# Recent developments in the theory of Fourier interpolation

Mateus Sousa
BCAM

XX Encuentros de Análisis Real y Complejo
Cartagena
2022

# The topics in the title

This presentation is based on:

I. Fourier uniqueness pairs of powers of integers (Joint with J.P.G. Ramos). To appear in *Journal of the European Mathematical Society*.

II. Perturbed interpolation formulae and applications (Joint with J. P. G. Ramos). Available at arXiv:2005.10337

# The topics in the title

This presentation is based on:

I. Fourier uniqueness pairs of powers of integers (Joint with J.P.G. Ramos). To appear in *Journal of the European Mathematical Society*.

II. Perturbed interpolation formulae and applications (Joint with J. P. G. Ramos). Available at arXiv:2005.10337

But some events may be changed for dramatization effects.

# The topics in the title

This presentation is based on:

I. Fourier uniqueness pairs of powers of integers (Joint with J.P.G. Ramos). To appear in *Journal of the European Mathematical Society*.

II. Perturbed interpolation formulae and applications (Joint with J. P. G. Ramos). Available at arXiv:2005.10337

But some events may be changed for dramatization effects.

**Interpolation formulas**

Sometimes a class of functions is nice enough for you to determine it just by sampling the functions over "small" sets.

# The topics in the title

This presentation is based on:

I. Fourier uniqueness pairs of powers of integers (Joint with J.P.G. Ramos). To appear in *Journal of the European Mathematical Society*.

II. Perturbed interpolation formulae and applications (Joint with J. P. G. Ramos). Available at arXiv:2005.10337

But some events may be changed for dramatization effects.

**Interpolation formulas**

Sometimes a class of functions is nice enough for you to determine it just by sampling the functions over ”small” sets.

**Perturbations**

Given an interpolation formula, how much can we twitch the interpolation nodes and still have similar properties?

**Theorem (Radchenko–Viazovska (2017))**

*There are functions $a_n \in \mathcal{S}(\mathbb{R})$ such that for any given even function $f : \mathbb{R} \to \mathbb{C}$ that belongs to the Schwartz class $\mathcal{S}(\mathbb{R})$ one has the following identity*

$$f(x) = \sum_{n=0}^{\infty} f(\sqrt{n}) a_n(x) + \sum_{n=0}^{\infty} \widehat{f}(\sqrt{n}) \widehat{a_n}(x),$$

*where the right-hand side converges absolutely.*

**Theorem (Cohn, Kumar, Miller, Radchenko, Viazovska (2019))**

*Let $(d, n_0)$ be either $(8, 1)$ or $(24, 2)$. There are functions $a_n, b_n \in \mathcal{S}(\mathbb{R})$ such that every $f \in \mathcal{S}_{rad}(\mathbb{R}^d)$ can be uniquely recovered by the sets of values*

$$\{f(\sqrt{2n}), f'(\sqrt{2n}), \widehat{f}(\sqrt{2n}), \widehat{f}'(\sqrt{2n})\},\ n \geq n_0,$$

*through the interpolation formula*

$$f(x) = \sum_{n \geq n_0} f(\sqrt{2n}) a_n(x) + \sum_{n \geq n_0} f'(\sqrt{2n}) b_n(x) + \sum_{n \geq n_0} \widehat{f}(\sqrt{2n}) \widehat{a_n}(x) + \sum_{n \geq n_0} \widehat{f}'(\sqrt{2n}) \widehat{b_n}(x),$$

*where the right-hand side converges absolutely.*

# Paley-Wiener Spaces

**Theorem (Shannon-Whittaker Interpolation - 1898)**

*Let $f$ be a function in $L^2(\mathbb{R})$ whose Fourier transform $\widehat{f}$ is supported on the interval $[-1/2, 1/2]$. Then*

$$f(x) = \sum_{k=-\infty}^{\infty} f(k) \frac{\sin \pi(x-k)}{\pi(x-k)},$$

*where convergence holds uniformly and in the $L^2(\mathbb{R})$ sense.*

**Theorem (Vaaler Interpolation - 1985)**

*Let $f \in L^2(\mathbb{R})$, and suppose that $\widehat{f}$ is supported on $[-1, 1]$. Then*

$$f(x) = \frac{\sin^2(\pi x)}{\pi^2} \sum_{k \in \mathbb{Z}} \left\{ \frac{f(k)}{(x-k)^2} + \frac{f'(k)}{x-k} \right\}.$$

*where convergence holds uniformly and in the $L^2(\mathbb{R})$ sense.*

# Questions

1. The Shannon-Whittaker result admits perturbations, namely, Kadec's 1/4-theorem. Does the Radchenko-Viazovska interpolation admits perturbations?

# Questions

1. The Shannon-Whittaker result admits perturbations, namely, Kadec's 1/4-theorem. Does the Radchenko-Viazovska interpolation admits perturbations?

