Pointwise descriptions of nearly incompressible vector fields with bounded curl

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Outline

1. Introduction
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   - Di Perna - Lions Theory
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2. Reimann vector fields
   - Quasisymmetry/Quasiconformality and Q class
   - Reimann fields produce Hölder and Sobolev flows
3. Euler equation in the plane
   - Euler flows are Hölder and Sobolev
   - Optimal regularity of flows
   - $\overline{Q}, R_\theta$

Introduction

Flow of a vector field

Given a vector field $b : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$, we call $X_{t,t_0}(\cdot) = X(t, t_0, \cdot) : [0, T] \times \mathbb{R}^n \to \mathbb{R}^n$ the flow of b if

$$\begin{cases} \frac{d}{dt} X(t, t_0, x) = b(t, t_0, X(t, t_0, x)) \\ X(t_0, t_0, x) = x. \end{cases}$$

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Existence and uniqueness of the flow

Cauchy-Lipschitz Theory: Existence and uniqueness (*b Lipschitz*)
Peano's Theorem : Existence (*b Continuous*)
Osgood's Theorem: Uniqueness (*b Lip-Log*)

Flow of a vector field

If $b(t,\cdot)$ has modulus of continuity η,

$$|b(t,x)-b(t,y)| \leq \eta(|x-y|) \qquad \text{where } \int_0 \frac{dr}{\eta(r)} = \infty$$

then X_t admits modulus of continuity $\omega(r) = \omega_{t,\eta}(r)$,

$$|X_t(x)-X_t(y)| \leq \omega(|x-y|), \quad \text{for } x, y \text{ close enough.}$$

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Modulus of continuity of the flow

1. Lipschitz vector fields produce Lipschitz flows:
   If $\eta(r) = Cr$, then $\omega(r) = e^{Ct}r$.
2. Lip-Log vector fields produce Hölder flows:
   If $\eta(r) = r\log\frac{1}{r}$, then $\omega(r) = r^{e^{-t}}$.

Di Perna- Lions Theory

Di Perna, Lions (1989): Extension of the notion of flow for Sobolev vector fields.

Well-posedness of Sobolev vector fields

$$b \in L^1(0,T;W^{1,p}_{loc}) \quad \text{for some } p \geq 1$$

$$\operatorname{div} b \in L^1(0,T;L^\infty)$$

$$\frac{|b(t,x)|}{1+|x|} \in L^1(0,T;L^1) + L^1(0,T;L^\infty)$$

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Properties of Di Perna- Lions flow

- **Linear Distortion of Lebesgue measure**

$$|X_t(E)| \leq C(t)\,|E| \qquad E \subset \mathbb{R}^n \text{ measurable}$$

- **Flow compatible with Linear Transport Equation**

Di Perna - Lions Theory

Linear Transport Equation

$$LTE : \begin{cases} \partial_t u + b \cdot \nabla u = 0 \\ u(0,x) = u_0(x) \end{cases}$$

- LTE ⇒ Flow: The i-th component of X_t^{-1} is the solution u of LTE with datum $u_0(x) = x_i$.
- Flow ⇒ LTE: The solution of LTE is

$$u(t,x) = u_0 \circ X_t^{-1}(x).$$

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Regularity of the flow

- measurable ✓
- differentiable in measure ✓ (Le Bris-Lyons)
- continuous ✗
- Sobolev ✗ (Jabin, Alberti-Crippa-Mazzucato)

Borderline

Difference in the two classical theories

- **Cauchy-Lipschitz Theory**: If $b \in W^{1,\infty}$ then the flow is Lipschitz
- **Di Perna - Lions Theory**: If $b \in W^{1,p}$ for any $p < +\infty$ then the flow may not be even Sobolev

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Principal Examples

- Geometric Function Theory (Reimann vector fields)
- Fluid Mechanics (Euler equations)

Reimann vector fields

Reimann vector fields

We say that a continuous vector field $b : \mathbb{R}^n \to \mathbb{R}^n$ is of Reimann's type and write $b \in \mathcal{Q}$ iff

$$\left| \frac{\langle b(x+h) - b(x), h \rangle}{|h|^2} - \frac{\langle b(x+k) - b(x), k \rangle}{|k|^2} \right| \leq C$$

for all $x \in \mathbb{R}^n$, and $|h| = |k| \neq 0$.

