

Subgaussian Kahane-Salem-Zygmund inequalities in Banach spaces

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Random polynomials

- For a multi-index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}^n$ and $z = (z_1, \dots, z_n) \in \mathbb{C}^n$, we let $|\alpha| := |\alpha_1| + \dots + |\alpha_n|$ and $z^\alpha := z_1^{\alpha_1} \dots z_n^{\alpha_n}$. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$, $\alpha! := \alpha_1! \dots \alpha_n!$, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$.
- If $P: \mathbb{C}^n \rightarrow \mathbb{C}$ is a polynomial given by

$$P(z) = \sum_{\alpha \in \mathbb{N}_0^n} c_\alpha z^\alpha, \quad z \in \mathbb{C}^n,$$

then its degree $\deg(P) := \max\{|\alpha|; c_\alpha \neq 0\}$.

- For $n \in \mathbb{N}$ and $m \in \mathbb{N}_0$ we denote by $\mathcal{T}_m(\mathbb{T}^n)$ the space of all trigonometric polynomials

$$P(z) = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha, \quad z \in \mathbb{T}^n$$

on the n -dimensional torus \mathbb{T}^n with $\deg(P) := \max\{|\alpha|; c_\alpha \neq 0\} \leq m$.

- The origin of trigonometric random polynomials goes back to the **Salem** and **Zygmund** seminal Acta Math. (1954) paper. They proved that there exist positive constants c_1, c_2 such that for almost all $t \in [0, 1]$ one has

$$c_1 \sqrt{n \log n} \leq \sup_{x \in [0, 2\pi]} \left| \sum_{j=0}^{n-1} r_j(t) e^{ijx} \right| \leq c_2 \sqrt{n \log n}.$$

- The study of random polynomials was continued by **Kahane** (1960) in the setting of random polynomials in several variables. In the recent decades the multidimensional variants of the Kahane-Salem-Zygmund inequalities (**KSZ-inequalities** for short) have been of central importance in modern analysis, as, e.g., Fourier analysis, analytic number theory, or holomorphy in high dimensions. The multidimensional **KSZ-inequality** states:

Theorem There exists a positive constant C such that, for each $m, n \in \mathbb{N}$ with $m \geq 2$ and any trigonometric polynomial $\sum_{\alpha \in \mathbb{Z}^n} c_\alpha z^\alpha$ in $\mathcal{T}_m(\mathbb{T}^n)$ there exists a choice of signs $\varepsilon_\alpha = \pm 1$ for which

$$\sup_{z \in \mathbb{T}^n} \left| \sum_{|\alpha| \leq m} \varepsilon_\alpha c_\alpha z^\alpha \right| \leq C \sqrt{n \log m} \left(\sum_{|\alpha| \leq m} |c_\alpha|^2 \right)^{\frac{1}{2}}.$$

Extended variant of Kahane-Salem-Zygmund inequality

Theorem For every $a > 0$, for each integers $m, n \geq 2$ and for all families of $(\varepsilon_\alpha)_\alpha$ of independent Bernoulli variables on a probability measure space $(\Omega, \mathcal{A}, \mathbb{P})$ and $(c_\alpha)_\alpha \subset \mathbb{C}$, where $\alpha \in \mathbb{Z}^n$, $|\alpha| \leq m$ and not all of them equal to zero, we have

$$\mathbb{P}\left\{\omega \in \Omega; \sup_{z \in \mathbb{T}^n} \left| \sum_{|\alpha| \leq m} \varepsilon_\alpha(\omega) c_\alpha z^\alpha \right| \geq a \sqrt{n \log m} \left(\sum_{|\alpha| \leq m} |c_\alpha|^2 \right)^{\frac{1}{2}} \right\} \leq C(a),$$

where $C(a) := 4(\pi^2 / (m^{\frac{a^2}{16}} - 1))^n$.

