim_{t\to\infty} \frac{1}{\rho_A(t)} = 0\right).$$

We consider two functions (variants of those studied by Dmitriev, by Maligranda and Mastyło and by Pustylnik) that describe the "position" of an intermediate space within the couple.

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We consider two functions (variants of those studied by Dmitriev, by Maligranda and Mastyło and by Pustylnik) that describe the "position" of an intermediate space within the couple.

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$$\lim_{t\to 0} \psi_A(t) = \lim_{t\to\infty} \frac{\psi_A(t)}{t} = 0 \quad \left(\text{resp., } \lim_{t\to 0} \frac{t}{\rho_A(t)} = \lim_{t\to\infty} \frac{1}{\rho_A(t)} = 0\right).$$

$\bar{A}_{\theta,q}$ and $\bar{A}_{[\theta]}$ are both of class $C_K(\theta;\bar{A})$ and of class $C_J(\theta;\bar{A})$.

For the measure $\chi_{\mathcal{A}}$ we have obtained:

Theorem

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let Y be a Banach space. Assume that X is an intermediate space with respect to $\bar{X}$. For any $T \in \mathcal{A}^{sur}(\Sigma(\bar{X}), Y)$,

(i) If $\chi_{\mathcal{A}}(T_{X_0,Y}) = 0$,

$$\chi_{\mathcal{A}}(T_{X,Y}) \leq \chi_{\mathcal{A}}(T_{X_1,Y}) \cdot \lim_{t\to\infty} \frac{\psi_X(t)}{t}.$$

(ii) If $\chi_{\mathcal{A}}(T_{X_1,Y}) = 0$,

$$\chi_{\mathcal{A}}(T_{X,Y}) \leq \chi_{\mathcal{A}}(T_{X_0,Y}) \cdot \lim_{t\to 0} \psi_X(t).$$

(iii) If $\chi_{\mathcal{A}}(T_{X_i,Y}) > 0$ for $i = 0, 1$, then

$$\chi_{\mathcal{A}}(T_{X,Y}) \leq 2\chi_{\mathcal{A}}(T_{X_0,Y}) \cdot \psi_X\left(\frac{\chi_{\mathcal{A}}(T_{X_1,Y})}{\chi_{\mathcal{A}}(T_{X_0,Y})}\right).$$

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(i) If $\chi_{\mathcal{A}}(T_{X_0,Y})=0$,

$$\chi_{\mathcal{A}}(T_{X,Y})\le\chi_{\mathcal{A}}(T_{X_1,Y})\cdot\lim_{t\to\infty}\frac{\psi_X(t)}{t}.$$

(ii) If $\chi_{\mathcal{A}}(T_{X_1,Y})=0$,

$$\chi_{\mathcal{A}}(T_{X,Y})\le\chi_{\mathcal{A}}(T_{X_0,Y})\cdot\lim_{t\to0}\psi_X(t).$$

(iii) If $\chi_{\mathcal{A}}(T_{X_i,Y})>0$ for $i=0,1$, then

$$\chi_{\mathcal{A}}(T_{X,Y})\le 2\chi_{\mathcal{A}}(T_{X_0,Y})\cdot\psi_X\left(\frac{\chi_{\mathcal{A}}(T_{X_1,Y})}{\chi_{\mathcal{A}}(T_{X_0,Y})}\right).$$

Writing down the last theorem for the ideal $\mathcal{L}$ of all bounded operators, we recover a previous result for the (ball) measure of non-compactness, established by Cobos, Cwikel and Matos.

As a consequence, we deduce the following well-known theorem due to Lions-Peetre:

THEOREM (Lions and Peetre, 1964)

Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let Y be a Banach space. Assume that X is of class $C_K(\theta; \bar{X})$. For any $T \in \mathcal{L}(\Sigma(\bar{X}), Y)$ such that

$$T : X_i \to Y \text{ is compact for } i = 0 \text{ or } i = 1,$$

then

$$T : X \to Y \text{ is compact.}$$

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$$T : X_i \to Y \text{ is compact for } i = 0 \text{ or } i = 1,$$

then

$$T : X \to Y \text{ is compact.}$$

We also obtain a Lions-Peetre type result for p-compactness ($1 \leq p < \infty$):

Corollary

Let $1 \leq p < \infty$. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let Y be a Banach space. Assume that $T \in \Pi_p^d(\Sigma(\bar{X}), Y)$. Let X be of class $C_K(\theta; \bar{X})$. If either

$$T : X_0 \to Y \text{ or } T : X_1 \to Y \text{ is } p\text{-compact,}$$

then

$$T : X \to Y \text{ is } p\text{-compact.}$$

The following corollary is a straightforward consequence of our estimates for $\chi_{\mathcal{A}}$ and arbitrary intermediate spaces.

