Transference and restriction of Fourier multipliers on Orlicz spaces

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XX EARCO
Encuentros de Análisis Real y Complejo
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Multipliers on L^p

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Aim: Similar questions for multipliers between Orlicz spaces

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(Amemiya norm) $|||f|||_\Phi = \inf_{k>0} \frac{1}{k}(1 + \rho_\Phi(kf))$.

Δ_2-condition

A Young function Φ is said to satisfy Δ_2-condition (globally) if there exists a constant $K > 0$ such that

$$\Phi(2x) \leq K\Phi(x), \quad x \geq 0. \tag{2}$$

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Let Φ_1 and Φ_2 be Young functions, and let m be a bounded measurable function defined on G. The function m is said to be a (Φ_1, Φ_2)-multiplier on G if there exists $C > 0$ such that

$$N_{\Phi_2}(T_m(f)) \leq CN_{\Phi_1}(f) \tag{5}$$

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Moreover $\|T_m\|_{L^{\Phi_1}\to L^{\Phi_2}} = \|m\|_{(\Phi_1,\Phi_2)}$.

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If Φ is a Young function satisfying Δ_2 condition then $A(G)$ is dense in $L^{\Phi}(G)$.

Basic Examples

As usual we denote $\hat{\mu}(x) = \int_{\hat{G}} \gamma^{-1}(x) d\mu(\gamma)$ for the Fourier transform of a regular Borel measure μ defined in $\hat{G}$.

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Let Φ_1,Φ_2 and Φ_3 be Young functions.
(i) Assume that there exists $C>0$ such that

$$\Phi_2(x)\leq C\Phi_1(x), \quad x>0. \tag{6}$$

If $m(x)=\hat{\mu}(x)$ for some regular Borel measure μ defined on $\hat{G}$ then $m\in\mathscr{M}_{\Phi_1,\Phi_2}(G)$. Moreover $\|m\|_{(\Phi_1,\Phi_2)}\leq C\|\mu\|_1$.
(ii) Assume that

$$\Phi_1^{-1}(x)\Phi_2^{-1}(x)\leq x\Phi_3^{-1}(x), \quad x\geq 0. \tag{7}$$

If $m(x)=\hat{g}(x)$ for some $g\in L^1(\hat{G})\cap L^{\Phi_2}(\hat{G})$ then $m\in\mathscr{M}_{\Phi_1,\Phi_3}(G)$ and

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More Examples

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Let Φ, Φ_i for $i=1,2$ be Young functions and $m \in \mathcal{M}_{\Phi_1,\Phi_2}(G)$.
(i) If $\varphi \in L^1(G)$ then $\varphi * m \in \mathcal{M}_{\Phi_1,\Phi_2}(G)$. Moreover

$$\|\varphi * m\|_{(\Phi_1,\Phi_2)} \leq \|\varphi\|_1 \|m\|_{(\Phi_1,\Phi_2)}$$

(ii) If $\psi \in L^1(\hat{G})$ then $\hat{\psi} m \in \mathcal{M}_{\Phi_1,\Phi_2}(G)$. Moreover

$$\|\hat{\psi} m\|_{(\Phi_1,\Phi_2)} \leq \|\psi\|_1 \|m\|_{(\Phi_1,\Phi_2)}.$$

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satisfies $N_{\Phi_2}(T_m(f))\leq CN_{\Phi_1}(f)$ for any $f\in\mathscr{S}(\mathbb{R})$, which in case that Φ_1 satisfies Δ_2 is equivalent to the fact that T_m extends to a bounded operator from $L^{\Phi_1}(\mathbb{R})$ into $L^{\Phi_2}(\mathbb{R})$.

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- $\mathscr{M}_{p,q}(\mathbb{R}) = \mathscr{M}_{q',p'}(\mathbb{R})$ where $1/p + 1/p' = 1$.
- $\mathscr{M}_{2,2}(\mathbb{R}) = L^\infty(\mathbb{R})$.

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There are a lot of results known about (p, q)-multipliers corresponding to $\Phi_1(x) = x^p$ and $\Phi_2(x) = x^q$ and denoted by $\mathscr{M}_{p,q}(\mathbb{R})$.