Remark. For all $a > 4\left(\frac{\log(2\pi^2)}{\log 2} + 1\right)^{1/2}$ one has $C(a) < 1$, so for this a , we get

$$\mathbb{P}\left\{\omega \in \Omega; \sup_{z \in \mathbb{T}^n} \left| \sum_{|\alpha| \leq m} \varepsilon_\alpha(\omega) c_\alpha z^\alpha \right| \leq a \sqrt{n \log m} \left(\sum_{|\alpha| \leq m} |c_\alpha|^2 \right)^{\frac{1}{2}} \right\} > 0.$$

Theorem (H. P. Boas (2000)) Let $1 \leq p \leq \infty$ and $m, n \in \mathbb{N}$ with $m, n \geq 2$. Then there exists a choice of signs $(\varepsilon_\alpha)_{|\alpha|=m}$, $\varepsilon_\alpha = \pm 1$ such that

- If $1 \leq p \leq 2$, then

$$\sup_{z \in B_{\ell_p^{(n)}}} \left| \sum_{|\alpha|=m} \varepsilon_\alpha \frac{m!}{\alpha!} z^\alpha \right| \leq C \sqrt{mn \log m} (m!)^{1-1/p}.$$

- If $2 \leq p \leq \infty$, then

$$\sup_{z \in B_{\ell_p^{(n)}}} \left| \sum_{|\alpha|=m} \varepsilon_\alpha \frac{m!}{\alpha!} z^\alpha \right| \leq C \sqrt{mn \log m} n^{(1/2-1/p)m} (m!)^{1/2},$$

where $C > 0$ does not depend on m nor on n .

Abstract random Kahane-Salem-Zygmund inequalities

- **Definition** (A. Defant–M. M.) Let $X \subset L^0(\Omega, \mathcal{A}, \mu)$ be a Banach function lattice over a measure space $(\Omega, \mathcal{A}, \mu)$. A sequence $(\gamma_i)_{i \in \mathbb{N}} \subset X$ is said to satisfy the **KSZ-inequality** of type $(X, (S^n), \psi)$ provided there exist a function $\psi: \mathbb{N} \rightarrow (0, \infty)$ and a sequence $(\|\cdot\|_n)_{n \in \mathbb{N}}$ of semi-norms on \mathbb{K}^n such that, for every choice of finitely many vectors $(a_i(j))_{j=1}^N \in \ell_\infty^N$, $1 \leq i \leq K$ the following inequality holds:

$$\left\| \sup_{1 \leq j \leq N} \left\| \sum_{i=1}^K a_i(j) \gamma_i \right\| \right\|_X \leq \psi(N) \sup_{1 \leq j \leq N} \|(a_i(j))_{i=1}^K\|_{S^K},$$

- Of special importance for the applications we have in mind, are Rademacher random variables and the much larger class of subgaussian random variables.

KSZ-inequalities by lattice constants

- If X is a Banach lattice, then for each $n \in \mathbb{N}$, the M -constant $\mu_n(X)$ is defined by

$$\mu_n(X) := \sup \left\{ \left\| \sup_{1 \leq j \leq n} |x_j| \right\|_X : \|x_j\|_X \leq 1, \text{ for } 1 \leq j \leq n \right\}.$$

- Properties:** $(\mu_n(X))_n$ is a non-decreasing sequence with $\mu_n(X) \in [1, n]$ for each $n \in \mathbb{N}$; $(\mu_n(X))_n$ is a submultiplicative sequence, that is,

$$\mu_{mn}(X) \leq \mu_m(X)\mu_n(X), \quad m, n \in \mathbb{N};$$

$\left(\frac{\mu_n(X)}{n}\right)$ is non-increasing sequence (Abramovich–Lozanovskii (1973)).

- $\lim_{n \rightarrow \infty} \frac{\mu_n(X)}{n} \in \{0, 1\}$. This implies $\mu_n(X) = n$ for each n whenever $\lim_{n \rightarrow \infty} \frac{\mu_n(X)}{n} = 1$.
- Theorem** (Abramovich–Lozanovskii (1973)) If $\lim_{n \rightarrow \infty} \frac{\mu_n(X)}{n} = 0$, then all odd duals of X are KB -spaces.