Corollary

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let Y be a Banach space. Assume that X is an intermediate space with respect to $\bar{X}$. Given $T \in \mathcal{A}^{sur}(\Sigma(\bar{X}), Y)$, it follows that

$T: X \to Y$ is a surjectively $\mathcal{A}$-compact operator

whenever one of the following assertions holds:

- $T: X_0 \to Y$ and $T: X_1 \to Y$ are surjectively $\mathcal{A}$-compact operators.
- $T: X_0 \to Y$ is surjectively $\mathcal{A}$-compact and $\lim_{t\to\infty} \frac{\psi_X(t)}{t} = 0$.
- $T: X_1 \to Y$ is surjectively $\mathcal{A}$-compact and $\lim_{t\to 0} \psi_X(t) = 0$.

In the previous example

For $[\mathcal{L}, \|\cdot\|]$, $\bar{X} = (\ell_\infty(2^n), \ell_\infty)$, $X = c_0$, $Y = \ell_\infty$ and $T =$ Identity, we have

$$T : X_0 \to Y \text{ is compact (i.e., surjectively } \mathcal{L}\text{-compact)},$$

but neither

$$T : X_1 \to Y \;\text{ nor }\; T : X \to Y \text{ is compact (i.e., surjectively } \mathcal{L}\text{-compact)}.$$

It holds

$$\overline{X_0 \cap X_1}^{X_1} = \overline{\ell_\infty(2^n) \cap \ell_\infty}^{\ell_\infty} = \overline{\ell_\infty(2^n)}^{\ell_\infty} = c_0,$$

that is, $X_1^\circ = X$. Moreover, $\psi_X(t) = t$ for all $t > 0$, and so $\lim_{t\to\infty} \frac{\psi_X(t)}{t} > 0$.

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that is, $X_1^\circ = X$. Moreover, $\psi_X(t) = t$ for all $t > 0$, and so $\lim_{t\to\infty} \frac{\psi_X(t)}{t} > 0$.

Proposition (Cobos, Cwikel and Matos, 2001)

Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let X be a rank-one interp. space.

(a) If $\lim_{t\to 0} \psi_X(t) > 0$, then $X_0^\circ \hookrightarrow X$.

(b) If $\lim_{t\to\infty} \frac{\psi_X(t)}{t} > 0$, then $X_1^\circ \hookrightarrow X$.

Theorem

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let Y be a Banach space. Suppose that $T \in \mathcal{A}^{sur}(\Sigma(\bar{X}), Y)$ and X is a rank-one interpolation space with respect to $\bar{X}$. When

$$T\colon X_0 \to Y \text{ is a surjectively } \mathcal{A}\text{-compact operator,}$$

then at least one of the following conditions is fulfilled:

(i) $T\colon X \to Y$ is surjectively $\mathcal{A}$-compact.

(ii) $X_1^\circ \hookrightarrow X$.

Theorem

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let Y be a Banach space. Suppose that $T \in \mathcal{A}^{sur}(\Sigma(\bar{X}), Y)$ and X is a rank-one interpolation space with respect to $\bar{X}$. When

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then at least one of the following conditions is fulfilled:

(i) $T : X \to Y$ is surjectively $\mathcal{A}$-compact.

(ii) $X_1^\circ \hookrightarrow X$.

Furthermore, if $X_1^\circ = X_1$,

$$T : X \to Y \text{ is surjectively } \mathcal{A}\text{-compact}$$

if and only if at least one of the next conditions holds:

(i') $T : X_1 \to Y$ is surjectively $\mathcal{A}$-compact.

(ii') $\lim_{t \to \infty} \frac{\psi_X(t)}{t} = 0.$

Concerning measure $n_{\mathcal{A}}$, we have established:

Theorem

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let X be a Banach space and let $\bar{Y} = (Y_0, Y_1)$ be a Banach couple. Assume that Y is an intermediate space with respect to $\bar{Y}$. For any $T \in \mathcal{A}^{inj}(X, \Delta(\bar{Y}))$,

(i) If $n_{\mathcal{A}}(T_{X,Y_0}) = 0$,
$$n_{\mathcal{A}}(T_{X,Y}) \leq n_{\mathcal{A}}(T_{X,Y_1}) \cdot \lim_{t\to 0} \frac{t}{\rho_Y(t)} .$$

