Universidad de Murcia
Departamento Matemáticas

Compactness and Variational Analysis.

José Orihuela

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Coauthors and main publications

- J. Orihuela and M. Ruiz Galán *A coercive and nonlinear James's weak compactness theorem* Nonlinear Analysis 75 (2012) 598-611.
- B. Cascales, J. Orihuela and A. Pérez: *One-sided James Compactness Theorem*, J. Math. Anal. Appli. 445, Issue 2, 1267-1283 (2017).
- J. Orihuela: *Conic James' Compactness Theorem*, Journal of Convex Analysis (2018)(3), 1335–1344.
- F. Delbaen and J. Orihuela *Mackey's constraints for James's Compactness Theorem and Risk Measures*, Journal Math. Anal. Appli, 485-1 (2020), https://doi.org/10.1016/j.jmaa.2019.123764
- F. Delbaen and J. Orihuela *On the range of the subdifferential in non reflexive Banach spaces*, Journal Functional Anal., 281-2, (2021) https://doi.org/10.1016/j.jfa.2020.108915
- F. Delbaen and J. Orihuela *A multiset version of James's Theorem*, Journal Functional Analysis, to appear (2022).
- F. Delbaen and J. Orihuela *A nonlinear James's* w^**-compactness theorem.* Preprint 2022.

Delbaen and Schachermayer questions

Question (W. Schachermayer)

Let us fix a proper function

$$\alpha : \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P}) \to (-\infty, +\infty]$$

When the minimization problem

$$\min\{\alpha(X) + \mathbb{E}[Y \cdot X] : X \in \mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})\}$$

has solution for all $Y \in \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$?

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Question (F. Delbaen)

Let C be a convex, bounded and closed, but not weakly compact subset of the Banach space $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ with $0 \notin C$.
Is it possible to find a linear functional $Y \in \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ not attaining its minimum on C but that stays strictly positive on C?

Theorem (F. Delbaen and J. Orihuela)

Let A be a convex, closed, bounded but non weakly compact subset of a Banach space E such that $0 \notin A$. Let us fix a non-void open set Ω in the Makey dual $(E^*, \tau(E^*, E))$.
Then there is a continuous linear form $x_0^* \in \Omega$ which doest not attains supremum on A and such that

$$\sup x_0^*(A) < 0$$

Maximizing $\{x^*(x) - \alpha(x) : x \in E\}$

Theorem (F. Delbaen and J. Orihuela)

Let E be a Banach space,

$$\alpha : E \to (-\infty, +\infty]$$

be a proper and bounded below function such that $\partial\alpha(E)$ has non empty interior in E^* for the Mackey topology $\tau(E^*, E)$, then the level sets

$$\{\alpha \leq c\}$$

are relatively weakly compact for all $c \in \mathbb{R}$. If in addition the function α has a domain with non-empty norm interior, the Banach space must be reflexive.

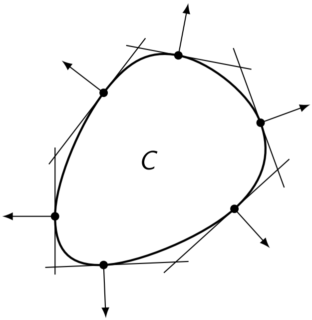
Let X be a (real) Banach space.

Theorem (James, 1964)

Let $C \subset X$ be a bounded, convex and closed set such that

$$\forall x^* \in X^* \quad \exists c \in C \text{ with } \langle x^*, c \rangle = \sup(x^*, C)$$

Then, C is w-compact.



A new measure of non-weak compactness

Theorem

Let A be a bounded subset of a Banach space E. Then A is weakly relatively compact if, and only if, for every bounded sequence $\{x_n^*\}_{n\geq 1}$ in E^* we have

$$\mathrm{dist}_{\|\cdot\|_A}(L\{x_n^*\}, \mathrm{co}\{x_n^* : n \geq 1\}) = 0.$$

where we are denoting with $L\{x_n^*\}$ the set of all w^*-cluster points of the bounded sequence $\{x_n^*\}$ in E^*,
and

$$\|x^*\|_A := \sup\{|x^*(a)| : a \in A\}$$

for every $x^* \in E^*$.

