# Decreasing rearrangement on average operators


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Baena-Miret, S. and Carro, M. J. [2022], 'On weak-type $(1, 1)$ for averaging type operators', *Preprint*.

# Background and motivation

## Definition

Given an operator $T$, (quasi-)normed function spaces $\mathbb{X}$ and $\mathbb{Y}$, and $C > 0$,

$$T : \mathbb{X} \to \mathbb{Y}, \quad C \qquad \Longleftrightarrow \qquad \|Tf\|_{\mathbb{Y}} \leq C \|f\|_{\mathbb{X}} .$$

Further, $T$ is called *sublinear* if

$$|T(f+g)| \leq |Tf| + |Tg|, \qquad \forall f, g \in \mathbb{X}.$$

# Background and motivation

## Definition

A *weight* is a locally integrable function $v : \mathbb{R}^n \to (0, \infty)$.

## Definition (F. Riesz, 1910)

Given $p \geq 1$, the *weighted Lebesgue space* is

$$L^p(v) = \left\{ f \in \mathcal{M}(\mathbb{R}^n) \, : \, \|f\|_{L^p(v)} = \left( \int_{\mathbb{R}^n} |f(x)|^p v(x) \, dx \right)^{\frac{1}{p}} < \infty \right\}.$$

If $v = 1$, then $L^p(v) = L^p(\mathbb{R}^n)$.

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Riesz, F. [1910], 'Untersuchungen über systeme integrierbarer funktionen', *Mathematische Annalen* **69**, 449–497.

# Background and motivation

## Definition (G.G. Lorentz, 1950/1951)

Given $p \geq 1$, the *weighted weak Lebesgue space* is

$$L^{p,\infty}(v) = \left\{ f \in \mathcal{M}(\mathbb{R}^n) \,:\, \|f\|_{L^{p,\infty}(v)} = \sup_{y>0} y \lambda_f^v(y)^{\frac{1}{p}} < \infty \right\},$$

where

$$\lambda_f^v(y) = v\left(\{x \in \mathbb{R}^n : |f(x)| > y\}\right) = \int_{\{|f(x)|>y\}} v(x)\,dx, \qquad y > 0.$$

If $v = 1$, then $L^{p,\infty}(v) = L^{p,\infty}(\mathbb{R}^n)$ and $\lambda_f^v(y) = \lambda_f(y)$.

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Lorentz, G. G. [1950], 'Some new functional spaces', *Ann. of Math. (2)* **51**, 37–55.
Lorentz, G. G. [1951], 'On the theory of spaces $\Lambda$', *Pacific J. Math.* **1**, 411–429.

## Definition (G.G. Lorentz, 1950/1951)

Given $p \geq 1$,

$$L^{p,1}(v) = \left\{ f \in \mathcal{M}(\mathbb{R}^n) \,:\, \|f\|_{L^{p,1}(v)} = \int_0^\infty t^{\frac{1}{p}} f_v^*(t) \,\frac{dt}{t} < \infty \right\},$$

is a *weighted Lorentz space* where

$$f_v^*(t) = \inf \left\{ y > 0 : \lambda_f^v(y) \leq t \right\}, \qquad t > 0.$$

If $p = 1$, then $L^{1,1}(v) = L^1(v)$. Further, if $v = 1$, then $L^{p,1}(v) = L^{p,1}(\mathbb{R}^n)$ and $f_v^*(t) = f^*(t)$.

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Lorentz, G. G. [1950], 'Some new functional spaces', *Ann. of Math. (2)* **51**, 37–55.
Lorentz, G. G. [1951], 'On the theory of spaces $\Lambda$', *Pacific J. Math.* **1**, 411–429.

# Background and motivation

**Definition (G.H. Hardy and J.E. Littlewood, 1930)**

The *Hardy-Littlewood maximal function* is

$$Mf(x) = \sup_{Q \subseteq \mathbb{R}^n} \left( \frac{1}{|Q|} \int_Q |f(y)|\, dy \right) \chi_Q(x), \qquad \forall x \in \mathbb{R}^n, \quad f \in L^1_{\mathrm{loc}}(\mathbb{R}^n).$$

**Definition (B. Muckenhoupt, 1972)**

$$u \in A_1 \qquad \Longleftrightarrow \qquad Mu(x) \leq Cu(x), \qquad \text{a.e. } x \in \mathbb{R}^n,$$

and $||u||_{A_1}$ is the infimum of such constants $C$.