- $sign(\xi) \in \mathscr{M}_{p,p}(\mathbb{R})$ for $1 < p < \infty$.
- $|2\pi\xi|^{-\alpha} \in \mathscr{M}_{p,q}(\mathbb{R})$ for $0 < \alpha < 1, 1/q = 1/p - \alpha$.
- $\mathscr{M}_{p,q}(\mathbb{R}) = \mathscr{M}_{q',p'}(\mathbb{R})$ where $1/p + 1/p' = 1$.
- $\mathscr{M}_{2,2}(\mathbb{R}) = L^\infty(\mathbb{R})$.
- $\mathscr{M}_{1,1}(\mathbb{R}) = \{\hat{\mu} : \mu \in M(\mathbb{R})\}$

$G=\mathbb{R}$

A bounded measurable function m defined in $\mathbb{R}$ is (Φ_1,Φ_2)-multiplier on $\mathbb{R}$ if there exists $C>0$ such that

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- $\mathscr{M}_{p,q}(\mathbb{R})=\{0\}$ for $p>q$.

The dilation operator D_λ

Denote $D_\lambda(f)(x) = f(\lambda x)$ for $\lambda > 0$.

$$C_\Phi(\lambda) = \|D_\lambda\|_{L^\Phi(\mathbb{R}) \to L^\Phi(\mathbb{R})} = \sup\{N_\Phi(D_\lambda(f)) : N_\Phi(f) \leq 1\}$$

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$\alpha(\Phi) > 0$ implies Φ satisfies Δ_2 and $\beta(\Phi) < 1$ implies Φ satisfies ∇_2.

New results

Theorem

Let Φ_1, Φ_2 be Young functions satisfying Δ_2.
If $\mathscr{M}_{\Phi_1,\Phi_2}(\mathbb{R}) \neq \{0\}$ then $\beta(\Phi_1) \geq \alpha(\Phi_2)$.

Corollary

Let $\Phi_{p,q}(t) = \max\{t^p, t^q\}$. If $\max\{p_2, q_2\} < \min\{p_1, q_1\}$ then $\mathscr{M}_{\Phi_{p_1,q_1},\Phi_{p_2,q_2}}(\mathbb{R}) = \{0\}$.

The Bohr group

It is well-known that $\hat{\mathrm{D}}$ is the Bohr compactification of D. We use the notation $AP(\mathbb{R})$ for the set of all continuous almost periodic functions on $\mathbb{R}$, that is to say uniform limits of polynomials $\sum_{k=1}^{n} \alpha_k e^{2\pi i x_k t}$ where $x_k \in \mathbb{R}$ and $\alpha_k \in \mathbb{C}$.

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Recall now the Besicovich-Orlicz spaces for almost periodic functions: If $f \in AP(\mathbb{R})$ and Φ is a Young function we define

$$\tilde{\rho}_\Phi(f) = \overline{\lim}_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} \Phi(|f(x)|)dx = \overline{\lim}_{T\to\infty} \int_{-1/2}^{1/2} \Phi(|D_T f(x)|)dx$$

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$$\|f\|_{B^\Phi} = \inf\{k > 0 : \tilde{\rho}_\Phi(f/k) \le 1\}.$$

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A basic fact to use for the Bohr group is that if μ is any measure defined on $\mathbb{R}$ having support on a finite number of points, then $\hat{\mu} \in AP(\mathbb{R})$ and

$$\|\hat{\mu}\|_{B_\Phi(\mathbb{R})} = \|\mu\|_{L^\Phi(\hat{\mathrm{D}})}. \tag{9}$$

Multipliers for $G = \mathbb{D}$

Let Φ_1, Φ_2 be Young functions. A bounded function $m \in \mathscr{M}_{\Phi_1,\Phi_2}(\mathbb{D})$ if there exists a constant $C > 0$ such that

$$N_{\Phi_2}(\sum \alpha_t m(t)\chi_t) \leq C N_{\Phi_1}(\sum \alpha_t \chi_t) \quad (10)$$

for any $\alpha = \sum \alpha_t \chi_t$ (finite sum) .

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Assume that Φ_2 satisfies ∇_2 and m is a bounded function on $\mathbb{R}$. The following are equivalent:
(i) $m \in \mathscr{M}_{\Phi_1,\Phi_2}(\mathrm{D})$.
(ii) There exists a constant K such that

$$|\sum_{t\in\mathbb{R}} m(t)\mu(t)\lambda(t)dx| \leq C\|\hat{\mu}\|_{B_{\Phi_1}}\|\hat{\lambda}\|_{B_{\Psi_2}} \tag{11}$$

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Main results 1

Theorem

Let m be a bounded continuous function on $\mathbb{R}$ and let Φ_1, Φ_2 be Young functions such that Φ_2 satisfies ∇_2 and

$$\sup_{\lambda > 1} C_{\Phi_1}(\lambda) C_{\Phi_2}(1/\lambda) < +\infty. \tag{12}$$

If $m \in \mathscr{M}_{\Phi_1,\Phi_2}(\mathrm{D})$ then $m \in \mathscr{M}_{\Phi_1,\Phi_2}(\mathbb{R})$.