Connection with Pryce's arguments for the general case

Theorem

Let E be a Banach space, A a bounded subset of E with $A = -A$, $\{x_n^*\}_{n\geq 1}$ a bounded sequence in the dual space E^*, and D its norm-closed linear span in E^*. Then there exists a subsequence $\{x_{n_k}^*\}_{k\geq 1}$ of $\{x_n^*\}_{n\geq 1}$ such that

$$S_A\left(x^* - \liminf_k x_{n_k}^*\right) = S_A\left(x^* - \limsup_k x_{n_k}^*\right) =$$

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$$= \mathrm{dist}_{\|\cdot\|_A}(x^*, L\{x_{n_k}^*\})$$

for all $x^* \in D$.

Theorem (James-Pryce undetermined function technique)

Let X be a nonempty set, $\{h_j\}_{j\geq 1}$ a bounded sequence in $\ell^\infty(X)$, and $\delta > 0$ such that

$$S_X\left(h - \limsup_j h_j\right) = S_X\left(h - \liminf_j h_j\right) \geq \delta,$$

whenever $h \in \mathrm{co}_\sigma\{h_j : \ j \geq 1\}$. Then there exists a sequence $\{g_i\}_{i\geq 1}$ in $\ell^\infty(X)$ with

$$g_i \in co_\sigma\{h_j : \ j \geq i\}, \quad \textit{for all } i \geq 1,$$

and there exists $g_0 \in \mathrm{co}_\sigma\{g_i : \ i \geq 1\}$ such that for all $g \in \ell^\infty(X)$ with

$$\liminf_i g_i \leq g \leq \limsup_i g_i \quad \textit{on } X,$$

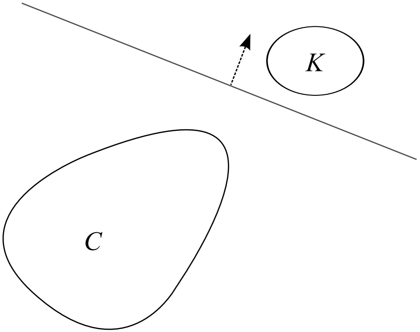
the function $g_0 - g$ doest not attain its supremum on X .

One-side James' Theorem

$C \subset E$ convex closed bounded
$K \subset E$ convex weakly compact
$C \cap K = \emptyset$

$x^* \in E^*$ with

$$\sup(x^*, C) < \inf(x^*, K)$$



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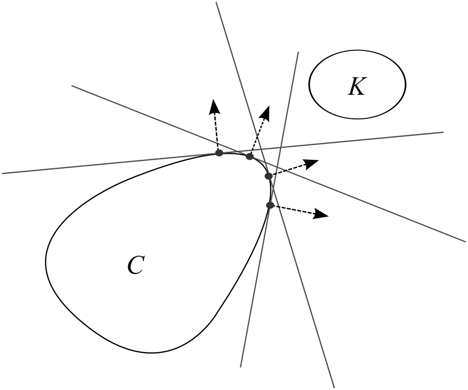

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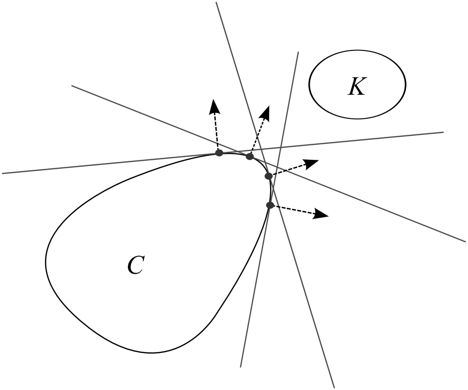

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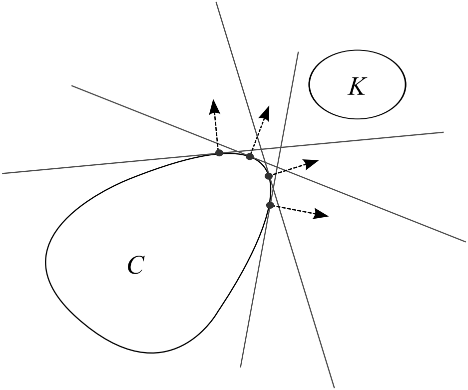
Thesis: C is weakly compact.