# Questions

1. The Shannon-Whittaker result admits perturbations, namely, Kadec's 1/4-theorem. Does the Radchenko-Viazovska interpolation admits perturbations?

2. Vaaler's result also admits perturbations. Does the CKMRV interpolation also admits perturbations?

# Questions

1. The Shannon-Whittaker result admits perturbations, namely, Kadec's 1/4-theorem. Does the Radchenko-Viazovska interpolation admits perturbations?

2. Vaaler's result also admits perturbations. Does the CKMRV interpolation also admits perturbations?

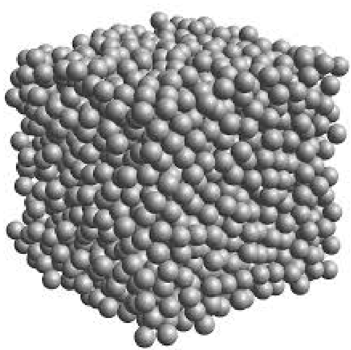
3. Is there a Hilbert space where these interpolation formulas follow just from orthogonality? One important remark is that the Schwartz class is not really necessary in the results, and it can be weakened to having some derivatives for the function, and also some derivatives for the Fourier transform.

# Sphere packing

## Definition (Sphere packing density)

*We call a sphere packing of* $\mathbb{R}^d$ *a collection* $\mathcal{P} = \{B_\gamma\}_{\gamma \in \Lambda}$ *in* $\mathbb{R}^d$ *of disjoint balls with the same radius, and we define the sphere packing density* $\Delta(\mathcal{P})$ *of* $\mathcal{P}$ *as the number*

$$\limsup_{r \to \infty} vol(\{\bigcup_{\gamma} B_\gamma\} \bigcap [-r/2, r/2]^d) r^{-d}$$



# Sphere packing

## Lattice

A lattice is a subgroup of the additive group $\mathbb{R}^d$ that is isomorphic to $\mathbb{Z}^d$ and also spans the real vector space $\mathbb{R}^d$. Alternatively, a lattice $\Lambda$ is any set of the form $\Lambda = A(\mathbb{Z}^d)$, where $A \in GL(d)$, and we define $\det(\Lambda) = \det(A)$. The dual lattice of $\Lambda$ is the set $\Lambda^* = (A^*)^{-1}(\mathbb{Z}^d)$

## Periodic packing density

A packing of the form $\mathcal{P} = \mathcal{P}(\Lambda, V) = \{B_{r/2}(x)\}_{x \in \Lambda + V}$, where $\Lambda \subset \mathbb{R}^d$ is a lattice, $V = \{v_1, \ldots, v_N\} \subset \mathbb{R}^d$ a set of vectors such that $v_i - v_j \notin \Lambda$ for $i \neq j$, and $r = r(\Lambda, V) = \min_{\lambda \in \Lambda, 1 \leq i \leq j \leq N} |\lambda - v_i - v_j|$, is called a periodic packing. The density of a periodic packing is given by

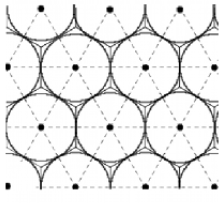
$$\Delta(\mathcal{P}) = N \frac{vol(B_d(r/2))}{det(\Lambda)}$$

## Proposition

$$\sup_{\mathcal{P} \text{ a packing in } \mathbb{R}^d} \Delta(\mathcal{P}) = \sup_{\mathcal{P} \text{ a periodic packing in } \mathbb{R}^d} \Delta(\mathcal{P})$$

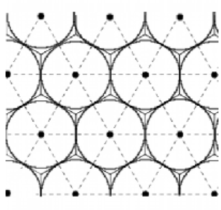
# Sphere packing

- László Fejes Tóth in 1940 proved that the honeycomb lattice solves the circle packing problem.



# Sphere packing

- László Fejes Tóth in 1940 proved that the honeycomb lattice solves the circle packing problem.



- In 1998, Thomas Hales, following an approach suggested by Tóth, announced a proof of the Kepler conjecture. This proof was computer assisted and the formal verification of the proof was completed in 2014.