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Reimann 1976

b Reimann $\Rightarrow X_t$ is $\eta_t -$ quasisymmetric

Quasisymmetry and Quasiconformality

Quasisymmetric map

$f : \mathbb{R}^n \to \mathbb{R}^n$ is η-**quasisymmetric** (η-QS) if

$$\frac{|f(y)-f(x)|}{|f(z)-f(x)|} \leq \eta\left(\frac{|y-x|}{|z-x|}\right) \qquad \forall x, y, z$$

for some $\eta : [0,\infty) \to [0,\infty)$.

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Quasiconformal map

A homeomorphism $f:\mathbb{R}^n\to\mathbb{R}^n$ is K**-quasiconformal** (K-QC) if $f\in W^{1,n}_{loc}$, $J(\cdot,f)\geq 0$ and

$$|Df(z)|^n\leq K\cdot J(z,f)$$

at almost every point z.

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Quasisymmetry and quasiconformality are equivalent.

Reimann’s Differential Characterization

Reimann’s Theorem

b Reimann $\iff$b is differentiable a.e., Sb$\in L^\infty$ and $\frac{|b(x)|}{|x|\log|x|} \leq C$

$$Sb(x) = \frac{Db(x) + Db(x)^T}{2} - \frac{\operatorname{div} b(x)}{n} Id$$

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$$\sup_{|h|=|k|} \left| \frac{\langle b(x+h)-b(x), h\rangle}{|h|^2} - \frac{\langle b(x+k)-b(x), k\rangle}{|k|^2} \right| \simeq 2|Sb(x)|$$

Reimann fields produce Hölder flows

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b Reimann $\Rightarrow X_t$ is K_t-QC $\quad \Rightarrow X_t \in C^{\alpha}_{loc}, \alpha = \alpha(K_t)$.

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K-QC maps are $W^{1,p}_{loc}$ for some $p = p(n, K) > n$.

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b Reimann $\Rightarrow X_t$ is K_t-QC $\quad \Rightarrow \; X_t \in W^{1,p}_{loc}$ for $p(n, K_t) > n$

Euler equation in the plane

Planar Euler equation in vorticity form

Euler equation

$$EE : \begin{cases} \omega_t + v \cdot \nabla\omega = 0, \\ \omega(0,\cdot) = \omega_0, \\ v = K * \omega \end{cases}$$

- $v(t,\cdot) : \mathbb{R}^2 \to \mathbb{R}^2$ velocity field
- $\omega(t,\cdot) : \mathbb{R}^2 \to \mathbb{R}$ vorticity
- K= Convolution Kernel

$$K(z) = K(x,y) = \frac{iz}{2\pi|z|^2} \equiv \frac{1}{2\pi}\frac{(-y,x)}{x^2+y^2}$$

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Biot-Savart Law

$$v = K * \omega \quad \Longleftrightarrow \quad \begin{cases} \mathrm{div}(v) = 0 \\ \mathrm{curl}(v) = \omega \end{cases} \quad \Longleftrightarrow \quad \partial_z v = \frac{i\omega}{2}$$

Euler Flows are Hölder

Yudovich (1963)

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If ω is an Yudovich solution, then $\omega(t, \cdot) \in L^\infty$

$$\Rightarrow \partial_z v = \frac{i\omega}{2} \in L^\infty$$

$$\Rightarrow \partial_{\bar{z}} v \in BMO$$

$$\Rightarrow v \text{ is } Zygmund$$

$$\Rightarrow v \text{ is } Lip - Log$$

$$\Rightarrow v \text{ has flow } X_t \in C^{\alpha(t)}$$

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Bahouri-Chemin (1993)

$\alpha(t) \leq e^{-t\|\omega_0\|_\infty}$

Euler Flows are Sobolev

Clop-Jylhä (2019)

If $\omega \in L^\infty(L^\infty)$ is an Yudovich solution, and $v = K * \omega$, then

$$X_t \in W^{1,p}_{loc}$$

for $1 < p < \dfrac{2}{t\|\omega_0\|_\infty}$

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- For only small t
- Optimal if $t \to 0$
- t large???