One-sided plus Mackey's constraints Case

Theorem (Joint work with Freddy Delbaen)

Let A be a convex, closed and bounded subset of a Banach space E which is assumed non to be weakly compact with $0 \notin A$. Let us fix a relatively weakly compact subset D in $(E, \sigma(E, E^*))$ together with an absolutely convex and weakly compact subset W in $(E, \sigma(E, E^*))$ and a functional $z_0^* \in E^*$ with

$$\inf z_0^*(A) > 0, \inf z_0^*(D) > 0 \text{ and } \epsilon > 0.$$

Then there is a linear form $z^* \in E^*$ such that

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$$\inf z_0^*(A) > 0, \inf z_0^*(D) > 0 \text{ and } \epsilon > 0.$$

Then there is a linear form $z^* \in E^*$ such that

1. $a \to \langle z_0^* + z^*, a \rangle$ *does not attain its infimum on A,*
2. $\sup_{w \in W} |z^*(w)| < \epsilon$, *and*
3. $\inf(z_0^* + z^*)(A) > 0, \inf(z_0^* + z^*)(D) > 0$

Theorem (Delbaen - Orihuela)

Let $u_1 : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a Fatou coherent monetary utility function. Suppose that u_1 is not the essential infimum function. The following are equivalent:

1. *u_1 is a Lebesgue monetary utility function*
2. *$u_1 \Box u_2$ is Fatou for all Fatou coherent utility functions u_2*
3. *$u_1 \Box u_2$ is Lebesgue for all Fatou coherent utility functionu_2*

The Unbounded Case

Theorem (Delbaen - Orihuela)

Let C be a closed, convex unbounded subset in the Banach space E and D be weakly compact subset of E such that every bounded set $Z \in E^*$ satisfies that

$$\sup\{z^*(c) : c \in C, z^* \in Z\} < +\infty$$

whenever $\sup\{z^*(d) : d \in D, z^* \in D\} < 0$.

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1. $\sup_{w \in W} |z^*(w)| < \epsilon$, *and*
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1. $\sup_{w \in W} |z^*(w)| < \epsilon$, *and*
2. $\sup\{(z_0^* + z^*)(d) : d \in D\} < 0$, *and so*
3. $\sup\{(z_0^* + z^*)(c) : c \in C\} < +\infty$ *but this supremum is not attained.*

Maximizing $\{x^*(x) - \alpha(x) : x \in E\}$

Theorem (Delbaen - Orihuela)

Let E be a Banach space, $\alpha : E \to (-\infty, +\infty]$ be a proper and bounded below function.

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Let E be a Banach space, $\alpha : E \to (-\infty, +\infty]$ be a proper and bounded below function.
If $\partial\alpha(E)$ has non empty interior in E^* for the Mackey topology $\tau(E^*, E)$, then the level sets $\{\alpha \le c\}$ are relatively weakly compact for all $c \in \mathbb{R}$.

Corollary

Let E be a real Banach space and let $\alpha : E \longrightarrow \mathbb{R} \cup \{+\infty\}$ be a bounded below function such that $\mathrm{dom}(\alpha)$ has nonempty norm–interior and for all $x^* \in U$ there exists $x_0 \in E$ with

$$\alpha(x_0) - x^*(x_0) = \inf_{x \in E} \left(\alpha(x) - x^*(x)\right), \tag{1}$$

where U is a non void $\tau(E^*, E)$-open set,

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where U is a non void $\tau(E^*, E)$-open set, then E is a reflexive Banach space. Moreover, the minimization problem (1) has a solution for all $x^* \in E^*$.

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where U is a non void $\tau(E^*, E)$-open set, then E is a reflexive Banach space. Moreover, the minimization problem (1) has a solution for all $x^* \in E^*$.
In particular, if we have a monotone and symmetric map $\Phi : E \longrightarrow E^*$ such that $\Phi(E)$ has non empty interior for the Mackey topology $\tau(E^*, E)$, the Banach space E must be reflexive and $\Phi(E) = E^*$

Last news on James' theorem

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Theorem (A Dichotomous James' Theorem, Delbaen - Orihuela)

Let $A \subset E$ be a bounded subset of a weakly sequentially complete Banach space E. If every $x^* \in E^*$ either attains its supremum or infimum on A, then A is weakly relatively compact.