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Hardy, G. H. and Littlewood, J. E. [1930], 'A maximal theorem with function theoretic applications', *Acta Math.* **54**(1), 81–116.

Muckenhoupt, B. [1972], 'Weighted norm inequalities for the Hardy maximal function', *Trans. Amer. Math. Soc.* **165**, 207–226.

# Background and motivation

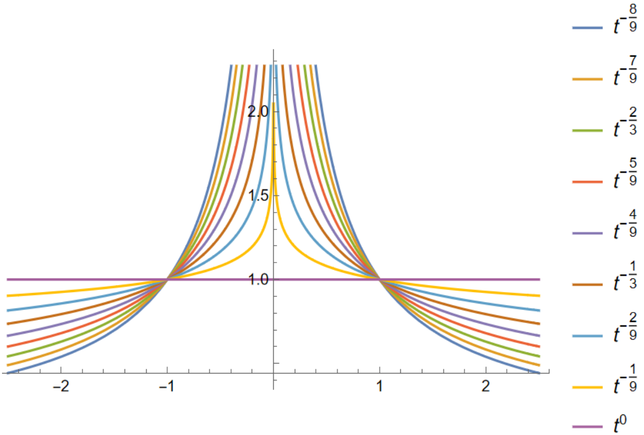


Figure 1: Examples of $A_1$ weights for $n = 1$.

# Background and motivation

Let $\{T_\theta\}_\theta$ be a family of sublinear operators indexed in a probability measure space $(\Omega, \mathcal{A}, P)$ such that the boundedness

$$T_\theta : L^1(u) \longrightarrow L^{1,\infty}(u), \qquad \forall u \in A_1, \qquad \varphi(\|u\|_{A_1}),$$

with $\varphi$ being an *admissible* function. The aim of this talk is to address the following question: what can we say about the decreasing rearrangement of the average operator

$$T_A f(x) = \int_\Omega T_\theta f(x) dP(\theta), \qquad x \in \mathbb{R}^n,$$

whenever is well defined? To do so, we first will see whenever

$$T_A : L^1(u) \longrightarrow L^{1,\infty}(u), \qquad \forall u \in A_1, \qquad \Phi(\|u\|_{A_1}).$$

# Average operators: easy example

First, let $0 < \theta < 1$ and set

$$T_\theta f(x) = \frac{\int_0^1 f(y)dy}{|x-\theta|}\chi_{(0,1)}(x), \qquad x \in \mathbb{R}.$$

Clearly $T_\theta$ satisfies

$$T_\theta : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n)$$

with $C \leq 2$, but

$$T_A f(x) = \int_0^1 T_\theta f(x) d\theta \equiv \infty, \qquad \forall x \in \mathbb{R}.$$

# Average operators: Fourier multipliers

## Definition (Fourier transform)

$$\hat{f}(\xi) = \int_{\mathbb{R}} f(x) e^{-2\pi i x \xi}\, dx, \qquad \xi \in \mathbb{R}, \qquad f \in L^1(\mathbb{R}).$$

Let $m \in L^\infty(\mathbb{R})$, i.e.,

$$\|m\|_\infty := \|m\|_{L^\infty(\mathbb{R})} = \operatorname{ess\,sup} |m| < \infty.$$

## Definition (Fourier multiplier)

$$\widehat{T_m f}(\xi) = m(\xi)\hat{f}(\xi), \qquad \xi \in \mathbb{R}, \qquad f \in L^2(\mathbb{R}),$$

and $m$ is called a *multiplier*.

