Differentiable extensions with uniformly continuous derivatives

Carlos Mudarra

University of Jyväskylä

JYVÄSKYLÄN YLIOPISTO
UNIVERSITY OF JYVÄSKYLÄ

Based on joint works with Daniel Azagra and Erwan Le Gruyer

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• Let X be a Banach space. $C^{1,\omega}(X)$ is the space of functions $F : X \to \mathbb{R}$ such that F is Fréchet differentiable in X, and $DF : X \to X^*$ is uniformly continuous with

$$M_\omega(DF) := \sup_{x,y\in X;\, x\neq y} \frac{\|DF(x) - DF(y)\|_*}{\omega(\|x-y\|)} < \infty.$$

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• A modulus of continuity $\omega : [0,\infty) \to [0,\infty)$ is a concave increasing function with $\omega(0) = 0$ and $\omega(\infty) = \infty$.

Problem ($C^{1,\omega}$ extension of 1-jets)

Let X be a Hilbert space. Let $E \subset X$ be arbitrary, let $(f, G) : E \to \mathbb{R} \times X$ be a 1-jet, and let ω be a modulus of continuity.

- Find necessary and sufficient conditions on (f, G) for the existence of an $F \in C^{1,\omega}(X)$ such that $(F, \nabla F) = (f, G)$ on E.

- Construct such extension F (if it exists), estimate the seminorm

$$M_\omega(\nabla F) := \sup_{x,y \in X;\, x \neq y} \frac{|\nabla F(x) - \nabla F(y)|}{\omega(|x-y|)},$$

and compare it to the $C^{1,\omega}$-trace seminorm of (f, G) on E :

$$\|(f, G)\|_{E,\omega} := \inf \left\{ M_\omega(\nabla H) \ : \ H \in C^{1,\omega}(X),\ (H, \nabla H) = (f, G) \text{ on } E \right\}.$$

Theorem (Whitney 1934-Glaeser 1958)

Let $E \subset \mathbb{R}^n$, *let* $(f, G) : E \to \mathbb{R} \times \mathbb{R}^n$ *be a* 1*-jet. There exists* $F \in C^{1,\omega}(\mathbb{R}^n)$ *such that* $(F, \nabla F) = (f, G)$ *on* E *if and only if there is some* $M > 0$ *for which*

$$|f(y) - f(z) - \langle G(z), y - z\rangle| \le M|y - z|\omega(|y - z|),$$

$$|G(y) - G(z)| \le M\omega(|y - z|)$$

for every $y, z \in E$. *Moreover,* F *can be taken so that* $M_\omega(\nabla F) \le k(n)M$.

We can assume that E is closed, and then consider a *Whitney decomposition* $\{Q\}_{Q\in\mathcal{W}}$ of $\mathbb{R}^n \setminus E$ and the usual *Whitney partition of unity* $\{\varphi_Q\}_{Q\in\mathcal{W}}$ associated with $\{Q\}_{Q\in\mathcal{W}}$. For a suitable sequence of points $\{p_Q\}_{Q\in\mathcal{W}} \subset E$, the function defined by

$$F(x) = \begin{cases} f(x) & \text{if } x \in E \\ \sum_{Q\in\mathcal{W}} \left(f(p_Q) + \langle G(p_Q), x - p_Q\rangle\right) \varphi_Q(x) & \text{if } x \in \mathbb{R}^n \setminus E \end{cases}$$

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is of class $C^{1,\omega}(\mathbb{R}^n)$ and $(F, \nabla F) = (f, G)$ on E.

Moreover $M_\omega(\nabla F) \leq k(n)\|(f,G)\|_{E,\omega}$, where $\lim_n k(n) = \infty$.

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• As a consequence of our solution to an extension problem for $C^{1,1}$ convex functions, we proved the Wells-Le Gruyer's theorem, via simple and explicit formulas.