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The hypothesis of weakly sequentially completenes can not be removed

Theorem (James - Pryce - Delbaen - Orihuela)

Let $\{h_j\}_{j\geq 1}$ *be a uniformly bounded sequence in* $\mathbb{R}^{X\cup Y}$. *Let*

$$0 < A < 1 \leq K$$

be positive real numbers such that for all $h_0 \in co_\sigma\{h_j : j \geq 1\}$*:*

$$0 < A \leq S_X(h_0 - \limsup_j h_j) = S_X(h_0 - \liminf_j h_j) \leq K < \infty$$

and

$$0 < A \leq S_Y(h_0 - \limsup_j h_j) = S_Y(h_0 - \liminf_j h_j) \leq K < \infty.$$

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Then there is a pseudo-subsequence $\{g_i\}_{i\geq 1}$ *of* $\{h_j\}_{j\geq 1}$*, and* $g_0 \in co_\sigma\{g_i : i \geq 1\}$*, such that for every* $\hat{g}$ *satisfying for every* $x \in X$

$$\liminf g_i(x) \leq \hat{g}(x) \leq \limsup g_i(x),$$

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$$\liminf g_i(x) \leq \hat{g}(x) \leq \limsup g_i(x),$$

we have that $g_0 - \hat{g}$ *does not attain its supremum neither on* X *nor on* Y.

Multivalued James's Theorem

Corollary (Delbaen - Orihuela)

Let A and B be closed, bounded and convex subsets of Banach space E. If there are vectors $x_0^{**} \in \overline{A}^{\sigma(E^{**},E^*)} \setminus E$, $y_0^{**} \in \overline{B}^{\sigma(E^{**},E^*)} E^{**} \setminus E$ with

$$[x_0^{**}, y_0^{**}] \cap E = \emptyset,$$

then there exist $x^* \in E^*$ such that x^* does not attain its supremum neither on A or on B.

Multivalued James's Theorem

Theorem (Delbaen - Orihuela)

Let $A_1, A_2, \cdots A_p$ be closed, bounded and convex subsets of Banach space E. If there are vectors $x_1^{**}, x_2^{**}, \cdots x_p^{**} \in E^{**} \setminus E$ with

$$x_i^{**} \in \overline{A_i}^{\sigma(E^{**},E^*)} \setminus A_i : i = 1, 2, \cdots, p$$

and

$$\mathrm{co}(\{x_i^{**} : i = 1, 2, \cdots, p\}) \cap E = \emptyset,$$

then there exists $x^* \in E^*$ such that x^* does not attain its supremum on any A_i for $i = 1, 2, \cdots, p$.

Multivalued James's Theorem

Theorem (Delbaen - Orihuela)

Let $\{A_1, A_2, \cdots A_p\}$ be a finite family of closed, bounded, convex but not weakly compact subsets of a weakly sequentially complete Banach space E. Then there exist $x^* \in E^*$ such that x^* does not attain its supremum on any A_i for $i = 1, 2, \cdots, p$.

Nonlinear w*-James's compactness Theorem

Theorem (Delbaen - Orihuela)

Let E be a Banach space without copies of ℓ^1 together with a w-K analytic subset $A \subset E^*$. Let $B \subset E^*$ be such that $A \subset B \subset \overline{A}^{\|\cdot\|}$ and $D \subset E^*$ convex and weakly compact set with*

$$(-D) \cap \overline{\mathrm{co}(B \cup \{0\})}^{\|\cdot\|} = \emptyset. \tag{2}$$

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Let us assume that for $x \in E$ with $x(d^*) < 0$ for every $d^* \in D$, we have

$$\sup\{x(c^*) : c^* \in B\} = x(b^*) \tag{3}$$

for some $b^* \in B$.

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for some $b^* \in B$. Then we have that

$$\overline{\mathrm{co}(B)}^{\omega^*} \subset \overline{\mathrm{co}(B)}^{\|\cdot\|} + \Lambda_D.$$

Nonlinear w*-James's compactness Theorem

Corollary

Let E be a Banach space without copies of ℓ^1 and $B \subset E^*$ be a norm closed convex and w*-K *analytic set, with $0 \in B$ and such that $B + B \subset B$. Let us assume there is a weakly compact convex set $D \subset B$ with $(-D) \cap B = \emptyset$ such that for $x \in E$ with $x(d^*) < 0$ for every $d^* \in D$, we have*

$$\sup\{x(c^*) : c^* \in B\} = 0 \tag{4}$$

Then B is going to be w*-*closed, i.e.:*

$$\overline{B}^{\omega^*} \subset B.$$

Nonlinear w*-James's compactness Theorem

Theorem (Delbaen - Orihuela)

Let E be a Banach space without copies of $\ell^1(\mathbb{N})$ and let

$$\alpha : E^* \longrightarrow \mathbb{R} \cup \{+\infty\}$$

be a convex proper and norm lower semicontinuous map with a w*-K-analytic subset $A \subset$ dom (α) with dom $(\alpha) \subset \overline{A}^{\|\cdot\|}$, and such that

$$\text{for all } x \in E, \quad x - \alpha \text{ attains its supremum on } E^*.$$

Then α is w*-lower semicontinuous and for every $\mu \in \mathbb{R}$, the sublevel set $\alpha^{-1}((-\infty, \mu])$ is w*-compact.