(ii) If $n_{\mathcal{A}}(T_{X,Y_1}) = 0$,
$$n_{\mathcal{A}}(T_{X,Y}) \leq n_{\mathcal{A}}(T_{X,Y_0}) \cdot \lim_{t\to \infty} \frac{1}{\rho_Y(t)} .$$

(iii) If $n_{\mathcal{A}}(T_{X,Y_i}) > 0$ for $i = 0, 1$, then
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(iii) If $n_{\mathcal{A}}(T_{X,Y_i}) > 0$ for $i = 0, 1$, then
$$n_{\mathcal{A}}(T_{X,Y}) \le \frac{2n_{\mathcal{A}}(T_{X,Y_0})}{\rho\Big(n_{\mathcal{A}}(T_{X,Y_0})/n_{\mathcal{A}}(T_{X,Y_1})\Big)} .$$

Writing down the last theorem for the ideal $\mathcal{L}$ of all bounded operators, we obtain a similar estimate to that one proved by Cobos, Cwikel and Matos for the (ball) measure of non-compactness.

Our formulas for $n_{\mathcal{A}}$ allow to deduce the Lions-Peetre result in the dual situation:

Theorem (Lions and Peetre, 1964)

Let X be a Banach space and let $\bar{Y} = (Y_0, Y_1)$ be a Banach couple. Assume that Y is of class $C_J(\theta; \bar{Y})$. For any $T \in \mathcal{L}(X, \Delta(\bar{Y}))$ such that

$$T : X \to Y_i \text{ is compact for } i = 0 \text{ or } i = 1,$$

then

$$T : X \to Y \text{ is compact.}$$

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$$T : X \to Y_i \text{ is compact for } i = 0 \text{ or } i = 1,$$

then

$$T : X \to Y \text{ is compact.}$$

As an application, we also obtain a Lions-Peetre type result on interpolation of quasi p-nuclear operators:

Corollary

Let $1 \leq p < \infty$. Let X be a Banach space and let $\bar{Y} = (Y_0, Y_1)$ be a Banach couple. Assume that $T \in \Pi_p(X, \Delta(\bar{Y}))$. Let Y be of class $C_J(\theta; \bar{Y})$. If either

$$T : X \to Y_0 \text{ or } T : X \to Y_1 \text{ is quasi } p\text{-nuclear,}$$

then

$$T : X \to Y \text{ is a quasi } p\text{-nuclear.}$$

The next corollary follows directly from our estimates for $n_{\mathcal{A}}$ and general intermediate spaces.

Corollary

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let X be a Banach space and let $\bar{Y} = (Y_0, Y_1)$ be a Banach couple. Assume that Y is an intermediate space with respect to $\bar{Y}$. Given $T \in \mathcal{A}^{inj}(X, \Delta(\bar{Y}))$, it follows that

$$T\colon X \to Y \text{ is an injectively } \mathcal{A}\text{-compact operator}$$

whenever one of the following assertions holds:

- ◇ $T\colon X \to Y_0$ and $T\colon X \to Y_1$ are injectively $\mathcal{A}$-compact operators.
- ◇ $T\colon X \to Y_0$ is injectively $\mathcal{A}$-compact and $\lim_{t\to 0} \frac{t}{\rho_Y(t)} = 0$.
- ◇ $T\colon X \to Y_1$ is injectively $\mathcal{A}$-compact and $\lim_{t\to \infty} \frac{1}{\rho_Y(t)} = 0$.

Theorem

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let X be a Banach space and let $\bar{Y} = (Y_0, Y_1)$ be a Banach couple. Suppose that $T \in \mathcal{A}^{inj}(X, \Delta(\bar{Y}))$ and Y is a rank-one interpolation space with respect to $\bar{Y}$. When

$$T : X \to Y_0 \text{ is an injectively } \mathcal{A}\text{-compact operator,}$$

then at least one of the following conditions is fulfilled:

(i) $T : X \to Y$ is injectively $\mathcal{A}$-compact.

(ii) $Y \hookrightarrow Y_1^{\sim}$.

Theorem

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let X be a Banach space and let $\bar{Y} = (Y_0, Y_1)$ be a Banach couple. Suppose that $T \in \mathcal{A}^{inj}(X, \Delta(\bar{Y}))$ and Y is a rank-one interpolation space with respect to $\bar{Y}$. When

$$T\colon X \to Y_0 \text{ is an injectively } \mathcal{A}\text{-compact operator,}$$

then at least one of the following conditions is fulfilled:

(i) $T\colon X \to Y$ is injectively $\mathcal{A}$-compact.