Proposition (A. Defant–M. M.) Let X be a Banach lattice over $(\Omega, \mathcal{A}, \nu)$ and let $\psi: \mathbb{N} \rightarrow [1, \infty)$ be given by $\psi(n) := \mu_n(X)$ for each $n \in \mathbb{N}$. Then every sequence $(\gamma_i)_{i \in \mathbb{N}}$ of random variables in X satisfies the KSZ-inequality of type $(X, (S^n), \psi)$ with $S^n := (\mathbb{K}^n, \|\cdot\|_n)$, that is,

$$\left\| \sup_{1 \leq j \leq N} \left\| \sum_{i=1}^K a_i(j) \gamma_i \right\|_X \right\| \leq \psi(N) \sup_{1 \leq j \leq N} \|(a_i(j))_{i=1}^K\|_{S^K}, \quad (a_i(j))_{j=1}^N \in \ell_\infty^N$$

$S^n := (\mathbb{K}^n, \|\cdot\|_n)$, where the semi norm $\|\cdot\|_n$ (resp., norm $\|\cdot\|_n$, whenever the γ_i are linearly independent) are defined by

$$\|z\|_n := \|z_1 \gamma_1 + \dots + z_n \gamma_n\|_X, \quad z = (z_1, \dots, z_n) \in \mathbb{K}^n.$$

M -constants for some class of Orlicz spaces

Let $\Phi: \mathbb{R}_+ \rightarrow \mathbb{R}_+$ be an Orlicz function (that is, a convex, increasing and continuous positive function with $\Phi(0) = 0$). The Orlicz space $L_\Phi \subset L^0(\Omega, \mathcal{A}, \mu)$ is defined to be the space of all f such that $\int_\Omega \Phi(\lambda|f|) d\mu < \infty$ for some $\lambda > 0$, and it is equipped with the norm

$$\|f\|_\Phi := \inf \left\{ \lambda > 0 : \int_\Omega \Phi\left(\frac{|f|}{\lambda}\right) d\mu \leq 1 \right\}.$$

For $1 \leq r < \infty$, the exponential Orlicz function $\varphi_r(t) := e^{t^r} - 1$, $t \geq 0$.

Proposition. [A. Defant–M. M.] Let L_Φ be an Orlicz space over a probability measure space $(\Omega, \mathcal{A}, \nu)$ with $\Phi(t) = e^{\varphi(t)} - 1$ for all $t \geq 0$, where φ is an Orlicz function on \mathbb{R}_+ such that, for some $\gamma > 0$, $\varphi(st) \leq \gamma\varphi(s)\varphi(t)$ for all $s \in (0, 1]$ and $t > 0$. Then, for each $n \in \mathbb{N}$, one has

$$\mu_n(L_\Phi) \leq \frac{K}{\varphi^{-1}(\varphi(1)/(1 + \log n))},$$

where $K = (e - 1)\gamma\varphi(1)$.

Corollary For $r \in [1, \infty)$ let L_{φ_r} be an Orlicz space over a probability measure space $(\Omega, \mathcal{A}, \nu)$ with $\varphi_r(t) = e^{t^r} - 1$ for all $t \geq 0$. Then for each $n \in \mathbb{N}$ one has

$$\mu_n(L_{\varphi_r}) \leq (e - 1)(1 + \log n)^{\frac{1}{r}}.$$

KSZ-inequalities for subgaussian random variables

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space, and f a random variable. If f is **real-valued**, then f is said to be **subgaussian**, whenever there exists $s \geq 0$ such that

$$\mathbb{E} \exp(\lambda f) \leq \exp\left(\frac{s^2 \lambda^2}{2}\right), \quad \lambda \in \mathbb{R},$$

and if f is **complex-valued**, whenever there exists $s \geq 0$ such that

$$\mathbb{E} \exp(\operatorname{Re}(zf)) \leq \exp\left(\frac{s^2 |z|^2}{2}\right), \quad z \in \mathbb{C}.$$

The best such s is denoted by $\operatorname{sg}(f)$.

- A real-valued sequence (f_n) is called subgaussian if there is $s \geq 0$ such that for any $x = (x_n) \in \ell_2$ of norm one, the random variable $f = \sum_{n=1}^{\infty} x_n f_n$ is subgaussian. The best possible number s is denoted by $sg((f_n))$.
- A complex-valued sequence (f_n) is said to be subgaussian, whenever $(\operatorname{Re} f_n)$ and the imaginary parts $(\operatorname{Im} f_n)$, is subgaussian.