# Average operators: Fourier multipliers

## Definition

$m$ is of *bounded variation* if

$$V(m) = \sup \sum_{i=1}^{N} |m(x_i) - m(x_{i-1})| < \infty,$$

for all possible choices of points such that $-\infty < x_0 < x_1 < \cdots < x_N < \infty$.

$$V(m) < \infty \qquad \Longrightarrow \qquad m \in L^\infty(\mathbb{R}).$$

Let $m$ be a bounded variation function on $\mathbb{R}$ that is right-continuous at every point $x \in \mathbb{R}$ and $\lim_{x \to -\infty} m(x) = 0$.

Then,

$$m(\xi) = \int_{-\infty}^{\xi} dm(t) = \int_{\mathbb{R}} \chi_{(-\infty,\xi)}(t)\, dm(t) = \int_{\mathbb{R}} \chi_{(t,\infty)}(\xi)\, dm(t), \quad \forall \xi \in \mathbb{R},$$

and $dm$ (*Lebesgue-Stieltjes measure*) is finite.

# Average operators: Fourier multipliers

Consider

$$T_m f(x) = \int_{\mathbb{R}} m(\xi)\hat{f}(\xi)e^{2\pi i x\xi}d\xi, \qquad x \in \mathbb{R},$$

for every Schwartz function $f$. A formal computation shows that

$$T_m f(x) = \int_{\mathbb{R}} H_t f(x) dm(t), \qquad \forall x \in \mathbb{R},$$

where

$$H_t f(x) := T_{\chi_{(t,\infty)}} f(x) = \int_t^\infty \hat{f}(\xi)e^{2\pi i x\xi}d\xi, \qquad x \in \mathbb{R}.$$

# Average operators: Fourier multipliers

Since [1]

$$\forall p > 1, \forall t \in \mathbb{R}: \qquad H_t : L^p(\mathbb{R}) \to L^p(\mathbb{R}), \qquad C,$$

we have

- using the Minkowski's integral inequality, that is

$$\|T_m f\|_{L^p(\mathbb{R}^n)} \lesssim \int_{\mathbb{R}} \|H_t f\|_{L^p(\mathbb{R}^n)} \, dm(t),$$

- and the density of the Schwartz functions on $L^p(\mathbb{R})$,

that

$$\forall p > 1: \qquad T_m : L^p(\mathbb{R}) \to L^p(\mathbb{R}).$$

[1] Duoandikoetxea, J. [2001], *Fourier analysis*, Vol. 29 of *Graduate Studies in Mathematics*, American Mathematical Society, Providence, RI. Translated and revised from the 1995 Spanish original by David Cruz-Uribe.

However, even though we also have

$$H_t : L^1(\mathbb{R}) \to L^{1,\infty}(\mathbb{R}),$$

we cannot deduce (at least not immediately) that

$$T_m : L^1(\mathbb{R}) \to L^{1,\infty}(\mathbb{R}),$$

(due to the lack of the Minkowski's integral inequality for the space $L^{1,\infty}(\mathbb{R})$).

# Average operators: Fourier multipliers

## Definition (D. Hilbert, 1904)

The *Hilbert transform* is defined by

$$Hf(x) = \frac{1}{\pi} \lim_{\varepsilon \to 0^+} \int_{|x-y|>\varepsilon} \frac{f(y)}{x-y}\, dy, \qquad x \in \mathbb{R}, \qquad f \in \mathcal{C}^\infty(\mathbb{R}).$$

Since $Hf = T_m f$ with

$$m(\xi) = -i \operatorname{sgn} \xi \quad \text{and} \quad \chi_{(t,\infty)}(\xi) = \frac{\operatorname{sgn}(\xi - t) + 1}{2}, \qquad \forall \xi \in \mathbb{R},$$

then

$$\forall t \in \mathbb{R}: \qquad H_t f(x) = \frac{1}{2}\left[f(x) + ie^{2\pi itx} H(e^{-2\pi it\cdot} f)(x)\right], \qquad \forall x \in \mathbb{R}.$$

Hilbert, D. [1904], 'Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen', *Nach. Akad. Wissensch. Gottingen. ath.phys. Klasse* **3**, 213–259.