$C^{1,1}$ extensions of jets

Theorem (Wells 1973, Le Gruyer 2009, Azagra-Le Gruyer-M. 2017)

Let E be a subset of a Hilbert space X, and let $(f, G) : E \to \mathbb{R} \times X$ be a jet. There exists $F \in C^{1,1}(X)$ with $(F, \nabla F) = (f, G)$ on E if and only if there exists $M > 0$ such that

$$f(z) \leq f(y) + \frac{1}{2}\langle G(y) + G(z), z - y\rangle + \frac{M}{4}|y - z|^2 - \frac{1}{4M}|G(y) - G(z)|^2$$

for all $y, z \in E$. Moreover,

$$F = \mathrm{conv}(g) - \frac{M}{2}|\cdot|^2,$$

$$g(x) = \inf_{y \in E}\left\{f(y) + \langle G(y), x - y\rangle + \frac{M}{2}|x - y|^2\right\} + \frac{M}{2}|x|^2, \quad x \in X,$$

defines a $C^{1,1}(X)$ function with $(F, \nabla F) = (f, G)$ on E, and $\mathrm{Lip}(\nabla F) \leq M$.

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The function F can be taken so as to satisfy

$$\mathrm{Lip}(\nabla F) = \inf\left\{\mathrm{Lip}(\nabla H) \ : \ H \in C^{1,1}(X), \ (H, \nabla H) = (f, G) \text{ on } E\right\}.$$

For $g : X \to \mathbb{R}$, the convex envelope of g is

$$\begin{aligned} \mathrm{conv}(g)(x) &= \sup\{h(x) \ : \ h : X \to \mathbb{R} \ \text{convex, lsc, } \ h \leq g\} \\ &= \sup\{h(x) \ : \ h : X \to \mathbb{R} \ \text{affine, lsc, } \ h \leq g\}. \end{aligned}$$

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Let $M > 0$. For any jet (f, G) defined on E, denote $(\widetilde{f}, \widetilde{G}) = \left(f + \frac{M}{2}|\cdot|^2, G + M\,\mathrm{Id}\right)$. Then

$$(f, G) \text{ has a } C^{1,1} \text{ extension } F \text{ and } \operatorname{Lip}(\nabla F) \leq M \iff$$
$$(\widetilde{f}, \widetilde{G}) \text{ has a } C^{1,1} \text{ convex extension } \widetilde{F} \text{ and } \operatorname{Lip}(\nabla \widetilde{F}) \leq 2M.$$

Corollary (Kirszbraun's theorem via an explicit formula; Azagra-Le Gruyer-M.; 2017)

Let X, Y *two Hilbert spaces,* $E \subset X$ *and* $G : E \to Y$ *a Lipschitz mapping. Define* $\widetilde{G} : X \to Y$ *by:*

$$\widetilde{G}(x) := \nabla_Y(\mathrm{conv}(g))(x, 0) \quad \textit{for every} \quad x \in X; \quad \textit{where}$$

$$g(x, y) = \inf_{z \in E} \left\{ \langle G(z), y \rangle_Y + \frac{\mathrm{Lip}(G)}{2} |x - z|_X^2 \right\} + \frac{\mathrm{Lip}(G)}{2} |x|_X^2 + \mathrm{Lip}(G) |y|_Y^2$$

for every $(x, y) \in X \times Y$*. Then* $\widetilde{G} = G$ *on* E *and* $\mathrm{Lip}(\widetilde{G}, X) = \mathrm{Lip}(G, E)$.

$C^{1,\omega}$ extensions of jets, arbitrary ω

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- Wells' and Le Gruyer's proof cannot be adapted to $C^{1,\omega}$.
- Unlike for $C^{1,1}$, there is no relation between $C^{1,\omega}$ and $C^{1,\omega}_{\mathrm{conv}}$.
- We need a different approach: paraconvex analysis.

Definition (Paraconvexity)

Let X be a Banach space and $\varphi : [0, \infty) \to [0, \infty)$. We say that $F : X \to \mathbb{R}$ is φ-paraconvex if

$$F(\lambda x + (1-\lambda)y) - \lambda F(x) - (1-\lambda)F(y) \leq \lambda(1-\lambda)\varphi(\|x-y\|)$$

for all $x, y \in X$ and all $\lambda \in [0, 1]$.