Examples

- Every sequence (γ_n) of independent, real (resp., complex) normal gaussian variables is subgaussian with $sg((\gamma_n)) = 1$.
- Every sequences (ε_n) of independent Rademacher variables is subgaussian with $sg((\varepsilon_n)) = 1$.

Theorem (A. Defant–M. M.) Let $(\gamma_i)_{i \in \mathbb{N}}$ be a (real or complex) subgaussian sequence of random variables over $(\Omega, \mathcal{A}, \mathbb{P})$ with $s = \text{sg}((\gamma_i))$. The following statements are true for each $K, N \in \mathbb{N}$ and all $a_1, \dots, a_K \in \ell_\infty^N$ with $a_i = (a_i(j))_{j=1}^N$, $1 \leq i \leq K$:

(1) There is a constant $C_2 = C(s) > 0$ such that

$$\left\| \sum_{i=1}^K \gamma_i a_i \right\|_{L_{\varphi_2}(\ell_\infty^N)} \leq C_2 (1 + \log N)^{\frac{1}{2}} \sup_{1 \leq j \leq N} \|(a_i(j))_{i=1}^K\|_{\ell_2^K}.$$

(2) If in addition $M = \sup_i \|\gamma_i\|_\infty < \infty$, then for every $r \in (2, \infty)$ there is a constant $C_r = C(r, s, M) > 0$ such that

$$\left\| \sum_{i=1}^K \gamma_i a_i \right\|_{L_{\varphi_r}(\ell_\infty^N)} \leq C_r (1 + \log N)^{\frac{1}{r}} \sup_{1 \leq j \leq N} \|(a_i(j))_{i=1}^K\|_{\ell_{r'}^K}.$$

- Here $\ell_{p,\infty}$ for $p \in (1, \infty)$ denotes the Marcinkiewicz sequence space of all scalar sequences $x = (x_k)_k \in \omega(\mathbb{N})$ equipped with the norm

$$\|x\|_{p,\infty} := \sup_{n \in \mathbb{N}} \frac{x_1^* + \dots + x_n^*}{n^{1-\frac{1}{p}}},$$

where (x_k^*) denotes the decreasing rearrangement of the sequence $(|x_k|)$.

Abstract type Kahane–Salem–Zygmund inequalities

Let P be an m -homogeneous random Bernoulli polynomial over a probability measure $(\Omega, \mathcal{A}, \mathbb{P})$ given by

$$P(\omega, z) := \sum_{|\alpha|=m} \varepsilon_\alpha(\omega) c_\alpha z^\alpha, \quad \omega, \Omega, z \in \mathbb{C}^n.$$

Theorem (F. Bayart (2012)) For an arbitrary n -dimensional Banach space $X_n = (\mathbb{C}^n, \|\cdot\|)$ and for every $r \in [2, \infty)$ one has

$$\mathbb{E} \left(\sup_{z \in B_{X_n}} |P(\cdot, z)| \right) \leq C_r (n(1 + \log m))^{\frac{1}{r}} \sup_{|\alpha|=m} |c_\alpha| \left(\frac{\alpha!}{m!} \right)^{\frac{1}{r'}} \sup_{z \in B_{X_n}} \left(\sum_{k=1}^n |z_k|^{r'} \right)^{\frac{m}{r}},$$

where $C_r > 0$ is a constant only depending on r .

Given a real number $1 \leq \lambda < \infty$. A Banach space X **λ -embeds** into a Banach space Y whenever there exists an isomorphic embedding T of X into Y such

$$\|T\|_{X \rightarrow Y} \|T^{-1}\|_{T(X) \rightarrow X} \leq \lambda.$$

In this case, we call T a **λ -embedding** of X into Y .

Theorem [A. Defant–M. M.] For every $r \in [2, \infty)$ there is a constant $C_r > 0$ such that, for every Banach space E , for every λ -embedding $I: E \rightarrow \ell_\infty^N$, and for every choice of $x_1, \dots, x_K \in E$, we have

$$\left\| \sum_{i=1}^K \gamma_i x_i \right\|_{L_{\varphi_r}(E)} \leq C_r \|I^{-1}\| (1 + \log N)^{\frac{1}{r}} \sup_{1 \leq j \leq N} \|(I(x_i)(j))_{i=1}^K\|_{S_r^K},$$