# Average operators: Fourier multipliers

**Proposition (A.K. Lerner, S. Ombrosi and C. Pérez, 2009)**

$$H : L^1(u) \longrightarrow L^{1,\infty}(u), \quad \forall u \in A_1, \quad C\|u\|_{A_1}\left(1+\log_+\|u\|_{A_1}\right).$$

Then

$$\forall t \in \mathbb{R} : \quad H_t : L^1(u) \longrightarrow L^{1,\infty}(u), \quad \forall u \in A_1, \quad \tilde{C}\|u\|_{A_1}\left(1+\log_+\|u\|_{A_1}\right).$$

**Goal**

$$T_m : L^1(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1, \qquad C\,\|u\|_{A_1}^2\left(1+\log_+\|u\|_{A_1}\right)^2.$$

Lerner, A. K., Ombrosi, S. and Pérez, C. [2009], '$A_1$ bounds for Calderón-Zygmund operators related to a problem of Muckenhoupt and Wheeden', *Math. Res. Lett.* **16**(1), 149–156.

# Extrapolation theory of Rubio de Francia

## Theorem (Rubio de Francia, J. L., 1982/1984)

*Let $T$ be a sublinear operator such that*

$$\exists p_0 > 1 : \qquad T : L^{p_0}(v) \to L^{p_0}(v), \quad \forall v \in A_{p_0}, \quad \varphi(\|v\|_{A_{p_0}}).$$

*Then,*

$$\forall p > 1 : \qquad T : L^{p}(v) \to L^{p}(v), \quad \forall v \in A_{p}, \quad \Phi(\|v\|_{A_{p}}).$$

Given $p > 1$,

$$v \in A_p \qquad \iff \qquad ||v||_{A_p} := \sup_Q \left( \frac{1}{|Q|} \int_Q v \right) \left( \frac{1}{|Q|} \int_Q v^{\frac{1}{1-p}} \right)^{p-1} < +\infty.$$

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Rubio de Francia, J. L. [1982], 'Factorization and extrapolation of weights', *Bull. Amer. Math. Soc. (N.S.)* **7**(2), 393–395.

Rubio de Francia, J. L. [1984], 'Factorization theory and $A_p$ weights', *Amer. J. Math.* **106**(3), 533–547.

Let $T$ be a sublinear operator such that

$$\exists p_0 > 1 : \qquad T : L^{p_0}(v) \to L^{p_0}(v), \quad \forall v \in A_{p_0}, \quad \varphi(\|v\|_{A_{p_0}}).$$

Then,

$$T : L^1(\mathbb{R}^n) \to L^{1,\infty}(\mathbb{R}^n).$$

## Theorem (M.J. Carro, L. Grafakos and J. Soria, 2015)

*Let $T$ be an operator such that*

$$\exists p_0 > 1 : \qquad T : L^{p_0,1}(v) \longrightarrow L^{p_0,\infty}(v), \qquad \forall v \in \widehat{A}_{p_0}, \qquad \varphi(\|v\|_{\widehat{A}_{p_0}}).$$

*Then, for any measurable set $E \subseteq \mathbb{R}^n$,*

$$\|T\chi_E\|_{L^{1,\infty}(u)} \leq C\|u\|_{A_1}^{1-\frac{1}{p_0}} \varphi\left(\|u\|_{A_1}^{\frac{1}{p_0}}\right) u(E), \qquad \forall u \in A_1.$$

Given $p \geq 1$,

$$\widehat{A}_p = \left\{v \in L^1_{\text{loc}}(\mathbb{R}^n) : v = (Mh)^{1-p}u \text{ such that } h \in L^1_{\text{loc}}(\mathbb{R}^n) \text{ and } u \in A_1\right\},$$

with

$$\|v\|_{\widehat{A}_p} = \inf\left\{\|u\|_{A_1}^{\frac{1}{p}} : v = (Mh)^{1-p}u\right\}.$$

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Carro, M. J., Grafakos, L. and Soria, J. [2015], 'Weighted weak-type $(1,1)$ estimates via Rubio de Francia extrapolation', *J. Funct. Anal.* **269**(5), 1203–1233.