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- Let X be a Hilbert space. If ω is a modulus of continuity, and $\varphi = \int \omega$, then $F = -(\varphi \circ |\cdot|)$ is 2φ-paraconvex.

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• F is φ-paraconvex $\not\Rightarrow$ $F + C(\varphi \circ \|\cdot\|)$ is convex (unless $\varphi(t) = ct^2$ and X is Hilbert).

• Let X be a Banach space, $F : X \to \mathbb{R}$ locally bounded, and $\varphi = \int \omega$. If F and $(-F)$ are $C\varphi$-paraconvex, then $F \in C^{1,\omega}(X)$ and $M_\omega(DF) \leq MC$, where M is an absolute constant.

Extension results

Let ω be a modulus of continuity, $\varphi = \int \omega$, and $(f, G) : E \to \mathbb{R} \times X^*$ a jet. We define the seminorm:

$$A_\omega(f, G) := \sup_{\substack{x \in X;\, y,z \in E; \\ \|x-y\|+\|x-z\|>0}} \frac{|f(y) + \langle G(y), x - y\rangle - f(z) - \langle G(z), x - z\rangle|}{\varphi(\|x - y\|) + \varphi(\|x - z\|)}.$$

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• $A_\omega(f, G)$ is comparable to the smallest $M > 0$ for which

$$f(z) \leq f(y) + \frac{1}{2}\langle G(y) + G(z), z - y \rangle + M\varphi(\|y - z\|) - 2M\varphi^* \left(\frac{\|G(y) - G(z)\|_*}{2M} \right) \qquad (W^{1,\omega})$$

for all $y, z \in E$. Here $\varphi^*(t) = \int_0^t \omega^{-1}(t)\, dt$.

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• $A_\omega(f,G)$ is comparable to the smallest $M > 0$ for which

$$f(z) \leq f(y) + \frac{1}{2}\langle G(y)+G(z), z-y\rangle + M\varphi(\|y-z\|) - 2M\varphi^*\left(\frac{\|G(y)-G(z)\|_*}{2M}\right) \qquad (W^{1,\omega})$$

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• $A_\omega(f,G)$ is comparable to the smallest $M > 0$ for which (f, G) satisfies the Whitney-Glaeser conditions.

Theorem (Azagra-M.; 2019)

A 1-jet (f,G) defined on a subset E of a Hilbert space X has an extension $(F,\nabla F)$ with $F \in C^{1,\omega}(X)$ if and only if $A_\omega(f,G) < \infty$. Moreover, we can take F such that

$$A_\omega(F,\nabla F) \leq 2\, A_\omega(f,G).$$

In addition, when $\omega(t) = t^\alpha$ with $0 < \alpha \leq 1$, we can arrange

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The extension F can be taken so that

$$M_\omega(\nabla F)\leq (16/\sqrt{15})\|(f,G)\|_{E,\omega} \quad \text{and}$$

$$M_\omega(\nabla F)\leq \tfrac{2^{2-2\alpha}}{\sqrt{1+\alpha}}\left(1+\tfrac{1}{\alpha}\right)^{\alpha/2}\|(f,G)\|_{E,\omega} \quad \text{when} \quad \omega(t)=t^\alpha.$$

Extension formulas

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- The extension is defined as a *$2M\varphi$-paraconvex envelope of g* :

$$F(x) = \sup\{h(x) \,:\, h \leq g,\ h \text{ is } 2M\varphi\text{-paraconvex}\}.$$

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- We have $F \in C^{1,\omega}(X)$, $(F, \nabla F) = (f, G)$ on E, etc

For $\omega(t) = \sqrt{t}$ and $E = \{-1, 0, 1\} \subset \mathbb{R}$, set $f(-1) = 1, f(0) = 0, f(1) = 1$ and $G \equiv 0$ on E. Then $A_\omega(f, G) = 3/\sqrt{2}$.

