Ecuación fraccionaria de Black–Scholes y cálculo funcional

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Trabajo conjunto con M. Warma

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Black–Scholes(–Merton) equation

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F. Black and M. Scholes. The pricing of options and corporate liabilities. J. Polit. Econ., 81:637–654, 1973. **Cited by 43863.**

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Black–Scholes equations in interpolation spaces

$$\begin{cases} u \in C^1((0,\infty);E), \quad u(t) \in \mathcal{D}(B), \quad t > 0, \\ u'(t) = Bu(t), \qquad t > 0, \\ \lim_{t\downarrow 0} u(t) = f \in E, \end{cases} \qquad (ACP_0)$$

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A $(L^1 - L^\infty)$-interpolation space E is said to have an **order continuous norm** if $\|f_n\|_E \to 0$ for every sequence of functions $E \supset |f_n| \downarrow 0$ a.e.

Theorem (Arendt, de Pagter 2002)

Let E be an $(L^1 - L^\infty)$ interpolation space. Then ACP_0 is well-posed if and only if E has order continuous norm.

Role of J

Set

$$(Ju)(x) := -xu'(x), \qquad J_E := J|_E.$$

J_E generates the exponentially bounded group $u(t,x) = (S_E(t)f)(x) = f(e^{-t}x)$ on any $(L^1 - L^\infty)$ interpolation space E.

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$$\underline{\eta}_E := -\lim_{t\to\infty} \frac{\log \|S_E(-t)\|}{t}, \quad \overline{\eta}_E := \lim_{t\to\infty} \frac{\log \|S_E(t)\|}{t},$$

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Theorem (Arendt, de Pagter 2002)

1. $\sigma(J_E) = \{\lambda \,|\, \underline{\eta}_E \le \Re\lambda \le \overline{\eta}_E\}$.
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$$B = J^2 \implies B \text{ generates a holomorphic semigroup } T_B.$$

Generalized Cesàro operator

Connection with Cesàro operator $\mathcal{C}_1$ and its adjoint $\mathcal{C}_1^*$

$$J = 1 - C_1^{-1} = (\mathcal{C}_1^*)^{-1}$$

$$(\mathcal{C}_1 f)(x) := \frac{1}{x}\int_0^x f(y)\,dy, \quad (\mathcal{C}_1^* f)(x) := \int_x^\infty \frac{f(y)}{y}\,dy, \quad x > 0.$$

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For $\alpha > 0$, one can define the generalized Cesàro operator $\mathcal{C}_\alpha, \mathcal{C}_\alpha^*$,

$$(\mathcal{C}_\alpha f)(x) := \frac{\alpha}{x^\alpha}\int_0^x (x-y)^{\alpha-1} f(y)\,dy = \frac{\Gamma(\alpha+1)}{x^\alpha}(D^{-\alpha} f)(x),$$

$$(\mathcal{C}_\alpha^* f)(x) := \alpha\int_x^\infty \frac{(x-y)^{\alpha-1}}{y^\alpha} f(y)\,dy = \Gamma(\alpha+1)(W^{-\alpha}(y^{-\alpha} f))(x).$$

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Generalized Black–Scholes equations

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With an arbitrary $\alpha > 0$, one obtains the following PDEs

$$u_t = \frac{1}{\Gamma(\alpha+1)^2} D^\alpha(x^\alpha D^\alpha(x^\alpha u)) - \frac{2}{\Gamma(\alpha+1)} D^\alpha(x^\alpha u) + u =: B_{1,\alpha} u,$$

$$u_t = \frac{1}{\Gamma(\alpha+1)^2} x^\alpha W^\alpha(x^\alpha W^\alpha u) =: B_{2,\alpha} u,$$

$$u_t = -\frac{1}{\Gamma(\alpha+1)^2} D^\alpha(x^{2\alpha} W^\alpha u) + \frac{1}{\Gamma(\alpha+1)} x^\alpha W^\alpha u =: B_{3,\alpha} u,$$

where D^α and W^α stand for the Riemann-Liouville and Weyl fractional derivatives of order α, respectively.