# Sphere packing

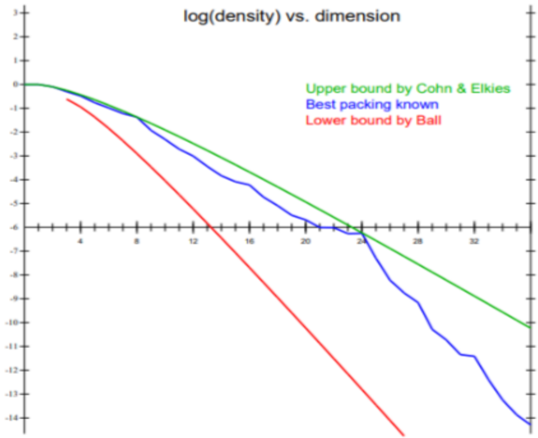
**Theorem (Cohn–Elkies–Gorbachev ($\sim$ 2000))**

*If $f : \mathbb{R}^d \to \mathbb{R}$ is a Schwartz function and $r$ is a positive scalar such that $f(0) = \widehat{f}(0) = 1$, $\widehat{f}(x) \geq 0$ for all $x \in \mathbb{R}^d$, and $f(u) \leq 0$ for $|u| \geq r$, then the density of a sphere packing in $\mathbb{R}^d$ is at most $vol(B_d(r/2))$*

# Sphere packing

**Theorem (Cohn–Elkies–Gorbachev ($\sim$ 2000))**

*If $f : \mathbb{R}^d \to \mathbb{R}$ is a Schwartz function and $r$ is a positive scalar such that $f(0) = \widehat{f}(0) = 1$, $\widehat{f}(x) \geq 0$ for all $x \in \mathbb{R}^d$, and $f(u) \leq 0$ for $|u| \geq r$, then the density of a sphere packing in $\mathbb{R}^d$ is at most $vol(B_d(r/2))$*



# Sphere packing

**Theorem (Cohn–Elkies–Gorbachev ($\sim$ 2000))**

*If $f : \mathbb{R}^d \to \mathbb{R}$ is a Schwartz function and $r$ is a positive scalar such that $f(0) = \widehat{f}(0) = 1$, $\widehat{f}(x) \geq 0$ for all $x \in \mathbb{R}^d$, and $f(u) \leq 0$ for $|u| \geq \rho$, then the density of a sphere packing in $\mathbb{R}^d$ is at most $vol(B_d(r/2))$*

# Sphere packing

**Theorem (Cohn–Elkies–Gorbachev ($\sim$ 2000))**

*If $f : \mathbb{R}^d \to \mathbb{R}$ is a Schwartz function and $r$ is a positive scalar such that $f(0) = \widehat{f}(0) = 1$, $\widehat{f}(x) \geq 0$ for all $x \in \mathbb{R}^d$, and $f(u) \leq 0$ for $|u| \geq \rho$, then the density of a sphere packing in $\mathbb{R}^d$ is at most $vol(B_d(r/2))$*

Assume $\mathcal{P}$ is a periodic packing given by $\Lambda$ and $V$. Then by Poisson summation we have

$$\frac{1}{det(\Lambda)} \sum_{k,j} \sum_{\lambda^* \in \Lambda^*} \widehat{f}(\lambda^*) e^{2\pi i (v_k - v_j)\cdot \lambda^*} = \sum_{k,j} \sum_{\lambda \in \Lambda} f(\lambda + v_k - v_j)$$

# Sphere packing

**Theorem (Cohn–Elkies–Gorbachev ($\sim$ 2000))**

*If $f : \mathbb{R}^d \to \mathbb{R}$ is a Schwartz function and $r$ is a positive scalar such that $f(0) = \widehat{f}(0) = 1$, $\widehat{f}(x) \geq 0$ for all $x \in \mathbb{R}^d$, and $f(u) \leq 0$ for $|u| \geq \rho$, then the density of a sphere packing in $\mathbb{R}^d$ is at most $vol(B_d(r/2))$*

Assume $\mathcal{P}$ is a periodic packing given by $\Lambda$ and $V$. Then by Poisson summation we have

$$\frac{1}{det(\Lambda)} \sum_{k,j} \sum_{\lambda^* \in \Lambda^*} \widehat{f}(\lambda^*) e^{2\pi i (v_k - v_j)\cdot \lambda^*} = \sum_{k,j} \sum_{\lambda \in \Lambda} f(\lambda + v_k - v_j)$$

By re-scaling if necessary, we can assume that $r(\Lambda, V) = \rho$. This means

$$N^2 \frac{1}{det(\Lambda)} \leq \frac{1}{det(\Lambda)} \sum_{\lambda^* \in \Lambda^*} \widehat{f}(\lambda^*) \left| \sum_j e^{2\pi i v_j \cdot \lambda^*} \right|^2 = \sum_{k,j} \sum_{\lambda \in \Lambda} f(\lambda + v_k - v_j) \leq N$$

By multiplying both sides by $vol(B_d(\rho/2))$ we have

$$\Delta(\mathcal{P}) \leq vol(B_d(\rho/2))$$

# Sphere packing

- If one wants to solve the sphere packing problem in a certain dimension via a lattice packing and the recently introduced bound, it follows from the proof of the Cohn–Elkies result that to find a extremal radius one has to look for functions that are radial and are zero over the lattice except for the origin, and also have derivatives equal to zero over the lattice, except for the first crossing.