(ii) $Y \hookrightarrow Y_1^{\sim}$.

Moreover, if $Y_1^{\sim} = Y_1$,

$$T\colon X \to Y \text{ is injectively } \mathcal{A}\text{-compact}$$

if and only if at least one of the next conditions holds:

(i') $T\colon X \to Y_1$ is injectively $\mathcal{A}$-compact.

(ii') $\lim_{t\to 0} \dfrac{t}{\rho_Y(t)} = 0$.

We have also established interpolation formulas for the measure of $T: X \to Y$ in terms of the measures of the restrictions $T: \Delta(\bar{X}) \to Y$ and $T: \Sigma(\bar{X}) \to Y$ (respectively, $T: X \to \Delta(\bar{Y})$ and $T: X \to \Sigma(\bar{Y})$), for $T \in \mathcal{A}^{sur}(\Sigma(\bar{X}), Y)$ (respectively, $T \in \mathcal{A}^{inj}(X, \Delta(\bar{Y}))$. In case of $\chi_{\mathcal{A}}$:

Theorem

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let Y be a Banach space. Assume that X is an intermediate space with respect to $\bar{X}$. For every $T \in \mathcal{A}^{sur}(\Sigma(\bar{X}), Y)$,

(i) When $\chi_{\mathcal{A}}(T_{\Delta(\bar{X}),Y}) = 0$,

$$\chi_{\mathcal{A}}(T_{X,Y}) \leq \chi_{\mathcal{A}}(T_{\Sigma(\bar{X}),Y}) \cdot \left(\lim_{t\to 0} \psi_X(t) + \lim_{t\to\infty} \frac{\psi_X(t)}{t} \right).$$

(ii) When $\chi_{\mathcal{A}}(T_{\Delta(\bar{X}),Y}) > 0$,

$$\chi_{\mathcal{A}}(T_{X,Y}) \leq 2 \left(\frac{\psi_X\left(\chi_{\mathcal{A}}(T_{\Sigma(\bar{X}),Y})/\chi_{\mathcal{A}}(T_{\Delta(\bar{X}),Y})\right)}{1/\chi_{\mathcal{A}}(T_{\Delta(\bar{X}),Y})} + \frac{\psi_X\left(\chi_{\mathcal{A}}(T_{\Delta(\bar{X}),Y})/\chi_{\mathcal{A}}(T_{\Sigma(\bar{X}),Y})\right)}{1/\chi_{\mathcal{A}}(T_{\Sigma(\bar{X}),Y})} \right).$$

Corollary

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let Y be a Banach space. Assume that X is an intermediate space with respect to $\bar{X}$ and $T \in \mathcal{A}^{sur}(\Sigma(\bar{X}), Y)$. When

$$T\colon \Delta(\bar{X}) \to Y \text{ is surjectively } \mathcal{A}\text{-compact and } \lim_{t\to 0} \psi_X(t) = \lim_{t\to\infty} \frac{\psi_X(t)}{t} = 0,$$

then

$$T\colon X \to Y \text{ is a surjectively } \mathcal{A}\text{-compact operator.}$$

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Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let Y be a Banach space. Assume that X is an intermediate space with respect to $\bar{X}$ and $T \in \mathcal{A}^{sur}(\Sigma(\bar{X}), Y)$. When

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then

$$T\colon X \to Y \text{ is a surjectively } \mathcal{A}\text{-compact operator.}$$

Corollary

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let Y be a Banach space. If X is of class $C_K(\theta; \bar{X})$ and $T \in \mathcal{A}^{sur}(\Sigma(\bar{X}), Y)$

$$T\colon X \to Y \text{ is surjectively } \mathcal{A}\text{-compact}$$

if and only if

$$T\colon \Delta(\bar{X}) \to Y \text{ is surjectively } \mathcal{A}\text{-compact.}$$

Interpolation results on p-compact operators can be deduced by applying above corollaries to the Banach ideal given by the dual ideal of p-summing operators.