Nonlinear w*-James's compactness Theorem

Theorem (Delbaen - Orihuela)

Let E be a Banach space and $B \subset E^*$, let A and D be weakly countably determined subsets of E^* with $B \subset \overline{A}^{\|\cdot\|}$, and D bounded, w*-closed and convex with $0 \notin D$. If for every $x \in E$, with $x(d^*) < 0$ for every $d^* \in D$, we have that

$$\sup\{x(c^*) : c^* \in B\} = x(b^*) \tag{5}$$

for some $b^* \in B$, then

$$\overline{\mathrm{co}(B)}^{\mathrm{w}^*} \subset \overline{\mathrm{co}(B) + \Lambda_D}^{\|\cdot\|}$$

Corollary

Let E be a Banach space and $B \subset E^*$ be a norm closed convex and weakly countably determined subset, with $0 \in B$ and such that $B + B \subset B$. Let us assume there is a bounded and w^*-closed set $D \subset B$ with $0 \notin D$ and such that for $x \in E$ with $x(d^*) < 0$ for every $d^* \in D$, we have

$$\sup\{x(c^*) : c^* \in B\} = 0 \tag{6}$$

Then we have that B is going to be w^*-closed, i.e.:

$$\overline{B}^{\omega^*} \subset B.$$

Nonlinear w*-James's compactness Theorem

Theorem (Delbaen - Orihuela)

Let E be a Banach space and let

$$\alpha : E^* \longrightarrow \mathbb{R} \cup \{+\infty\}$$

be a convex, proper and norm lower semicontinuous map with a weakly countably determined subset $A \subset \operatorname{dom}(\alpha)$ such that $\operatorname{dom}(\alpha) \subset \overline{A}^{\|\cdot\|}$. Let us assume that

$$\text{for all } x \in E, \quad x - \alpha \text{ attains its supremum on } E^*,$$

then α is w*-*lower semicontinuous and for every $\mu \in \mathbb{R}$, the sublevel set $\alpha^{-1}((-\infty, \mu])$ is* w*-*compact.*

THANKS A LOT FOR YOUR ATTENTION ... !!

Theorem (F. Delbaen and J. Orihuela)

Let $u_1 : \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{R}$ be a Fatou coherent monetary utility function. Suppose that u_1 is not the essential infimum function. The following are equivalent:

1. *u_1 is a Lebesgue monetary utility function*
2. *$u_1 \Box u_2$ is Fatou for all Fatou coherent utility functions u_2*
3. *$u_1 \Box u_2$ is Lebesgue for all Fatou coherent utility functionu_2*

Nonlinear w*-James's compactness Theorem

Theorem (γ-Conic Godefroy's Theorem)

Let E be a Banach space without copies of l^1. Let D be a convex and w^*-closed subset with $0 \notin D$ and $B \subset E^*$ a nonempty set satisfying that for each $x \in E$ such that $x(D) < 0$ there is $b^* \in B$ with $\langle x, b^* \rangle = \sup(x, B)$. Then, we have that

$$\overline{\mathrm{co}(B)}^{\omega^*} \subset \overline{\mathrm{co}(B) + \Lambda_D}^{\gamma(E^*,E)}.$$

Nonlinear w*-James's compactness Theorem

Theorem

Let E be a Banach space without copies of $\ell^1(\mathbb{N})$. If $\alpha : E^* \longrightarrow \mathbb{R} \cup \{+\infty\}$ is a convex, proper and $\gamma(E^*, E)$-lower semicontinuous map such that

$$\text{for all } x \in E, \quad x - \alpha \text{ attains its supremum on } E^*, \tag{7}$$

then α is w*-lower semicontinuous and for every $\mu \in \mathbb{R}$, the sublevel set $\alpha^{-1}((-\infty, \mu])$ is w*-compact.
If E is a Banach lattice without copies of $\ell^1(\mathbb{N})$and we assume that $\alpha(x^*) \leq 0$ for $x^* \in E^*_-$, then condition (7) can be relaxed to ask for

$$\text{for all } x \in E_+, \quad x - \alpha \text{ attains its supremum on } E^*, \tag{8}$$

and we also get the w*-lower semicontinuouty for α and the fact that its level sets are w*-compact.