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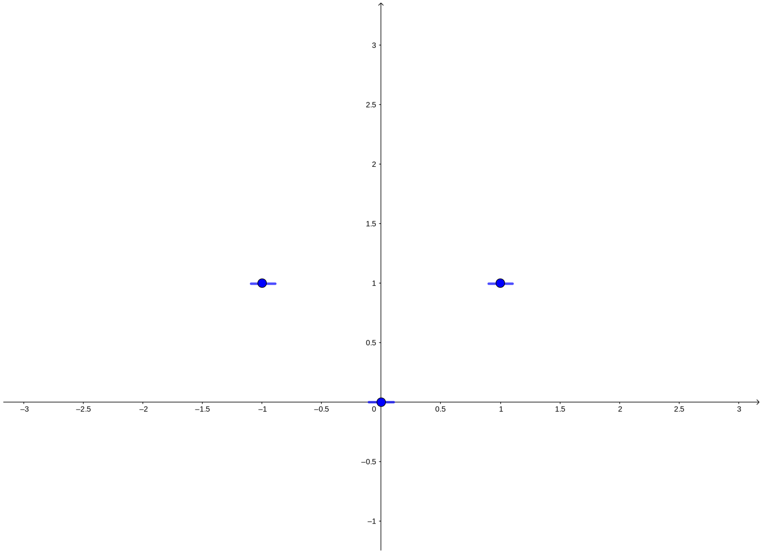
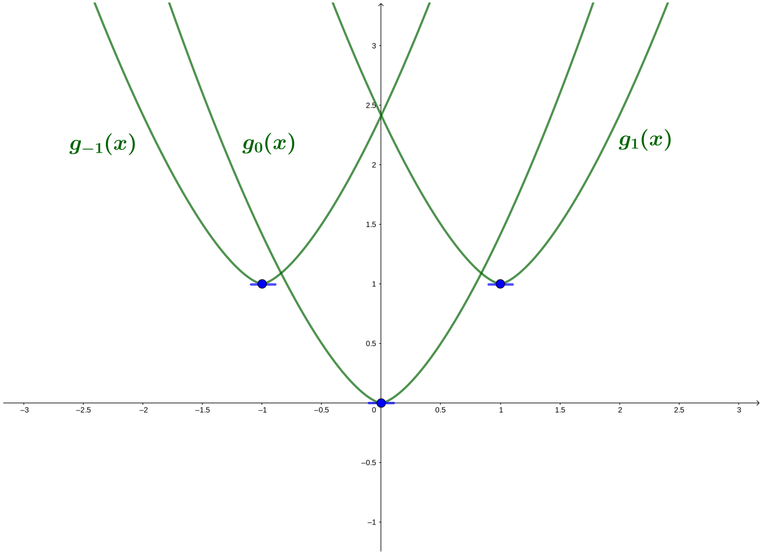
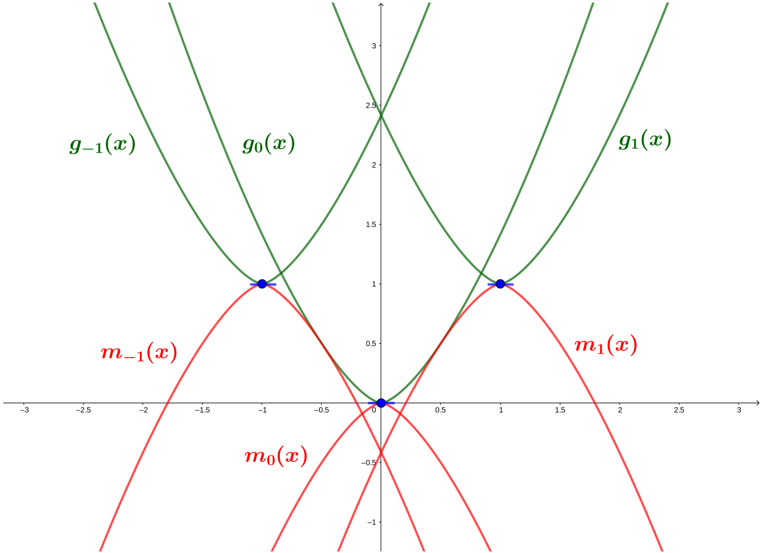