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Objective:

$$B_{i,\alpha} = g_{i,\alpha}(J)?$$

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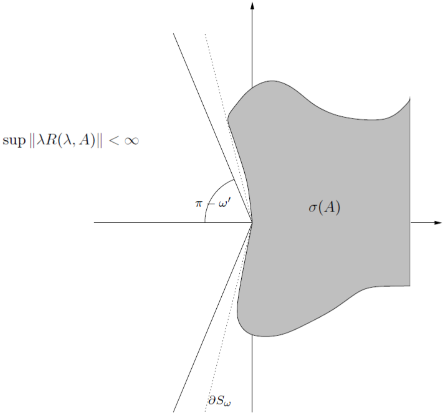
Sectorial operators

$A \in \mathcal{C}(X)$ is said to be a **sectorial operator** (of angle φ) if



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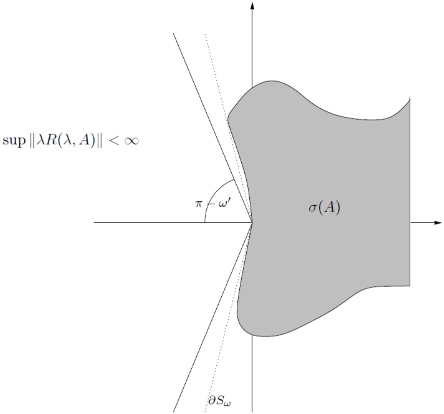

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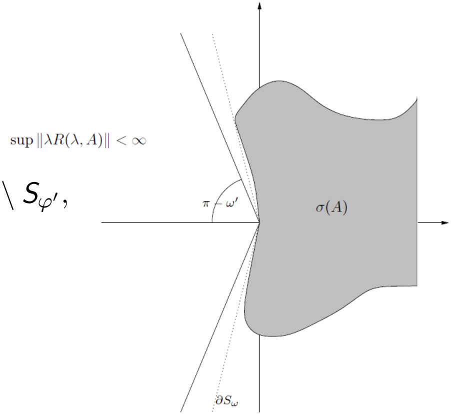
2. Fix $\varphi' \in (\varphi, \pi)$. Then

$$\|\lambda R(\lambda, A)\|_{\mathcal{L}(X)} \leq K_{\varphi'}, \quad \lambda \in \mathbb{C} \setminus S_{\varphi'},$$

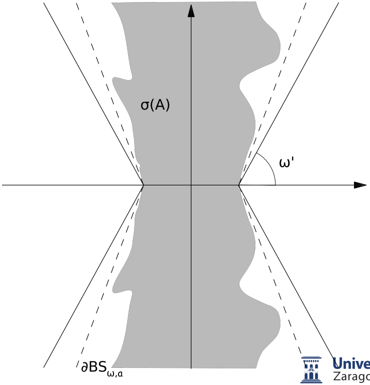
for some $K_{\varphi'} > 0$.



Bisectorial operators

$BS_{\omega,a} := (-a + S_{\pi-\omega}) \cap (a - S_{\pi-\omega})$
for $\omega \in (0, \pi/2]$, $a \geq 0$.

We say that
A is bisectorial, $A \in \mathrm{BSect}(\omega, a)$,
if both $a + A$, $a - A$ are
sectorial operators of angle $\pi - \omega$.



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Idea (Bade, 1953, McIntosh, 1986, Haase 2005): For $A \in \mathrm{BSect}(\omega, a)$ set

$$\mathcal{E}(A) := \left\{ f \in H^\infty(BS_{\varphi,a}) \; : \; \int_\Gamma \left| \frac{f(z)}{\min\{|\lambda - a|, |\lambda + a|\}} \right| |dz| < \infty, \varphi < \omega \right.$$

$$\left. \lim_{z \to a} f(z) = \lim_{z \to -a} f(z) = \lim_{z \to \infty} f(z) = 0. \right\},$$

Primary functional calculus of unbounded operators

Idea (Bade, 1953, McIntosh, 1986, Haase 2005): For $A \in \mathrm{BSect}(\omega, a)$ set

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and set

$$f(A) := \frac{1}{2\pi i} \int_\Gamma f(z) R(z, A)\, dz \in \mathcal{L}(X), \qquad f \in \mathcal{E}(A).$$

Definition (Haase, 2005)

Let $A \in \mathrm{BSect}(\omega, a)$ and $\varphi < \omega$. Let $f \in \mathcal{M}(BS_{\varphi,a})$ be such that there exists $e \in \mathcal{E}(A)$ for which

1. $ef \in \mathcal{E}(A)$.
2. $e(A)$ is injective.

Then we say that f is regularizable, $f \in \mathcal{M}(A)$ and set

$$f(A) := e(A)^{-1}(ef)(A) \in \mathcal{C}(X).$$

Generalized Black–Scholes through functional calculus

The generalized Black–Scholes equations

$$u_t = B_{1,\alpha} u = \frac{1}{\Gamma(\alpha+1)^2} D^\alpha(x^\alpha D^\alpha(x^\alpha u)) - \frac{2}{\Gamma(\alpha+1)} D^\alpha(x^\alpha u) + u,$$

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\end{aligned}
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can be respectively written as

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\begin{aligned}
B_{1,\alpha} &= (1-(\alpha\mathbb{B}(I-J,\alpha))^{-1})^2 =: g_{1,\alpha}(J),\\
B_{2,\alpha} &= (\alpha\mathbb{B}(J,\alpha))^{-2} =: g_{2,\alpha}(J),\\
B_{3,\alpha} &= (\alpha\mathbb{B}(J,\alpha))^{-1}(1-(\alpha\mathbb{B}(I-J,\alpha))^{-1}) =: g_{3,\alpha}(J),
\end{aligned}
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where $\mathbb{B}$ denotes the Beta-Euler function.