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Corollary

Let m be a bounded continuous function on $\mathbb{R}$ such that $m \in \mathscr{M}_{\Phi_1,\Phi_2}(\mathrm{D})$ and let Φ_1, Φ_2 be Young functions such that $\alpha(\Phi_1) > \beta(\Phi_2)$. Then $m \in \mathscr{M}_{\Phi_1,\Phi_2}(\mathbb{R})$.

Main results 2

Theorem

Let m be a bounded continuous function on $\mathbb{R}$ and let Φ_1, Φ_2 be Young functions satisfying that Φ_2 has ∇_2 condition and

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Corollary

Let m be a bounded continuous in $\mathbb{R}$ and Φ be a Young function satisfying ∇_2 and

$$\sup_{\lambda>0} C_{\Phi}(\lambda)C_{\Phi}(\frac{1}{\lambda}) < \infty. \tag{14}$$

Then $m \in \mathscr{M}_{\Phi}(\mathbb{R})$ iff $m \in \mathscr{M}_{\Phi}(\mathrm{D})$.

$G = \mathbb{Z}$

A bounded sequence $m = (m_n)_{n\in\mathbb{Z}}$ is (Φ_1, Φ_2)-multiplier on $\mathbb{Z}$ if there exists $C > 0$ such that

$$T_m(P)(t) = \sum_{k\in\mathbb{Z}} m_k \alpha_k e^{2\pi ikt} \tag{15}$$

satisfies $N_{\Phi_2}(T_m(P)) \leq CN_{\Phi_1}(P)$ for any $P(t) = \sum_{k\in\mathbb{Z}} \alpha_k e^{2\pi ikt} \in P(\mathbb{T})$, or equivalently, in case that Φ_1 satisfies Δ_2, extends to a bounded operator from $L^{\Phi_1}(\mathbb{T})$ to $L^{\Phi_2}(\mathbb{T})$.

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If Φ_2 satisfying ∇_2 and let $m=(m_n)$ be a bounded sequence on $\mathbb{Z}$. The following are equivalent:
(i) $m\in\mathscr{M}_{\Phi_1,\Phi_2}(\mathbb{Z})$.
(ii) There exists a constant K such that

$$|\sum_{n\in\mathbb{Z}} m_n\alpha_n\beta_n|\leq CN_{\Phi_1}(P)N_{\Psi_2}(Q) \tag{16}$$

for any $P(t)=\sum_{n\in\mathbb{Z}}\alpha_n e^{2\pi int}$ and $Q(t)=\sum_{n\in\mathbb{Z}}\beta_n e^{2\pi int}$ in $P(\mathbb{T})$.

Main results 3

Theorem

Let m be a bounded continuous function on $\mathbb{R}$ and Φ_1, Φ_2 be Young functions with Φ_2 satisfying ∇_2.
(i) Assume that

$$\sup_{0<\lambda<1} C_{\Phi_1}(\lambda) C_{\Phi_2}(1/\lambda) < \infty. \tag{17}$$

If $m \in \mathscr{M}_{\Phi_1,\Phi_2}(\mathbb{R})$ then $m_n = (m(n)) \in \mathscr{M}_{\Phi_1,\Phi_2}(\mathbb{Z})$.
(ii) Assume that

$$\sup_{\lambda>1} C_{\Phi_1}(\lambda) C_{\Phi_2}(1/\lambda) < \infty. \tag{18}$$

If $(D_{1/N} m(n)) \in \mathscr{M}_{\Phi_1,\Phi_2}(\mathbb{Z})$ for all $N \in \mathbb{N}$ with $\sup_N \|(D_{1/N} m)_n\|_{(\Phi_1,\Phi_2)} < \infty$ then $m \in \mathscr{M}_{\Phi_1,\Phi_2}(\mathbb{R})$.

DeLeeuw, K. *On L_p-multipliers.* Ann. Math. **91** [1965], pp 364–379.

Rao, M. M., Ren, Z.D. *Theory of Orlicz spaces.* CRM Press [1991].

Rudin, W. *Fourier Analysis on Groups.* Interscience, New York, [1962].

THANK YOU VERY MUCH FOR YOUR ATTENTION !