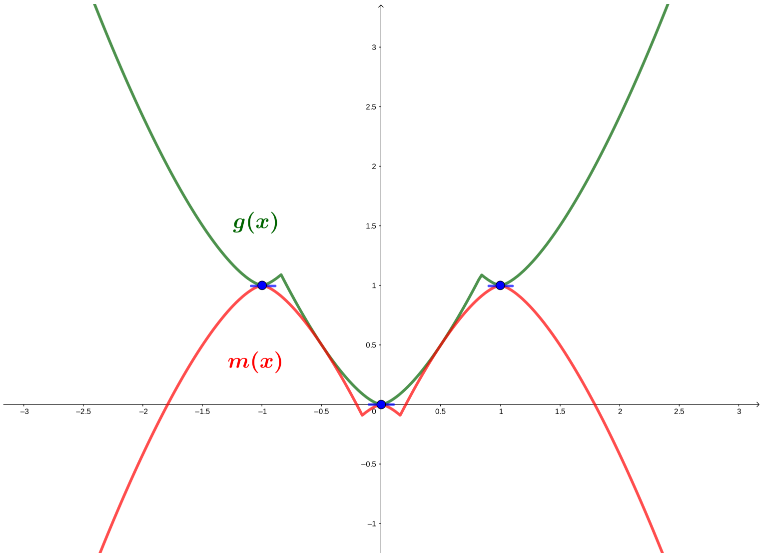
For every $y \in E$, set $g_y(x) := f(y) + G(y)(x-y) + \frac{2A(f,G)}{3}|x-y|^{3/2}$, $x \in \mathbb{R}$.



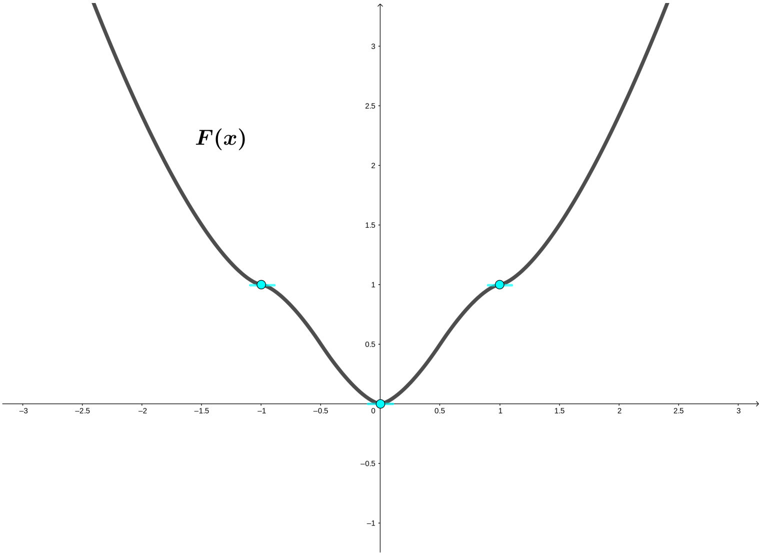
For every $y \in E$, set $m_y(x) := f(y) + G(y)(x-y) - \frac{2A(f,G)}{3}|x-y|^{3/2}$, $x \in \mathbb{R}$.



Define $g = \inf(g_y)_{y \in E}$ and $m = \sup(m_y)_{y \in E}$.



A suitable paraconvex envelope F of g defines a $C^{1,1/2}$ extension of (f, G).



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Fix $M > 0$, and a modulus of continuity ω, $\varphi = \int \omega$. Let $\mathcal{F}(M, \omega)$ be the family of functions of the form

$$X \ni z \longmapsto h(z) = \text{affine function} - \sum_{i=1}^{n} \lambda_i M\varphi(|z - p_i|),$$

where $p_i \in X$, $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$, and $n \in \mathbb{N}$.

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where $p_i \in X$, $\lambda_i \geq 0$, $\sum_{i=1}^{n} \lambda_i = 1$, and $n \in \mathbb{N}$.