## Theorem (S. B-M and M.J. Carro, 2022)

*Let* $(f, g)$ *be a pair of measurable functions such that*

$$||g||_{L^{1,\infty}(u)} \leq \varphi(\|u\|_{A_1})||f||_{L^1(u)}, \qquad \forall u \in A_1.$$

*Then,*

$$\forall p > 1: \qquad ||g||_{L^{p,\infty}(v)} \leq \Phi(\|v\|_{\widehat{A}_p})||f||_{L^{p,1}(v)}, \qquad \forall v \in \widehat{A}_p,$$

*where*

$$\Phi(r) = C_1\varphi(C_2 r^p)r^{p-1}(1 + \log_+ r)^{2-\frac{2}{p}}, \qquad r \geq 1.$$

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Baena-Miret, S. and Carro, M. J. [2022], 'On weak-type $(1, 1)$ for averaging type operators', *Preprint*.

## Pre-Goal

For every $E \subseteq \mathbb{R}^n$,

$$||T_m\chi_E||_{L^{1,\infty}(u)} \leq C \|u\|_{A_1} (1 + \log_+ \|u\|_{A_1})^2 u(E), \qquad \forall u \in A_1.$$

**Proof.** For $p > 1$, $L^{p,\infty}(v)$ is a Banach function space since there exists a norm $|| \cdot ||_{(p,\infty,v)}$ so that

$$||f||_{L^{p,\infty}(v)} \leq ||f||_{(p,\infty,v)} \leq \frac{p}{p-1}||f||_{L^{p,\infty}(v)}.$$

**Proof (continuation).**

$$\forall t \in \mathbb{R}: \qquad H_t : L^1(u) \longrightarrow L^{1,\infty}(u), \quad \forall u \in A_1, \quad C\|u\|_{A_1}\left(1+\log_+\|u\|_{A_1}\right).$$

$$\Downarrow$$

For every $1 < p \leq 2$

$$\forall t \in \mathbb{R}: \qquad H_t : L^{p,1}(v) \longrightarrow L^{p,\infty}(v), \qquad \forall v \in \widehat{A}_p, \qquad C\Phi(\|v\|_{\widehat{A}_p}),$$

with

$$\Phi(r) = r^{2p-1}(1+\log_+ r)^{3-\frac{2}{p}},\ r \geq 1.$$

# Average operators: Fourier multipliers

(Minkowski's integral inequality)

$$\Downarrow$$

For every $1 < p \leq 2$,

$$T_m f = \int_{\mathbb{R}} H_t f(\cdot)\, dm(t) : L^{p,1}(v) \longrightarrow L^{p,\infty}(v), \quad \forall v \in \widehat{A}_p, \quad C\frac{p}{p-1}\Phi(||v||_{\widehat{A}_p}).$$

$$\Downarrow$$

For every $E \subseteq \mathbb{R}^n$,

$$||T_m \chi_E||_{L^{1,\infty}(u)} \leq C\,\|u\|_{A_1}\,(1+\log_+\|u\|_{A_1})\Psi(\|u\|_{A_1})u(E), \quad \forall u \in A_1,$$

with

$$\Psi(\|u\|_{A_1}) = \inf_{1<p\leq 2} \frac{p}{p-1}\left[||u||_{A_1}(1+\log_+||u||_{A_1})^2\right]^{\frac{p-1}{p}} \approx 1+\log_+||u||_{A_1}.$$

# Average operators: Fourier multipliers

## Goal

$$T_m : L^1(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1, \qquad C\|u\|_{A_1}^2 \left(1 + \log_+ \|u\|_{A_1}\right)^2.$$

## Theorem (C. Domingo-Salazar and M.J. Carro, 2019)

*Let $m$ be a bounded variation function on $\mathbb{R}$. If*

$$\forall E \subseteq \mathbb{R}^n : \qquad \|T_m \chi_E\|_{L^{1,\infty}(u)} \leq C_u u(E),$$

*then*

$$T_m : L^1(u) \to L^{1,\infty}(u), \qquad 2^n C_u \|u\|_{A_1}.$$

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Carro, M. J. and Domingo-Salazar, C. [2019], 'Weighted weak-type $(1, 1)$ estimates for radial Fourier multipliers via extrapolation theory', *J. Anal. Math.* **138**(1), 83–105.