# Sphere packing

- If one wants to solve the sphere packing problem in a certain dimension via a lattice packing and the recently introduced bound, it follows from the proof of the Cohn–Elkies result that to find a extremal radius one has to look for functions that are radial and are zero over the lattice except for the origin, and also have derivatives equal to zero over the lattice, except for the first crossing.
- In march 14th of 2016: Maryna Viazovska announced the solution to the sphere packing problem in dimension 8, via the Cohn–Elkies approach. A week later, together with Henry Cohn, Abhinav Kumar, Stephen D. Miller, Danylo Radchenko, Viazovska announced the solution the problem in dimension 24.

# Sphere packing

- If one wants to solve the sphere packing problem in a certain dimension via a lattice packing and the recently introduced bound, it follows from the proof of the Cohn–Elkies result that to find a extremal radius one has to look for functions that are radial and are zero over the lattice except for the origin, and also have derivatives equal to zero over the lattice, except for the first crossing.
- In march 14th of 2016: Maryna Viazovska announced the solution to the sphere packing problem in dimension 8, via the Cohn–Elkies approach. A week later, together with Henry Cohn, Abhinav Kumar, Stephen D. Miller, Danylo Radchenko, Viazovska announced the solution the problem in dimension 24.

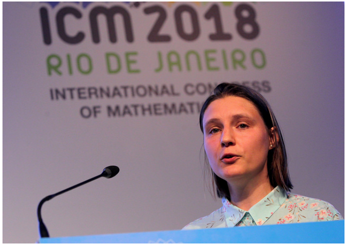



Figure: Maryna Viazovska talking at the ICM in Rio

# Sphere Packing

**Theorem (Viazovska - 2016)**

*There exists a radial Schwartz function* $f : \mathbb{R}^8 \to \mathbb{R}_+$ *such that*

(i) $f(0) = \widehat{f}(0) = 1$.

(ii) $\widehat{f}(u) \leq 0$ *for* $|u| \geq \sqrt{2}$.

*In particular, the* $E_8$ *lattice solves the sphere packing problem in* $\mathbb{R}^8$.

# Sphere Packing

**Theorem (Viazovska - 2016)**

*There exists a radial Schwartz function* $f : \mathbb{R}^8 \to \mathbb{R}_+$ *such that*

(i) $f(0) = \widehat{f}(0) = 1$.

(ii) $\widehat{f}(u) \leq 0$ *for* $|u| \geq \sqrt{2}$.

*In particular, the* $E_8$ *lattice solves the sphere packing problem in* $\mathbb{R}^8$.

**Theorem (Cohn, Kumar, Miller, Radchenko, Viazovska - 2016)**

*There exists a radial Schwartz function* $f : \mathbb{R}^{24} \to \mathbb{R}_+$ *such that*

(i) $f(0) = \widehat{f}(0) = 1$.

(ii) $\widehat{f}(u) \geq 0$ *for* $|u| \geq 1$.

*In particular, the Leech lattice solves the sphere packing problem for* $\mathbb{R}^{24}$.

# Sphere Packing

**Theorem (Viazovska - 2016)**

*There exists a radial Schwartz function* $f : \mathbb{R}^8 \to \mathbb{R}_+$ *such that*

(i) $f(0) = \widehat{f}(0) = 1$.

(ii) $\widehat{f}(u) \leq 0$ *for* $|u| \geq \sqrt{2}$.

*In particular, the* $E_8$ *lattice solves the sphere packing problem in* $\mathbb{R}^8$.

**Theorem (Cohn, Kumar, Miller, Radchenko, Viazovska - 2016)**

*There exists a radial Schwartz function* $f : \mathbb{R}^{24} \to \mathbb{R}_+$ *such that*

(i) $f(0) = \widehat{f}(0) = 1$.

(ii) $\widehat{f}(u) \geq 0$ *for* $|u| \geq 1$.

*In particular, the Leech lattice solves the sphere packing problem for* $\mathbb{R}^{24}$.

- These magical functions were built by ad hoc methods without the use of a more general interpolation formula. But the quest for such formulas remained.
- This led to the development of the Radchenko-Viazovska interpolation in 2017, and later the CKMRV interpolation in 2019.

# Some ideas behind the Radchenko-Viazovska interpolation

For $x \geq 0$, consider the linear functional $\phi_x$ on $\mathcal{S}_{even}$ given by

$$\phi_x(f) = f(x) - \sum_{n=0}^{\infty} a_n(x) f(\sqrt{n}) - \sum_{n=0}^{\infty} \widehat{a}_n(x) \widehat{f}(\sqrt{n})$$

We want to prove that $\phi_x(f) = 0$, and we may assume $f$ to be a compactly supported function in the Schwartz class, which is enough for density reasons. We may also assume that

$$f(x) = F(x^2)\, e^{-\pi x^2}$$

where $F$ is a $C^\infty$ function with compact support on $\mathbb{R}$. By the Fourier inversion formula we have

$$\begin{aligned} f(x) &= \int_{-\infty}^{\infty} \widehat{F}(s)\, e^{2\pi i s x^2 - \pi x^2}\, \mathrm{d}s \\ &= \int_{-\infty}^{\infty} \widehat{F}(s)\, \mathrm{e}_{i+2s}(x)\, \mathrm{d}s. \end{aligned}$$