Corollary

Let E be a Banach space and $B \subset E^*$ be a norm closed convex and weakly countably determined subset, with $0 \in B$ and such that $B + B \subset B$. Let us assume there is a weakly countably determined, convex, bounded and w**-closed set $D \subset B$ with $0 \notin D$ and such that for $x \in E$ with $x(d^*) < 0$ for every $d^* \in D$, we have*

$$\sup\{x(c^*) : c^* \in B\} = 0 \tag{9}$$

Then we have that B is going to be w**-closed, i.e.:*

$$\overline{B}^{\omega^*} \subset B.$$

Theorem

Let B, D be subsets of a dual Banach space E^* such that D is assumed to be a $\sigma(E^*, E)$-closed convex subset with $0 \notin D$. Given

$$x \in E \text{ such that : } x(d^*) < 0 \text{ for every } d^* \in D,$$

there is $b^* \in B$ with

$$\langle x, b^* \rangle = \sup \langle x, B \rangle,$$

and $B \subset \cup_{n=1}^{\infty} K_n$ for some family of w^*-compact convex subsets of E^*, then we have:

$$\overline{\mathrm{co}(B)}^{\omega^*} \subset \overline{\mathrm{co}(\cup_{n=1}^{\infty} K_n) + \Lambda_D}^{\|\cdot\|}.$$

Corollary

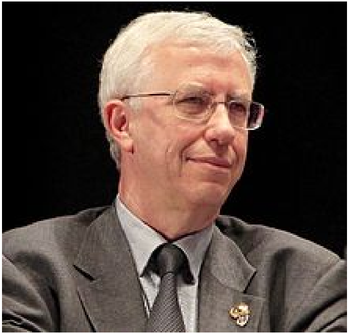
Let E be a Banach space without copies of ℓ^1 and $B \subset E^*$ be a norm closed convex and w*-*K analytic set, with* $0 \in B$ *and such that* $B + B \subset B$. *Let us assume there is a weakly compact convex set* $D \subset B$ *with* $(-D) \cap B = \emptyset$ *such that for* $x \in E$ *with* $x(d^*) < 0$ *for every* $d^* \in D$, *we have*

$$\sup\{x(c^*) : c^* \in B\} = 0 \tag{10}$$

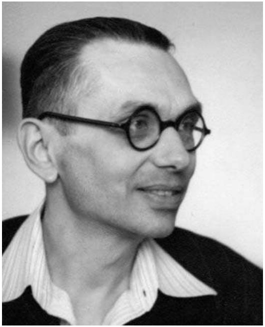
Then B *is going to be* w*-*closed, i.e.:*