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Transfer results

The functions of our generalized BS satisfy that

$$g_{i,\alpha}(z) \sim \begin{cases} (-z)^{2\alpha} & \text{if } i = 1, \\ z^{2\alpha} & \text{if } i = 2, \\ z^{\alpha}(-z)^{\alpha} & \text{if } i = 3, \end{cases} \qquad \text{as } z \to \infty.$$

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3. (Arendt, Zamboni, 2010), (Gomilko, Tomilov, 2015),...

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Main result

Theorem (O.-M., M. Warma)

Let $\omega \in (0, \pi/2]$, $a \geq 0$ and $\beta \in [0, \pi)$. Let $A \in \mathrm{BSect}(\omega, a)$, $g \in \mathcal{M}(A)$. If

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Then, $g(A)$ *is a sectorial operator of angle* β.

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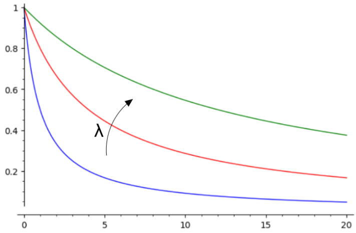
(Auscher, McIntosh, Nahmod, 1997)

Sketch of the problem: Let $\lambda R(\lambda, A^\alpha) = \left(\frac{\lambda}{\lambda - z^\alpha}\right)(A)$, with $\frac{\lambda}{\lambda - z^\alpha} \in \mathcal{E}$.



Extension of the scaling property

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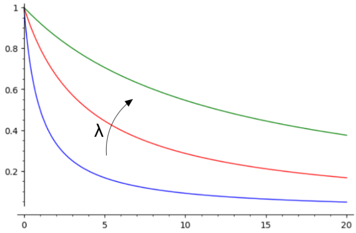

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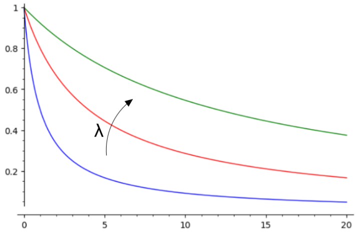

Idea:

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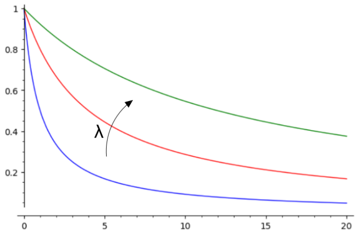

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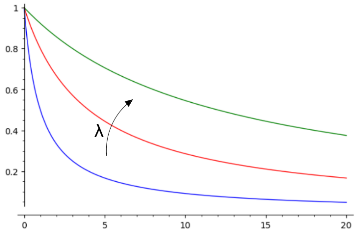
Extension of the scaling property

(Auscher, McIntosh, Nahmod, 1997)

Sketch
of the problem: Let $\lambda R(\lambda, g(A)) = g_\lambda(A)$
with $g_\lambda(z) = \frac{\lambda}{\lambda - g(z)} \in \mathcal{E}$.

$\|g_\lambda(A)\| \leq \int_\Gamma |g_\lambda(z)| \|R(z,A)\| \, |dz|$
is not uniformly bounded on λ.



Idea:

$$\|g_\lambda(A)\| \leq \|h_\lambda(A)\| + \int_\Gamma |g_\lambda(z) - h_\lambda(z)| \, \|R(z,A)\| \, |dz|.$$

Consequences

Corollary (O.-M., M. Warma)

Let A, g as before. In addition, assume that $\beta < \frac{\pi}{2}$. Then, $g(A)$ generates a bounded holomorphic semigroup of angle $\frac{\pi}{2} - \beta$.

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Corollary (O.-M., M. Warma)

Let $a \geq 0$, $A \in \mathrm{BSect}(\pi/2, a)$, and let $\alpha \in \mathbb{R}^+ \setminus \{1, 3, 5, ...\}$. Then $(-1)^n(A + aI)^\alpha$ is quasi-sectorial operator of angle $\pi \left|\frac{\alpha}{2} - n\right|$ for $\alpha \in (2n-1, 2n+1)$.

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Domain properties

Proposition (O.-M., M. Warma)

Let A, g be as above. If $g^{-1}(\infty) \cap M_A = \emptyset$, then $\overline{\mathcal{D}(g(A))} = X$. Otherwise,

$$\overline{\mathcal{D}(g(A))} = \bigcap_{d \in g^{-1}(\infty) \cap M_A} \overline{\mathcal{R}(dI - A)},$$

where $\mathcal{R}(\infty - A) := \mathcal{D}(A)$.