Where $\mathrm{e}_\tau(x) = e^{\pi i \tau x^2}$. If the interpolation formula were true for $\mathrm{e}_\tau$ when $\mathrm{Im}(\tau) > 0$, then formally we would have

$$\phi_x(f) = \int_{-\infty}^{\infty} \widehat{F}(s)\, \phi_x(\mathrm{e}_{i+2s})\, \mathrm{d}s = 0.$$

# Some ideas behind the Radchenko-Viazovska interpolation

## Proposition

*For any* $\varepsilon \in \{+,-\}$ *and* $x \in \mathbb{R}$ *the function*

$$F_\varepsilon(\tau,x) = \sum_{n=0}^{\infty} b_n^\varepsilon(x) e^{i\pi n\tau} = \frac{1}{2}\int_{-1}^{1} K_\varepsilon(\tau,z) e^{i\pi x^2 z} \mathrm{d}z,$$

*where the contour is the semicircle in the upper half-plane that passes through* $-1$ *and* $1$*, admits an analytic continuation to* $\mathfrak{h} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$*. Moreover, the analytic continuation satisfies the functional equations*

$$F_\varepsilon(\tau,x) - F_\varepsilon(\tau+2,x) = 0,$$

$$F_\varepsilon(\tau,x) + \varepsilon(-i\tau)^{-1/2} F_\varepsilon\Big(-\frac{1}{\tau}, x\Big) = e^{i\pi\tau x^2} + \varepsilon(-i\tau)^{-1/2} e^{i\pi(-1/\tau)x^2}.$$

# Some ideas behind the Radchenko-Viazovska interpolation

## Proposition

*For any* $\varepsilon \in \{+,-\}$ *and* $x \in \mathbb{R}$ *the function*

$$F_\varepsilon(\tau, x) = \sum_{n=0}^{\infty} b_n^\varepsilon(x) e^{i\pi n\tau} = \frac{1}{2}\int_{-1}^{1} K_\varepsilon(\tau, z) e^{i\pi x^2 z} \mathrm{d}z,$$

*where the contour is the semicircle in the upper half-plane that passes through* $-1$ *and* $1$, *admits an analytic continuation to* $\mathfrak{h} = \{\tau \in \mathbb{C} : \mathrm{Im}(\tau) > 0\}$. *Moreover, the analytic continuation satisfies the functional equations*

$$F_\varepsilon(\tau, x) - F_\varepsilon(\tau + 2, x) = 0,$$
$$F_\varepsilon(\tau, x) + \varepsilon(-i\tau)^{-1/2} F_\varepsilon\Big(-\frac{1}{\tau}, x\Big) = e^{i\pi\tau x^2} + \varepsilon(-i\tau)^{-1/2} e^{i\pi(-1/\tau)x^2}.$$

$K_\varepsilon(\tau, z)$ is a 2-periodic meromorphic function in both variables and has a simple pole at $z = \tau$ with residue $\frac{1}{i\pi}$ for all $\tau \in \mathfrak{h}$ and satisfies the following identities for $K_\varepsilon$:

$$K_\varepsilon(\tau, -1/z) = \varepsilon(-iz)^{3/2} K_\varepsilon(\tau, z),$$
$$K_\varepsilon(-1/\tau, z) = -\varepsilon(-i\tau)^{1/2} K_\varepsilon(\tau, z).$$

# Modularity in the Proof

Indeed $K_\varepsilon$ is the generating function of certain weakly modular forms of weight 3/2, i.e,

$$K_\varepsilon(\tau, z) = \sum_{n=0}^{\infty} g_n^\varepsilon(z) e^{i\pi n\tau},$$

where

$$g_n^\varepsilon(z+2) = g_n^\varepsilon(z) \quad \text{and} \quad (-iz)^{-3/2} g_n^\varepsilon(-1/z) = \varepsilon g_n^\varepsilon(z).$$

# Modularity in the Proof

Indeed $K_\varepsilon$ is the generating function of certain weakly modular forms of weight 3/2, i.e,

$$K_\varepsilon(\tau, z) = \sum_{n=0}^{\infty} g_n^\varepsilon(z) e^{i\pi n\tau},$$

where

$$g_n^\varepsilon(z+2) = g_n^\varepsilon(z) \quad \text{and} \quad (-iz)^{-3/2} g_n^\varepsilon(-1/z) = \varepsilon g_n^\varepsilon(z).$$

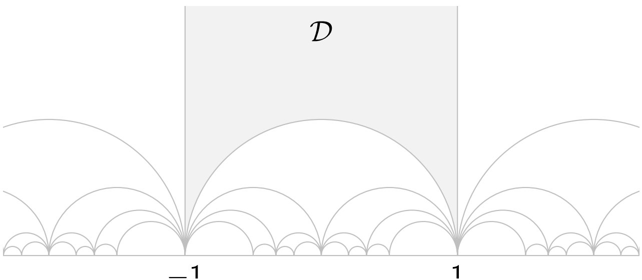



Figure: Due to the modularity, most of the work happens in $\mathcal{D}$, the fundamental domain for the group of the group generated by the transformations $s \mapsto s+2$ and $z \mapsto -1/s$.