In case of $n_{\mathcal{A}}$:

Theorem

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let X be a Banach space and let $\bar{Y} = (Y_0, Y_1)$ be a Banach couple. Assume that Y is an intermediate with respect to $\bar{Y}$. For every $T \in \mathcal{A}^{inj}(X, \Delta(\bar{Y}))$,

(i) When $n_{\mathcal{A}}(T_{X,\Sigma(\bar{Y})}) = 0$,

$$n_{\mathcal{A}}(T_{X,Y}) \leq 2 n_{\mathcal{A}}(T_{X,\Delta(\bar{Y})}) \cdot \left(\lim_{t\to 0} \frac{t}{\rho_Y(t)} + \lim_{t\to\infty} \frac{1}{\rho_Y(t)} \right).$$

(ii) When $n_{\mathcal{A}}(T_{X,\Sigma(\bar{Y})}) > 0$,

$$n_{\mathcal{A}}(T_{X,Y}) \leq 3 \left(\frac{n_{\mathcal{A}}(T_{X,\Sigma(\bar{Y})})}{\rho\Big(n_{\mathcal{A}}(T_{X,\Sigma(\bar{Y})})/n_{\mathcal{A}}(T_{X,\Delta(\bar{Y})})\Big)} + \frac{n_{\mathcal{A}}(T_{X,\Delta(\bar{Y})})}{\rho\Big(n_{\mathcal{A}}(T_{X,\Delta(\bar{Y})})/n_{\mathcal{A}}(T_{X,\Sigma(\bar{Y})})\Big)} \right).$$

Corollary

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let X be a Banach space and let $\bar{Y} = (Y_0, Y_1)$ be a Banach couple. Assume that Y is an intermediate with respect to $\bar{Y}$ and $T \in \mathcal{A}^{inj}(X, \Delta(\bar{Y}))$. When

$$T\colon X \to \Sigma(\bar{Y}) \text{ is injectively } \mathcal{A}\text{-compact and } \lim_{t\to 0} \frac{t}{\rho_Y(t)} = \lim_{t\to\infty} \frac{1}{\rho_Y(t)} = 0,$$

then

$$T\colon X \to Y \text{ is an injectively } \mathcal{A}\text{-compact operator.}$$

Corollary

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let X be a Banach space and let $\bar{Y} = (Y_0, Y_1)$ be a Banach couple. Assume that Y is an intermediate with respect to $\bar{Y}$ and $T \in \mathcal{A}^{inj}(X, \Delta(\bar{Y}))$. When

$$T: X \to \Sigma(\bar{Y}) \text{ is injectively } \mathcal{A}\text{-compact and } \lim_{t\to 0} \frac{t}{\rho_Y(t)} = \lim_{t\to\infty} \frac{1}{\rho_Y(t)} = 0,$$

then

$$T: X \to Y \text{ is an injectively } \mathcal{A}\text{-compact operator.}$$

Corollary

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let X be a Banach space and let $\bar{Y} = (Y_0, Y_1)$ be Banach couple. If Y is of class $C_J(\theta; \bar{Y})$ and $T \in \mathcal{A}^{inj}(X, \Delta(\bar{Y}))$

$$T: X \to Y \text{ is injectively } \mathcal{A}\text{-compact}$$

if and only if

$$T: X \to \Sigma(\bar{Y}) \text{ is injectively } \mathcal{A}\text{-compact.}$$

Interpolation results on quasi p-nuclear operators can be obtained by applying these corollaries to the Banach operator ideal of p-summing operators.

Corollary

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let $\bar{X} = (X_0, X_1)$ be a Banach couple and let Y be a Banach space. Suppose that $T \in \mathcal{A}^{sur}(\Sigma(\bar{X}), Y)$ and X is a rank-one interpolation space with respect to $\bar{X}$. When

$T\colon \Delta(\bar{X}) \to Y$ is a surjectively $\mathcal{A}$-compact operator,

then at least one of the following conditions is fulfilled:

(i) $T\colon X \to Y$ is surjectively $\mathcal{A}$-compact.

(ii) $X_0^\circ \hookrightarrow X$.

(iii) $X_1^\circ \hookrightarrow X$

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Corollary

Let $[\mathcal{A}, \alpha]$ be a Banach operator ideal. Let X be a Banach space and let $\bar{Y} = (Y_0, Y_1)$ be a Banach couple. Suppose that $T \in \mathcal{A}^{inj}(X, \Delta(\bar{Y}))$ and Y is a rank-one interpolation space with respect to $\bar{Y}$. When

$T\colon X \to \Sigma(\bar{Y})$ is an injectively $\mathcal{A}$-compact operator,

then at least one of the following conditions is fulfilled:

(i) $T\colon X \to Y$ is injectively $\mathcal{A}$-compact.
(ii) $Y \hookrightarrow Y_0^\sim$.
(iii) $Y \hookrightarrow Y_1^\sim$.

Thank you so much!

and

Congratulations, Fernando