Then the following formula does the same job:

$$\widetilde{F}(x) := \sup\{h(x) \,:\, h \leq g,\ h \in \mathcal{F}(M, \omega)\}.$$

$$X \ni z \longmapsto h(z) = \text{affine function} - \sum_{i=1}^{n} \lambda_i M\varphi(|z - p_i|),$$

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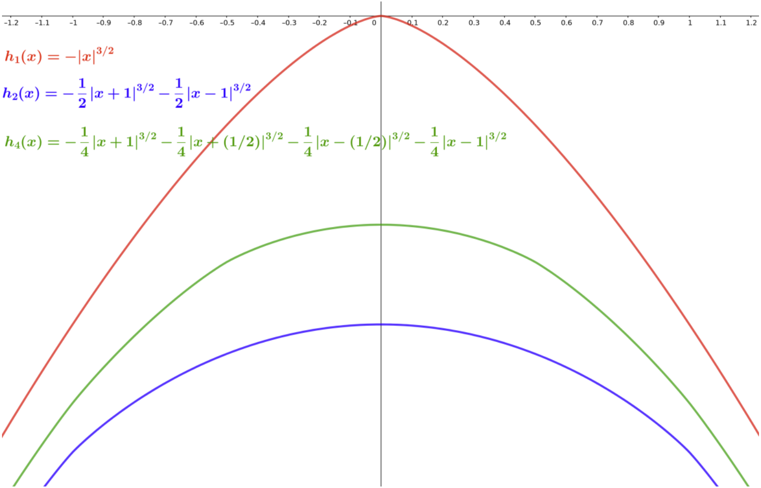

The subclass of bounded and Lipschitz functions

• If (f, G) is also bounded, there are extensions $F \in C^{1,\omega}(X)$ with $(F, \nabla F)$ bounded, and there is $C > 0$ absolute with

$$A_\omega(F, \nabla F) + \|F\|_\infty + \|\nabla F\|_\infty \leq C \left(A_\omega(f, G) + \|f\|_\infty + \|G\|_\infty \right).$$

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• If the sequence of jets $\{(f_n, G_n)\}_n$ is A_ω-uniformly bounded, and (f_n, G_n) converges to (f, G) uniformly on E, then the corresponding sequence of $C^{1,\omega}(X)$ extensions $(F_n, \nabla F_n)$ converges to $(F, \nabla F)$ uniformly on X.

• If (f, G) is also bounded, there are extensions $F \in C^{1,\omega}(X)$ with $(F, \nabla F)$ bounded, and there is $C > 0$ absolute with

$$A_\omega(F, \nabla F) + \|F\|_\infty + \|\nabla F\|_\infty \leq C \left(A_\omega(f, G) + \|f\|_\infty + \|G\|_\infty \right).$$

• If the sequence of jets $\{(f_n, G_n)\}_n$ is A_ω-uniformly bounded, and (f_n, G_n) converges to (f, G) uniformly on E, then the corresponding sequence of $C^{1,\omega}(X)$ extensions $(F_n, \nabla F_n)$ converges to $(F, \nabla F)$ uniformly on X.

• If f is Lipschitz and G bounded, we can find construct extensions $F \in C^{1,\omega}(X)$ with F Lipschitz, and there is $C > 0$ absolute with

$$A_\omega(F, \nabla F) + \mathrm{Lip}(F) \leq C \left(A_\omega(f, G) + \mathrm{Lip}(f) + \|G\|_\infty \right).$$

Results in more general Banach spaces

Let $(X, \|\cdot\|)$ be a Banach space, and let ω be a modulus of continuity. Let $\varphi = \int \omega$. Assume that $\varphi \circ \|\cdot\| \in C^{1,\omega}(X)$. (This implies that X is superreflexive)

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Then the previous results are true for $C^{1,\omega}(X)$, with:

1. Characterizations via the conditions $A_\omega(f, G)$ and $(W^{1,\omega})$;
2. Absolute control on the norm of the extension operators;
3. Explicit extension formulas via paraconvex envelopes;
4. Extension formulas via supremum of convex combinations of simple parabolas;
5. Versions for bounded and/or Lipschitz jets;
6. Continuous dependence on data.

Thank you for your attention!