# Perturbations

**Theorem (Ramos - S. 2020)**

*There is $\delta > 0$ so that, for each sequence of real numbers $\{\varepsilon_k\}_{k\geq 0}$ such that $\varepsilon_k \in (-1/2, 1/2), \varepsilon_0 = 0$, $\sup_{k\geq 0} |\varepsilon_k|(1+k)^{5/4}\log^3(1+k) < \delta$. The for every $k \geq 0$, there are sequences of functions $\{\theta_j\}_{j\geq 0}, \{\eta_j\}_{j\geq 0}$ with*

$$|\theta_j(x)| + |\eta_j(x)| + |\widehat{\theta}_j(x)| + |\widehat{\eta}_j(x)| \lesssim (1+j)^{\mathcal{O}(1)}(1+|x|)^{-10}$$

*such that for all $f \in \mathcal{S}_{even}(\mathbb{R})$*

$$f(x) = \sum_{j\geq 0}\left(f(\sqrt{j+\varepsilon_j})\theta_j(x) + \widehat{f}(\sqrt{j+\varepsilon_j})\eta_j(x)\right),$$

*where the convergence is uniform over $x \in \mathbb{R}$.*

# The main ingredient in the proofs

**Theorem (Ramos - S. 2020)**

*Let $b_n^{\pm} = a_n \pm \widehat{a_n}$, where $\{a_n\}_{n \geq 0}$ are the basis functions in the Radchenko-Viazovska interpolation. Then there is $c > 0$ such that for all positive integers $n \in \mathbb{N}$*

$$|b_n^{\pm}(x)| \lesssim n^{1/4} \log^{3/2}(1+n) e^{-c\frac{|x|}{\sqrt{n}}},$$

$$|(b_n^{\pm})'(x)| \lesssim n^{3/4} \log^{3/2}(1+n) e^{-c\frac{|x|}{\sqrt{n}}}.$$

# The main ingredient in the proofs

**Theorem (Ramos - S. 2020)**

*Let $b_n^{\pm} = a_n \pm \widehat{a_n}$, where $\{a_n\}_{n\geq 0}$ are the basis functions in the Radchenko-Viazovska interpolation. Then there is $c > 0$ such that for all positive integers $n \in \mathbb{N}$*

$$|b_n^{\pm}(x)| \lesssim n^{1/4} \log^{3/2}(1+n) e^{-c\frac{|x|}{\sqrt{n}}},$$

$$|(b_n^{\pm})'(x)| \lesssim n^{3/4} \log^{3/2}(1+n) e^{-c\frac{|x|}{\sqrt{n}}}.$$

1. Bounds of this kind were anavailable in the original papers. In fact to obtain the interpolation formulas they only need bounds of the form $|b_n^{\pm}(x)| = \mathrm{O}(n^{\alpha})$, for some $\alpha > 0$.

2. We performed an explicit computation of the best uniform constant bounding $|x|^k \, |b_n^{\pm}(x) + (b_n^{\pm})'(x)|$ in terms of $k$ and $n$. In order to obtain such a constant, we employ ideas from characterizations of Gelfand–Shilov spaces.

# Some comments and open questions

Recent progress on this theory includes

# Some comments and open questions

Recent progress on this theory includes

- Perturbed versions of the CKMRV (Ramos - S. 2020).

# Some comments and open questions

Recent progress on this theory includes

- Perturbed versions of the CKMRV (Ramos - S. 2020).
- Higher dimensional versions of the Radchenko-Viazovska formula (Stoller 2020), followed by perturbed versions (Ramos - Stoller 2021).

# Some comments and open questions

Recent progress on this theory includes

- Perturbed versions of the CKMRV (Ramos - S. 2020).
- Higher dimensional versions of the Radchenko-Viazovska formula (Stoller 2020), followed by perturbed versions (Ramos - Stoller 2021).
- Inteporlation formulas involving zeros of L-functions (Bondarenko - Radchenko - Seip 2020).