$$\overline{B}^{\omega^*} \subset B.$$

Lehman Brothers default



Mathematics and uncertainty



Kurt Gödel

Prof. Dr. Walter Schachermayer

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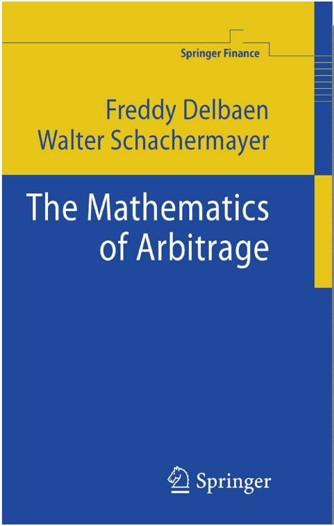
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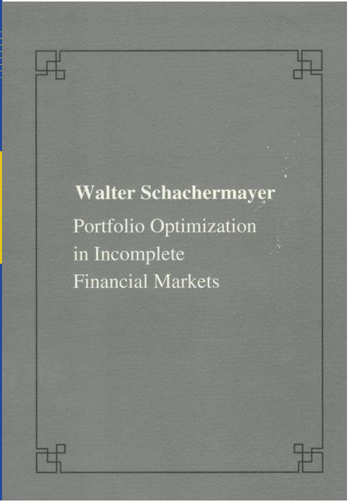
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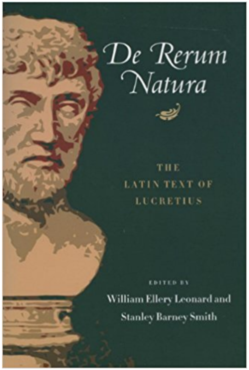
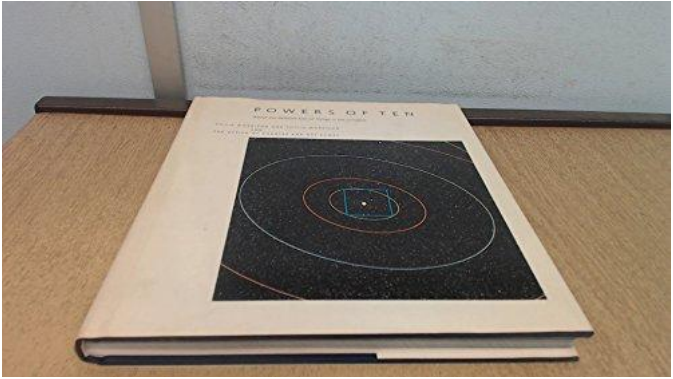
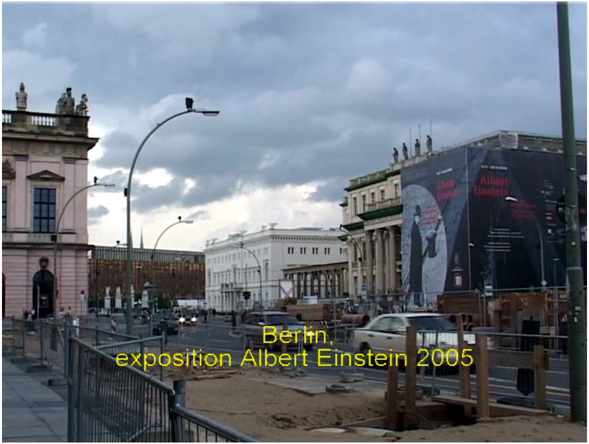


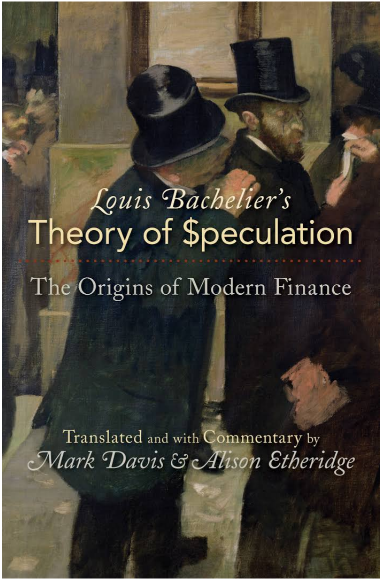
Titus Lucretius Carus (99-55 antes de Cristo)



Powers of ten.

Brownian Motion movie

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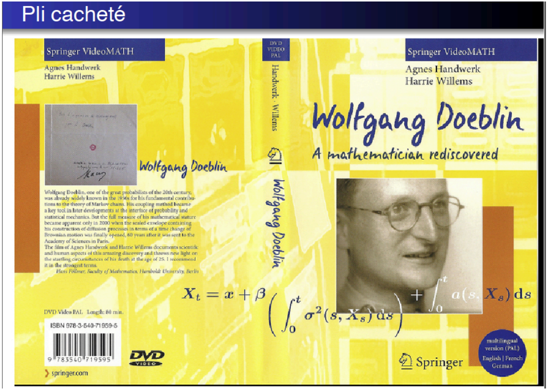
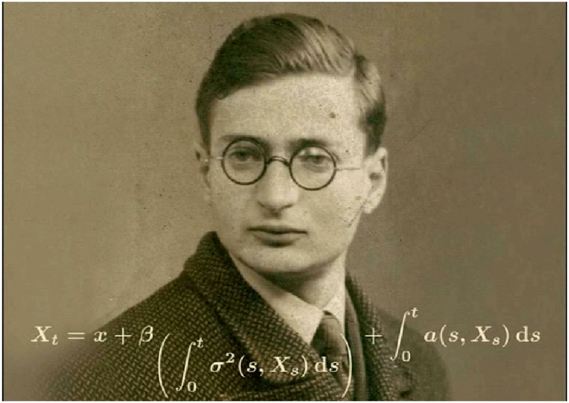
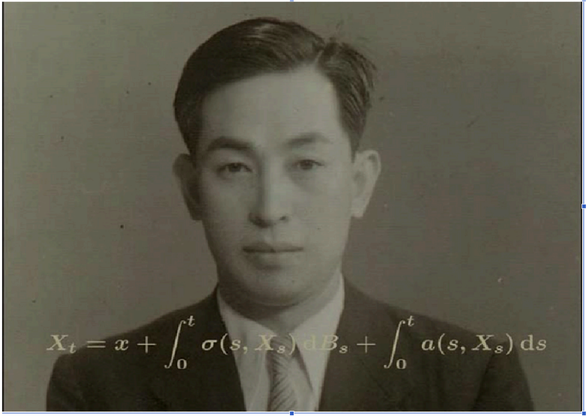