Theorem (A. Defant–M. M.) For every $2 \leq r < \infty$, there exists a constant $C_r > 0$ such that, for any choice of polynomials $P_1, \dots, P_K \in \mathcal{T}_m(\mathbb{C}^n)$, we have

$$\left\| \sup_{z \in \mathbb{T}^n} \left| \sum_{i=1}^K \varepsilon_i P_i(z) \right| \right\|_{L_{\varphi_2}} \leq C_2 (n(1 + \log m))^{\frac{1}{2}} \sup_{z \in \mathbb{T}^n} \|(P_i(z))_{i=1}^K\|_{\ell_2},$$

and for $2 < r < \infty$

$$\left\| \sup_{z \in \mathbb{T}^n} \left| \sum_{i=1}^K \varepsilon_i P_i(z) \right| \right\|_{L_{\varphi_r}} \leq C_r (n(1 + \log m))^{\frac{1}{r}} \sup_{z \in \mathbb{T}^n} \|(P_i(z))_{i=1}^K\|_{\ell_{r', \infty}}.$$

Theorem (Defant–M. M.) For every $r \in [2, \infty)$ there is a constant $C_r > 0$ such that for each $m \in \mathbb{N}_0$, $n \in \mathbb{N}$, every complex n -dimensional Banach space X , and every choice of polynomials $P_1, \dots, P_K \in \mathcal{P}_m(X)$, we have

$$\left\| \sup_{z \in B_X} \left\| \sum_{i=1}^K \gamma_i P_i(z) \right\| \right\|_{L_{\varphi_r}} \leq C_r (n(1 + \log m))^{\frac{1}{r}} \sup_{z \in B_X} \|(P_i(z))_{i=1}^K\|_{S_{r'}^K},$$

where $S_{r'}^K := \ell_2^K$ for $r = 2$ and $S_{r'}^K := \ell_{r', \infty}^K$ for $r \in (2, \infty)$.

The proof is based on the following result.

Proposition (A. Defant–M. M.) Let X be an n -dimensional Banach space, and $K \subset B_X$ a convex and compact subset, which satisfies a Markov–Fréchet inequality with exponent ν and constant M . For each $m \in \mathbb{N}$ there exists a subset $F \subset K$ such that

$$\|P\|_K \leq 2 \sup_{z \in F} |P(z)|_F, \quad P \in \mathcal{P}_m(X),$$

with $\text{card } F \leq N$, where $N = (1 + 2Mm^\nu)^n$ if X is real and $N = (1 + 2Mm^\nu)^{2n}$ if X is complex space. In other words the Banach space $\mathcal{P}_m(X)$, 2-embeds into ℓ_∞^N .

Given a Banach space X and a nonempty compact subset $K \subset B_X$.

Definition. We say that K satisfies a **Markov–Fréchet inequality** whenever there is an exponent $\nu \geq 0$, and a constant $M > 0$ such that for all $P \in \mathcal{P}(X)$ one has

$$\sup_{z \in K} \|\nabla P(z)\|_{X^*} \leq M(\deg P)^\nu \sup_{z \in K} |P(z)|,$$

where $\nabla P(z) \in X^*$ denotes the Fréchet derivative of P in $z \in K$. If this inequality only holds for a subclass \mathcal{P} of $\mathcal{P}(X)$, then we say that K satisfies a Markov–Fréchet inequality for \mathcal{P} with exponent ν and constant M .

Theorem (Harris (1997)) Let X be a complex Banach space. Then B_X satisfies a Markov–Fréchet inequality with constant e and exponent $\nu = 1$.

Random Dirichlet polynomials

Combining Bohr's vision of ordinary Dirichlet series and the mentioned results, we provide some new *KSZ*-inequalities for random Dirichlet polynomials.