# Some comments and open questions

Recent progress on this theory includes

- Perturbed versions of the CKMRV (Ramos - S. 2020).
- Higher dimensional versions of the Radchenko-Viazovska formula (Stoller 2020), followed by perturbed versions (Ramos - Stoller 2021).
- Inteporlation formulas involving zeros of L-functions (Bondarenko - Radchenko - Seip 2020).
- Nyquist-type ratio for what sequences can satisfy the property that $f(a_j)$ and $\widehat{f}(b_k)$ can be used to recover $f$ (Kulikov 2020)

$$|\{j : |a_j| \leq R_1\}| + |\{k : |b_k| \leq R_2\}| \geq 4R_1R_2 - C\log^{2+\varepsilon}(4R_1R_2).$$

# Some comments and open questions

Recent progress on this theory includes

- Perturbed versions of the CKMRV (Ramos - S. 2020).
- Higher dimensional versions of the Radchenko-Viazovska formula (Stoller 2020), followed by perturbed versions (Ramos - Stoller 2021).
- Inteporlation formulas involving zeros of L-functions (Bondarenko - Radchenko - Seip 2020).
- Nyquist-type ratio for what sequences can satisfy the property that $f(a_j)$ and $\widehat{f}(b_k)$ can be used to recover $f$ (Kulikov 2020)

$$|\{j : |a_j| \leq R_1\}| + |\{k : |b_k| \leq R_2\}| \geq 4R_1R_2 - C\log^{2+\varepsilon}(4R_1R_2).$$

- Interpolation results involving pairs of sequences of the type $a_j = \left(\frac{p}{2}j\right)^{1/p}$ and $b_k = \left(\frac{q}{2}k\right)^{1/q}$ where $1/p + 1/q < 1$ (Kulikov - Nazarov - Sodin (unpublished)).

# Questions

## Maximal perturbations conjecture

Let $f \in \mathcal{S}_{even}(\mathbb{R})$ be a real function. Then there is $\theta > 0$ so that, if $|\varepsilon_i| < \theta$, $\forall\, i \in \mathbb{N}$, $f$ can be uniquely recovered from its values

$$f(0), f(\sqrt{1+\varepsilon_1}), f(\sqrt{2+\varepsilon_2}), \ldots,$$

together with the values of its Fourier transform

$$\widehat{f}(0), \widehat{f}(\sqrt{1+\varepsilon_1}), \widehat{f}(\sqrt{2+\varepsilon_2}), \ldots.$$

# Questions

## Maximal perturbations conjecture

Let $f \in \mathcal{S}_{even}(\mathbb{R})$ be a real function. Then there is $\theta > 0$ so that, if $|\varepsilon_i| < \theta$, $\forall\, i \in \mathbb{N}$, $f$ can be uniquely recovered from its values

$$f(0), f(\sqrt{1+\varepsilon_1}), f(\sqrt{2+\varepsilon_2}), \ldots,$$

together with the values of its Fourier transform

$$\widehat{f}(0), \widehat{f}(\sqrt{1+\varepsilon_1}), \widehat{f}(\sqrt{2+\varepsilon_2}), \ldots.$$

## Maximal perturbations conjecture - Weak form

Let $f \in \mathcal{S}_{even}(\mathbb{R})$ be a real function. Then, for each $a > 0$, there is $\delta > 0$ so that, if $|\varepsilon_i| \leq \delta k^{-a}$, then $f$ can be uniquely recovered from its values

$$f(0), f(\sqrt{1+\varepsilon_1}), f(\sqrt{2+\varepsilon_2}), \ldots,$$

together with the values of its Fourier transform

$$\widehat{f}(0), \widehat{f}(\sqrt{1+\varepsilon_1}), \widehat{f}(\sqrt{2+\varepsilon_2}), \ldots.$$

# Uniqueness pairs

**Question**

Given a collection $\mathcal{C}$ of functions $f : \mathbb{R} \to \mathbb{C}$, what conditions can we impose on two sets $A, \widehat{A} \subset \mathbb{R}$ to ensure that the only function $f \in \mathcal{C}$ such that $f(x) = 0$ for every $x \in A$ and $\widehat{f}(\xi) = 0$ for every $\xi \in \widehat{A}$ is the zero function?

# Uniqueness pairs

**Question**

Given a collection $\mathcal{C}$ of functions $f : \mathbb{R} \to \mathbb{C}$, what conditions can we impose on two sets $A, \widehat{A} \subset \mathbb{R}$ to ensure that the only function $f \in \mathcal{C}$ such that $f(x) = 0$ for every $x \in A$ and $\widehat{f}(\xi) = 0$ for every $\xi \in \widehat{A}$ is the zero function?