# Admissible functions

## Definition

A differentiable function $\varphi : [1, \infty] \to [1, \infty]$ is called *admissible* if satisfies that $\varphi(1) = 1$ and that there exist some $\gamma_0, \gamma_1 > 0$ such that

$$\frac{\gamma_0}{x} \leq \frac{\varphi'(x)}{\varphi(x)} \leq \frac{\gamma_1}{x}, \qquad \forall x \geq 1.$$

## Example

For $\gamma_0 > 0$ and $\gamma_1 \geq 0$, then

$$\varphi(x) = x^{\gamma_0}(1 + \log_+ x)^{\gamma_1}, \qquad x \geq 1,$$

is admissible.

# From weighted weak type estimates to decreasing rearrangement

## Theorem (E. Agora, J. Antezana, S. B-M, M.J. Carro, 2022)

*Let $T$ be a sublinear operator and $\varphi$ be an admissible function. If*

$$T : L^1(u) \to L^{1,\infty}(u), \qquad \forall u \in A_1, \qquad C_1\varphi(\|u\|_{A_1}),$$

*then for every $f \in L^1_{loc}(\mathbb{R}^n)$,*

$$(Tf)^*(t) \leq C_2 \left( \frac{1}{t}\int_0^t f^*(s)\,ds + \int_t^\infty \frac{\varphi\left(1+\log\frac{s}{t}\right)}{1+\log\frac{s}{t}} f^*(s)\,\frac{ds}{s} \right), \qquad \forall t > 0.$$

Agora, E., Antezana, J., Baena-Miret, S. and Carro, M. J. [2022], 'From Weak-type Weighted Inequality to Pointwise Estimate for the Decreasing Rearrangement', *J. Geom. Anal.* **32**(2), 56.

## Corollary

*Let $m$ be a bounded variation function on $\mathbb{R}$ that is right-continuous at every point $x \in \mathbb{R}$ and $\lim_{x\to-\infty} m(x) = 0$. Then, for every $f \in L^1_{loc}(\mathbb{R}^n)$ and $t > 0$,*

$$(T_m f)^*(t) \\ \leq C \left( \frac{1}{t} \int_0^t f^*(s)\, ds + \int_t^\infty \left(1 + \log \frac{s}{t}\right) \left(1 + \log\left(1 + \log \frac{s}{t}\right)\right)^2 f^*(s) \frac{ds}{s} \right).$$

# Average operators

More generally, assuming that for a family of operators $\{T_\theta\}_\theta$ indexed in a probability measure space $(\Omega, \mathcal{A}, P)$

$$T_A f(x) = \int T_\theta f(x) dP(\theta), \qquad x \in \mathbb{R}^n,$$

is well defined, sublinear and that

$$\forall E \subseteq \mathbb{R}^n : \; \|T_A \chi_E\|_{L^{1,\infty}(u)} \leq C_u u(E) \implies T_A : L^1(u) \to L^{1,\infty}(u), \; C_n \|u\|_{A_1} C_u.$$

## Corollary

*If for an admissible function $\varphi$,*

$$T_\theta : L^1(u) \longrightarrow L^{1,\infty}(u), \qquad \forall u \in A_1, \qquad C\varphi(\|u\|_{A_1})$$

*then*

$$T_A : L^1(u) \longrightarrow L^{1,\infty}(u), \quad \forall u \in A_1, \quad C_1 \varphi(C_2||u||_{A_1})||u||_{A_1}(1 + \log ||u||_{A_1}).$$

## Corollary

*Under the previous conditions:*

$$(T_A f)^*(t) \leq C\left(\frac{1}{t}\int_0^t f^*(s)\,ds + \int_t^\infty \left(1+\log\left(1+\log\frac{s}{t}\right)\right)\varphi\left(1+\log\frac{s}{t}\right) f^*(s)\,\frac{ds}{s}\right).$$

¡Muchas gracias por vuestra atención!


Thanks for your attention!