Given a finite subset $A \subset \mathbb{N}$, we denote by \mathcal{D}_A the Banach space of all Dirichlet polynomials D given by

$$D(s) := \sum_{n \in A} a_n n^{-s}, \quad s \in \mathbb{C},$$

with $\{a_n\}_{n \in A} \subset \mathbb{C}$. Since each such Dirichlet polynomial defines a bounded and holomorphic function on the right half-plane in \mathbb{C} , the space \mathcal{D}_A forms a Banach space equipped with the norm

$$\|D\|_\infty := \sup_{\operatorname{Re} s > 0} \left| \sum_{n=1}^N a_n n^{-s} \right| = \sup_{t \in \mathbb{R}} \left| \sum_{n=1}^N a_n n^{-it} \right|.$$

Remark We note that the particular cases $a_n = 1$ and $a_n = (-1)^n$ play a crucial role within the study of the **Riemann zeta-function** $\zeta: \mathbb{C} \setminus \{1\} \rightarrow \mathbb{C}$. In fact, in recent times, techniques related to random inequalities for Dirichlet polynomials have gained more and more importance. This may be illustrated by a deep classical result of **Turán** (1962), which states that the truth of the famous Lindelöf's conjecture:

$$\zeta(1/2 + it) = \mathcal{O}_\varepsilon(t^\varepsilon), \quad t \in \mathbb{R},$$

with an arbitrarily small $\varepsilon > 0$, is equivalent to the validity of the inequality:

$$\left| \sum_{n=1}^N \frac{(-1)^n}{n^{it}} \right| \leq C N^{\frac{1}{2} + \varepsilon} (2 + |t|)^\varepsilon, \quad t \in \mathbb{R}$$

for an arbitrarily small $\varepsilon > 0$ and with $C = C(\varepsilon)$.

In order to formulate our main result we need two characteristics of the finite set $A \subset \mathbb{N}$ defining \mathcal{D}_A . As usual, for $x \geq 2$, we denote by $\pi(x)$ the number of all primes in the interval $[2, x]$, and by $\Omega(n)$ the number of prime divisors of $n \in \mathbb{N}$ counted accorded to their multiplicities. We define

$$\Pi(A) := \max_{n \in A} \pi(n), \quad \Omega(A) := \max_{n \in A} \Omega(n).$$

Theorem (A. Defant–M. M.) For every $r \in [2, \infty)$ there is a constant $C_r > 0$ such that for any finite set $A \subset \mathbb{N}$ and any choice of Dirichlet polynomials $D_1, \dots, D_K \in \mathcal{D}_A$, we have

$$\left\| \sup_{t \in \mathbb{R}} \left| \sum_{j=1}^K \gamma_j D_j(t) \right| \right\|_{L^{\varphi_r}} \leq C_r \left(1 + \Pi(A)(1 + 20 \log \Omega(A)) \right)^{\frac{1}{r}} \sup_{t \in \mathbb{R}} \left\| (D_j(t))_{j=1}^K \right\|_{S_r}.$$

Corollary. For every $r \in [2, \infty)$ there is a constant $C_r > 0$ such that such, for every Dirichlet random polynomial $\sum_{n \in A} \gamma_n a_n n^{-it}$ in \mathcal{D}_A one has

$$\left\| \sup_{t \in \mathbb{R}} \left| \sum_{n \in A} \gamma_n a_n n^{-it} \right| \right\|_{L_{\varphi_r}} \leq C_r \left(1 + \Pi(A)(1 + 20 \log \Omega(A)) \right)^{\frac{1}{r}} \|(a_n)_{n \in A}\|_{S_r}.$$

Idea of proof:

- We embed \mathcal{D}_A into a certain space of trigonometric polynomials, controlling the degree as well as the number of variables of the polynomials in this space. To achieve this, we use the so-called **Bohr lift**:

$$\mathcal{B}_A: \mathcal{D}_A \rightarrow \mathcal{T}_{\Omega(A)}(\mathbb{T}^{\Pi(A)}), \quad \sum_{n \in A} a_n n^{-s} \mapsto \sum_{\alpha: p^\alpha \in A} a_{p^\alpha} z^\alpha.$$

By Kronecker's theorem on Diophantine approximation we know that the continuous homomorphism

$$\beta: \mathbb{R} \rightarrow \mathbb{T}^{\Pi(A)}, \quad t \rightarrow \left(p_k^{it} \right)_{k=1}^{\Pi(A)}$$

has dense range. This implies that \mathcal{L}_A is an isometry into.