**Definition (Uniqueness pairs)**

*Given a collection $\mathcal{C}$ of functions $f : \mathbb{R} \to \mathbb{C}$, we say two sets $A, \widehat{A} \subset \mathbb{R}$ for a uniqueness pair for $\mathcal{C}$ if the only function $f \in \mathcal{C}$ such that $f(x) = 0$ for every $x \in A$ and $\widehat{f}(\xi) = 0$ for every $\xi \in \widehat{A}$ is the zero function.*

# Uniqueness pairs

**Question**

Given a collection $\mathcal{C}$ of functions $f : \mathbb{R} \to \mathbb{C}$, what conditions can we impose on two sets $A, \widehat{A} \subset \mathbb{R}$ to ensure that the only function $f \in \mathcal{C}$ such that $f(x) = 0$ for every $x \in A$ and $\widehat{f}(\xi) = 0$ for every $\xi \in \widehat{A}$ is the zero function?

**Definition (Uniqueness pairs)**

*Given a collection $\mathcal{C}$ of functions $f : \mathbb{R} \to \mathbb{C}$, we say two sets $A, \widehat{A} \subset \mathbb{R}$ for a uniqueness pair for $\mathcal{C}$ if the only function $f \in \mathcal{C}$ such that $f(x) = 0$ for every $x \in A$ and $\widehat{f}(\xi) = 0$ for every $\xi \in \widehat{A}$ is the zero function.*

**Question**

What uniqueness pairs can we think of? Clearly several of the interpolation results so far provide uniqueness pairs. What else?

# Uniqueness pairs

**Theorem (Ramos - S. 2019)**

*Let* $0 < \alpha, \beta < 1$ *and* $f \in \mathcal{S}(\mathbb{R})$. *Then*

(A) *If* $f(\pm \log(n+1))) = 0$ *and* $\widehat{f}(\pm n^{\alpha}) = 0$ *for every* $n \in \mathbb{N}$, *then* $f \equiv 0$.

(B) *There is a set* $A$ *such that when* $(\alpha, \beta) \in A$ *and* $f(\pm n^{\alpha}) = 0$ *and* $\widehat{f}(\pm n^{\beta}) = 0$ *for every* $n \in \mathbb{N}$, *then* $f \equiv 0$.

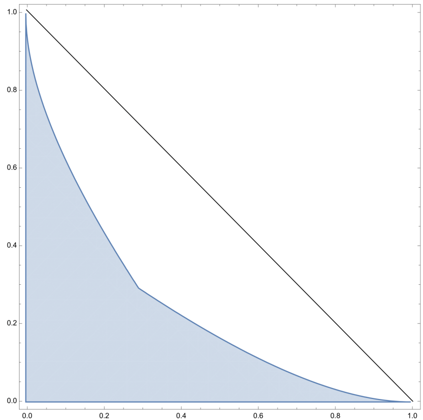
# Uniqueness pairs

**Theorem (Ramos - S. 2019)**

*Let $0 < \alpha, \beta < 1$ and $f \in \mathcal{S}(\mathbb{R})$. Then*

(A) *If $f(\pm \log(n+1))) = 0$ and $\widehat{f}(\pm n^{\alpha}) = 0$ for every $n \in \mathbb{N}$, then $f \equiv 0$.*

(B) *There is a set $A$ such that when $(\alpha, \beta) \in A$ and $f(\pm n^{\alpha}) = 0$ and $\widehat{f}(\pm n^{\beta}) = 0$ for every $n \in \mathbb{N}$, then $f \equiv 0$.*



# Questions and comments

- Any of the previous perturbation conjectures implies the following

**Uniqueness in the diagonal**

Let $\alpha \in (0, 1/2)$. There exists $c_\alpha > 0$ so that $\forall c < c_\alpha$, if $f \in \mathcal{S}_{even}(\mathbb{R})$ is a real function that vanishes together with its Fourier transform at $\pm c_\alpha n^\alpha$, then $f \equiv 0$.

Indeed from our result we can go up to $\alpha < 2/9$, which is worse than the range of our original result, which was $\alpha < 1 - \frac{\sqrt{2}}{2} = 0.29...$. Also there was no constant $c_\alpha$ in front of the vanishing points originally. Nevertheless, the fact one can obtain the uniqueness result from a perturbation one is interesting on itself.

- Any of the previous perturbation conjectures implies the following

**Uniqueness in the diagonal**

Let $\alpha \in (0, 1/2)$. There exists $c_\alpha > 0$ so that $\forall c < c_\alpha$, if $f \in \mathcal{S}_{even}(\mathbb{R})$ is a real function that vanishes together with its Fourier transform at $\pm c_\alpha n^\alpha$, then $f \equiv 0$.

Indeed from our result we can go up to $\alpha < 2/9$, which is worse than the range of our original result, which was $\alpha < 1 - \frac{\sqrt{2}}{2} = 0.29\ldots$. Also there was no constant $c_\alpha$ in front of the vanishing points originally. Nevertheless, the fact one can obtain the uniqueness result from a perturbation one is interesting on itself.

- The uniqueness conversation is now somewhat over, because F. Nazarov and M. Sodin have announced a essentially complete solution to this problem, which is yet unpublished.

# Muchas Gracias!