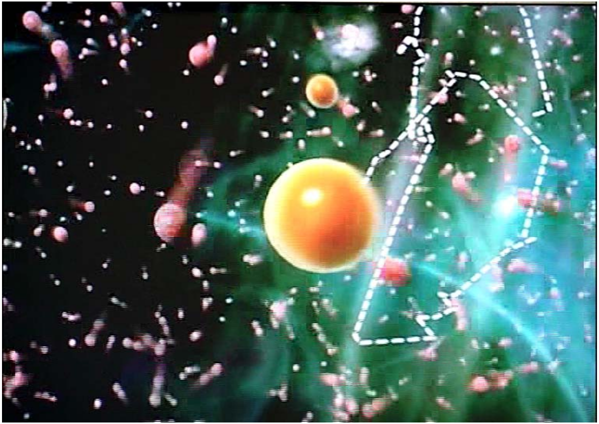
Doeblin-Itô (1935-2000)



Wolfgang Doeblin y su fórmula



Kiyoshi Itô y su fórmula



Path of a Brownian Motion

R. Merton, M. Sholes y F. Black

Paul Embrechets, ETH Zurich. *Extremal events researcher who advised on the risks of the copula formula what killed Wall Street in 2007*

$$\Pr[T_A<1, T_B<1] = \Phi_2(\Phi^{-1}(F_A(1)), \Phi^{-1}(F_B(1)), \gamma)$$

Here's what killed your 401(k) *David X. Li's Gaussian copula function as first published in 2000. Investors exploited it as a quick—and fatally flawed—way to assess risk. A shorter version*

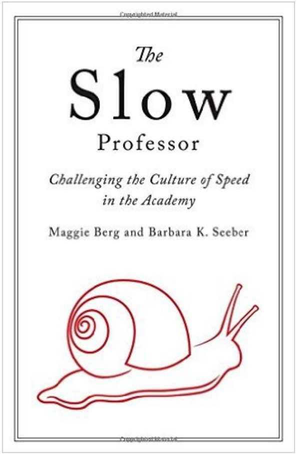
David X. Li
Illustration: David A. Johnson

En el mundo financiero muchos "quants" ven sólo números ante ellos y olvidan sobre la realidad concreta que dichos números representan. Piensan que pueden modelar con unos pocos años de datos y calculan probabilidades para sucesos que pueden ocurrir solamente una vez en 10.000 años. Entonces los inversores invierten sobre la base de dichas probabilidades, sin pararse a preguntarse si los números tienen algún sentido.

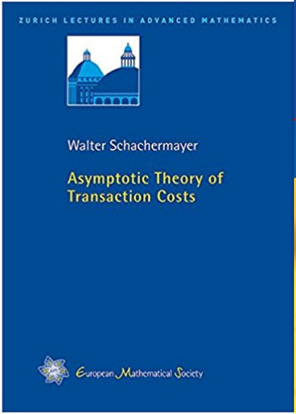
D. Li "The most dangerous part is when people believe everything coming out of it" _ (sobre su modelo....)

David X. Li

Felix Salmon: The formula that killed Wall Street



Slow Science Manifesto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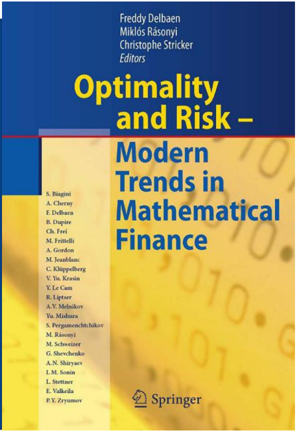
Walter Schachermayer
Asymptotic Theory of Transaction Costs
European Mathematical Society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