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Theorem (O.-M., M. Warma)

Let β, A, g be as above with $\beta \in \left(0, \frac{\pi}{2}\right)$. Then $\exp_{-w} \circ g \in \mathcal{M}_A$ and

$$T_g(w) = (\exp_{-w} \circ g)(A), \quad w \in S_{\pi/2-\beta}.$$

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Generalized Black–Scholes equation on order continuous norm interpolation spaces

Theorem (O.-M., M. Warma)

Let E be a $(L^1 - L^\infty)$-interpolation space with order continuous norm and let $n \in \mathbb{N}$ and $\alpha > 0$. Then the following assertions hold.

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1. *If $\overline{\eta}_E < 1$ and $\alpha \in \left(n - \frac{1}{2}, n + \frac{1}{2}\right)$, then (ACP_0) is well-posed with the operator $(-1)^{n+1} B_{1,\alpha}$.*

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In any case, identifying $u(t,x) = u(t)(x)$, we obtain that $u \in C^\infty((0,\infty) \times (0,\infty))$.

[ADP02, BHK09, Haa06, OMW22]

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The End

Explicit solutions

$$\left(T_{(-1)^{n+1}g_1^\alpha(A_E)}(w)f\right)(x) = \frac{1}{2\pi}\int_0^\infty \frac{f(s)}{s}\int_{-\infty}^\infty \left(\frac{s}{x}\right)^{iu} \exp\left(\left(-1\right)^{n+1} w\left(1-\frac{1}{\alpha\mathbb{B}(1-iu,\alpha)}\right)^2\right) duds,$$

$$\left(T_{(-1)^{n+1}g_2^\alpha(A_E)}(w)f\right)(x) = \frac{1}{2\pi}\int_0^\infty \frac{f(s)}{s}\int_{-\infty}^\infty \left(\frac{s}{x}\right)^{iu+\delta} \exp\left(\left(-1\right)^{n+1} w\left(\alpha\mathbb{B}(iu+\delta,\alpha)\right)^{-2}\right) duds,$$

$$\left(T_{g_3^\alpha(A_E)}(w)f\right)(x) = \frac{1}{2\pi}\int_0^\infty \frac{f(s)}{s}\int_{-\infty}^\infty \left(\frac{s}{x}\right)^{iu+\delta} \exp\left(\frac{w}{\alpha\mathbb{B}(\delta+iu,\alpha)}\left(1-\frac{1}{\alpha\mathbb{B}(1-\delta-iu,\alpha)}\right)\right) du$$

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Classical Black–Scholes

$$
\begin{aligned}
\left(T_{B_E}(w)f\right)(x) &= \frac{1}{2\pi}\int_0^{\infty}\frac{f(s)}{s}\int_{-\infty}^{\infty}\left(\frac{s}{x}\right)^{iu}\exp\left(-wu^2\right)\,duds \\
&= \frac{1}{\sqrt{4\pi w}}\int_0^{\infty}\exp\left(-\frac{(\log x-\log s)^2}{4w}\right)\frac{f(s)}{s}\,ds, \qquad x>0,
\end{aligned}
$$

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Regularity conditions and spectral inclusion

1. We say that a function f is regular at $d \in \{-a, a\}$ if $\lim_{z \to d} f(z) =: c_d \in \mathbb{C}$ exists and, for some $\varepsilon > 0$, $\varphi < \omega$,

$$\int_{\partial(BS_{\omega',a} \cap \{|z-d|<\varepsilon\}} \left| \frac{f(z) - c_d}{z - d} \right| |dz| < \infty, \quad \text{for all } \omega' \in \left(\varphi, \frac{\pi}{2}\right).$$

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Proposition

Let $A \in \mathrm{BSect}(\omega, a)$, and take $f \in \mathcal{M}_A$ to be quasi-regular at M_A. Then
$$\widetilde{\sigma}(f(A)) \subset f(\widetilde{\sigma}(A)).$$

Köthe dual

Theorem above does not hold for a $(L^1 - L^\infty)$-interpolation space E which has no order continuous norm. Consider the Köthe dual $E^\star$ of E, given by

$$E^\star := \Big\{ g : (0,\infty) \to \mathbb{C} \text{ measurable and } \int_0^\infty |f(x)g(x)|\, dx < \infty \quad \text{for all } f \in E \Big\}.$$

Every $g \in E^\star$ defines a bounded (order continuous) linear functional φ_g on E, given by

$$\langle f, \varphi_g \rangle_{E,E^\star} := \int_0^\infty f(x)g(x)\, dx \ \text{ for all } \ f \in E.$$