Optimal Sobolev regularity of the flows

- b Reimann: OK for all $t > 0$
  - $\mathcal{Q} \Rightarrow$ QS
  - $QS \Rightarrow$ QC
  - Astala 1994 (Optimal regularity QC)
  - $X_t \in W^{1,p}_{loc}$ if $p < \frac{2}{1-\exp\left(-2\int_0^t \|\partial_{\bar{z}} v\|_\infty ds\right)}$
- v Euler:
  - Conjecture : $X_t \in W^{1,p}_{loc}$ if $p < \frac{2}{1-\exp\left(-2\int_0^t \|\partial_z v\|_\infty ds\right)}$
  - CJ 2019 : Conjecture true if $t \to 0^+$
  - OPEN PROBLEM: t large???
  - Conjecture + Sobolev embedding $\implies$ $Bahouri - Chemin$

GOAL: To find analogue of $\mathcal{Q}$ for $\partial_z v \in L^\infty$ and its geometric interpretation.

$\overline{\mathcal{Q}}$ class

We say that a continuous planar vector field $v \in \overline{\mathcal{Q}}$ if

$$\left| \frac{\langle v(x+h) - v(x), \bar{h} \rangle}{|h|^2} - \frac{\langle v(x+k) - v(x), \bar{k} \rangle}{|k|^2} \right| \leq C$$

for all $x \in \mathbb{R}^2$, $t \in [0, T]$ and $|h| = |k| \neq 0$.

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CS 2021

Let $v : [0, T] \times \mathbb{R}^2 \to \mathbb{R}^2$ be continuous. TFAE:

1. $v \in \overline{\mathcal{Q}}$
2. It is true that:
   - $\frac{|v(t,x)|}{|x| \log |x|} \leq C$ when $|x| \to \infty$
   - v is differentiable a.e. and $\|\partial_z v(t, \cdot)\|_\infty \leq C$ for some $C \geq 0$

$\mathcal{R}_\theta$ class

Given $\theta \in [0, 2\pi]$, we say that a continuous planar vector field $v \in \mathcal{R}_\theta$, if

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R_0 class

- $\overline{\mathcal{Q}} = \bigcap_{0 \leq \theta \leq 2\pi} \mathcal{R}_\theta = \mathcal{R}_0 \cap \mathcal{R}_{\frac{\pi}{2}}$
- $R_\theta (\theta \neq 0)$, difficult to extend to $\mathbb{R}^n$, $n \geq 2$
- More room to play with $\mathcal{R}_0$

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Remarks:

- $\mathcal{R}_0$ is Lip-Log and grows like $|x| \log |x|$
- $\mathcal{R}_0$ is not differentiable a.e. in general
- $n = 2 \Rightarrow Dv - D^t v \equiv \text{curl}(v) = Im(\partial_z v)$

CS 2021

Let $n \geq 2$ and v be continuous.

1. $v \in \mathcal{R}_0 + \mathrm{div}(v) \in L^\infty \Rightarrow v$ differentiable a.e. and

$$Av = \left(\frac{Dv - D^t v}{2} + \frac{\mathrm{div}(v)}{n} \mathrm{Id} \right) \in L^\infty$$

2. If
   - $\frac{|v(x)|}{|x| \log |x|} \leq C$ when $|x| \to \infty$
   - $Av \in L^\infty$

   then $v \in \mathcal{R}_0$.

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   - $Av \in L^\infty$

   then $v \in \mathcal{R}_0$.

- If $\operatorname{div}(v) \in L^\infty$, then $v \in \mathcal{R}_0 \Leftrightarrow Dv - D^t v \in L^\infty$
- $\operatorname{div}(v) \in L^p, p > n$
- $n = 2 \Rightarrow Av \equiv \partial_z v \equiv \operatorname{div}(v) + i \operatorname{curl}(v)$

Thanks for your attention