- There is a subset $F \subset \mathbb{T}^{\Pi(A)}$ with $\text{card}(F) \leq N = (1 + 20\Omega(A))^{\Pi(A)}$ such that

$$I: \mathcal{T}_{\Omega(A)}(\mathbb{T}^{\Pi(A)}) \hookrightarrow \ell_{\infty}^N, \quad I(P) := (P(z_i))_{i \in F},$$

is a 2-isomorphic embedding. Combining all this we obtain the following embedding theorem.

In the following example we consider interesting subclass of Dirichlet polynomials of length N , each given by a particular finite subset $A \subset \mathbb{N}$:

Example. For $N \in \mathbb{N}$ and $2 \leq x \leq N$ define

$$A(N, x) := \{1 \leq n \leq N : \pi(n) \leq x\}.$$

Then $\mathcal{D}_{A(N, x)}$ is the space of all Dirichlet polynomials of length N , which only 'depend on $\pi(x)$ -many primes'. Using remarkable estimates for $\pi(x)$ due to Costa Periera (1985):

$$\frac{x \log 2}{\log x} < \pi(x), \quad x \geq 5 \quad \text{and} \quad \pi(x) < \frac{5x}{3 \log x}, \quad x > 1,$$

we get $\Pi(A(N, x)) \leq \pi(x) < \frac{5x}{3 \log x}$. Since for each $1 \leq n = p^\alpha \leq N$ with $\alpha \in \mathbb{N}^{\pi(x)}$ we have $2^{|\alpha|} \leq N$, it follows that

$$\Omega(A(N, x)) \leq \frac{\log N}{\log 2}.$$

With these estimates for $\Pi(A(N, x))$ and $\Omega(A(N, x))$ our *KSZ*-inequalities extend Queffélec's results (1995).

In the special case $x = N$, we denote by \mathcal{D}_N the Banach space of all Dirichlet polynomials of length N , in other words, $\mathcal{D}_N = \mathcal{D}_{A(N)}$ with $A(N) = \{1, \dots, N\}$. Then

$$\Pi(A(N)) < \frac{5N}{3 \log N}, \quad \Omega(A(N)) \leq \frac{\log N}{\log 2}.$$

It is worth noting that in the case $N = p_n$, the n th prime, one has $\Pi(A(N)) = n$.

Random multilinear forms in Banach spaces

Given Banach spaces X_1, \dots, X_m , the Banach space $\mathcal{L}_m(X_1, \dots, X_m)$ of all m -linear bounded forms L on $X_1 \times \dots \times X_m$ is equipped with the norm

$$\|L\| := \sup \{ |L(x_1, \dots, x_m)| : x_j \in B_{X_j}, 1 \leq j \leq m \}.$$

Theorem (A. Defant–M. M.) For every $r \in [2, \infty)$ there is a constant $C_r > 0$ such that, for every choice of finite dimensional Banach spaces X_j with $\dim X_j = n_j$, $1 \leq j \leq m$, and m -linear mappings $L_1, \dots, L_K \in \mathcal{L}_m(X_1, \dots, X_m)$, one has

$$\begin{aligned} & \left\| \sup_{(z_1, \dots, z_m) \in B_{X_1 \times \dots \times X_m}} \left| \sum_{i=1}^K \gamma_i L_i(z_1, \dots, z_m) \right| \right\|_{L_{\varphi_r}} \\ & \leq C_r \left(\sum_{j=1}^m n_j (1 + \log m) \right)^{\frac{1}{r}} \sup_{(z_1, \dots, z_m) \in B_{X_1 \times \dots \times X_m}} \left\| (L_i(z_1, \dots, z_m))_{i=1}^K \right\|_{S_r^K}, \end{aligned}$$

The proof of the above theorem is based on the following result.

Proposition (A. Defant–M. M.) Let X_j with $\dim X_j = n_j, 1 \leq j \leq m$ be finite dimensional (real or complex) Banach spaces. Then there is a subset $F \subset \prod_{j=1}^m B_{X_j}$ of cardinality

$$\text{card}(F) \leq \prod_{j=1}^m (1 + 2n_j)^{2n_j}$$

such that for every $L \in \mathcal{L}_m(X_1, \dots, X_m)$,

$$\|L\|_\infty \leq 2 \sup_{(z_1, \dots, z_m) \in F} |L(z_1, \dots, z_m)|.$$

If all Banach spaces X_j are real, we may replace the exponents $2n_j$ by n_j .