On the group generated by three and four anticonformal involutions of Riemann surfaces with maximal number of fixed curves

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ABSTRACT

Let X be a (compact) Riemann surface of genus $g > 1$. A symmetry α of X is an anticonformal involution of the Riemann surface X. The fixed point set of a symmetry α consists in a finite set of closed disjoint jordan curves in X, each one of these curves is called oval. The maximal number of ovals for three and four symmetries on surfaces of genus g was established by S. M. Natazon in [N1]. In this paper we determine the algebraic structure of the automorphisms groups generated by three and four symmetries having maximal number of ovals.

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1. Introduction

Let X be a (compact) Riemann surface of genus $g > 1$. A symmetry α of X is an anticonformal involution of the Riemann surface X. The fixed point set of a symmetry α consists in a finite set of closed disjoint jordan curves in X, each one of these curves is called oval. By a classical Theorem of Harnack ([H]) the number of ovals of a symmetry runs between 0 and $g + 1$.

The determination of the best bound for the total number of ovals for three and four symmetries for Riemann surfaces of a fixed genus has been obtained in [N1]. The

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study of the general case, for any finite set of symmetries, started in [N1] and [S] and was completely solved in [N2] and [G]

We are interested in the study of Riemann surfaces admitting a set of symmetries that have a maximal number of ovals between the Riemann surfaces with a given genus. If we consider Riemann surfaces with two or three symmetries, such maximal number of ovals is attained in the case that the symmetries commute (see [N1], [BCS] and Theorem 2.2 in the next Section). Our main result in this article is to show that, surprisingly, when we have more than three symmetries, the maximal number of ovals can be attained by non-commuting symmetries (Theorem 2.3).

In [N1], Sergei M. Natanzon proved that if $\{\alpha_i\}_{i=1,\ldots,k}$, $k = 3$ or 4, is a set of symmetries of a Riemann surface X of genus g then:

$$\text{if } k=3,\ |\alpha_1|+|\alpha_2|+|\alpha_3| \leq 2g+4 \qquad (1)$$

and

$$\text{if } k=4,\ |\alpha_1|+|\alpha_2|+|\alpha_3|+|\alpha_4| \leq 2g+8 \qquad (2).$$

Where $|\alpha_i|$ means the number of ovals of the symmetry α_i $(i = 1, 2, 3, 4)$ (Theorem 3.1 in [N1]). Moreover, if G is the group of automorphims generated by $\{\alpha_i\}_{i=1,\ldots,k}$, he established that:

1. For any odd $g > 1$ there exists a Riemann surface of genus g admitting three symmetries that satisfy the equality in (1) and with $G \backsimeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.
2. For each $g \geq 5$ such that $g \equiv 1 \bmod 4$ there exists a Riemann surface of genus g admitting four symmetries that satisfy the equality in (2) and with $G \backsimeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

In this article we are going to describe completely the groups generated by symmetries satisfying the equalities in (1) and (2).

More precisely we shall prove:

- If three symmetries $\{\alpha_1\}_{i=1,2,3}$ satisfy $|\alpha_1|+|\alpha_2|+|\alpha_3| = 2g+4$ then $\langle\alpha_1, \alpha_2, \alpha_3\rangle \backsimeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (Theorem 2.2)
- If four symmetries $\{\alpha_i\}_{i=1,\ldots,4}$ satisfy $|\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4| = 2g + 8$ then $\langle\alpha_1, \alpha_2, \alpha_3, \alpha_4\rangle \backsimeq \mathbb{D}_n \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$ (Theorem 2.3), where $\mathbb{D}_n$ is a dihedral group of $2n$ elements.

2. Groups generated by three and four symmetries with maximal number of ovals

In the case of surfaces with two symmetries, we have the following result of [BCS] and [N1]:

Proposition 2.1 (Corollary 3 in [BCS]) *Let X be a Riemann surface of genus $g > 1$ and α_1, α_2 be two non-commuting symmetries of X, then $|\alpha_1| + |\alpha_2| \leq g + 2$.*

As a consequence, if X has two symmetries α_1, α_2 such that $|\alpha_1| + |\alpha_2| = 2g + 2$ (i. e. the maximal number) then $\langle \alpha_1, \alpha_2 \rangle \backsimeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

For three symmetries with maximal number of ovals we have the following result:

Theorem 2.2 *Let X be a Riemann surface of genus $g > 1$, admitting three symmetries $\alpha_1, \alpha_2, \alpha_3$ with $|\alpha_1| + |\alpha_2| + |\alpha_3| = 2g + 4$, then $\langle \alpha_1, \alpha_2, \alpha_3 \rangle \backsimeq \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

Proof. We need only to proof that the three symmetries $\alpha_1, \alpha_2, \alpha_3$ commute. If α_i, α_j do not commute for some $i \neq j$ then $|\alpha_i| + |\alpha_j| \leq g + 2$ and by the Harnack Theorem $|\alpha_k| \leq g + 1$, for $i \neq k \neq j$. Hence $|\alpha_1| + |\alpha_2| + |\alpha_3| \leq g + 1 + g + 2 \leq 2g + 3$ that contradicts the hypotesis $|\alpha_1| + |\alpha_2| + |\alpha_3| = 2g + 4$. □

Theorem 2.3 *Let X be a Riemann surface of genus $g > 1$ admitting the symmetries $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with $|\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4| = 2g + 8$, then $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \backsimeq \mathbb{D}_n \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$. Moreover for every integer $n \geq 2$, if $g \geq 2n + 1$ and $g \equiv 1 \mod 2n$ then there is a Riemann surface X_g of genus g such that X_g has four symmetries $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with $|\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4| = 2g + 8$ and $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \backsimeq \mathbb{D}_n \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.*

Proof. Assume that $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are four symmetries of a Riemann surface X and $|\alpha_1| \geq |\alpha_2| \geq |\alpha_3| \geq |\alpha_4|$. In the proof of the Theorem 3.1 in [N1] it is shown that $\alpha_1\alpha_2 = \alpha_2\alpha_1$, $\alpha_2\alpha_3 = \alpha_3\alpha_2$, $\alpha_1\alpha_4 = \alpha_4\alpha_1$, $\alpha_1\alpha_3 = \alpha_3\alpha_1$ if $|\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4| > 2g + 6$. Now we shall show that $\alpha_2\alpha_4 = \alpha_4\alpha_2$ if $|\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4| = 2g + 8$ (and in consequence $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \backsimeq \mathbb{D}_n \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$).

Let Γ be a Fuchsian group uniformizing X, i.e. $X = D/\Gamma$ where D is the complex unit disc, and let Γ' be a non-euclidean crystallographic (NEC) group containing Γ and uniformizing the orbifold $X/\langle \alpha_2, \alpha_4 \rangle$. For the terminology, notation and basic results of NEC group see [BEGG]. In the proof of Theorem 3.1 of [N1] it is shown that if $\alpha_2\alpha_4 \neq \alpha_4\alpha_2$ the order of $\langle \alpha_2, \alpha_4 \rangle$ must be 8 and there is no link periods in the signature of Γ' (i.e. there is no corner points in the orbifold $X/\langle \alpha_2, \alpha_4 \rangle$).

Assume that Γ' has signature $(g'; \pm; [m_1, ..., m_r]; \{(-), \overset{k}{\ldots}, (-)\})$. Since $\#\langle \alpha_2, \alpha_4 \rangle = 8$, by the Riemann-Hurwitz formula we have:

$$\frac{2g - 2}{\eta g' - 2 + r + k + \sum(1 + \frac{1}{m_i})} = 8, \text{ where } \eta = 1 \text{ or } 2.$$

Hence

$$k \leq \tfrac{g+7}{4}.$$

By Theorem 2 in [BCS], $|\alpha_2| + |\alpha_4| \leq 2k$, thus $|\alpha_2| + |\alpha_4| \leq \frac{g+7}{2}$.

Since $|\alpha_1| + |\alpha_2| + |\alpha_3| \leq 2g + 4$, then $|\alpha_4| \geq 4$ and $|\alpha_3| \leq |\alpha_2| \leq \frac{g-1}{2}$. Hence $|\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4| \leq \frac{g+7}{2} + \frac{g-1}{2} + g + 1 = 2g + 4$ that is a contradiction. Consequently $\alpha_2\alpha_4 = \alpha_4\alpha_2$ and $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \cong \mathbb{D}_n \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Given an integer $n \geq 2$ and $g \geq 2n+1$, $g \equiv 1 \mod 2n$, we shall construct a Riemann surface X_g such that X_g admits four symmetries $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with $|\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4| = 2g+8$ and $\langle \alpha_1, \alpha_2, \alpha_3, \alpha_4 \rangle \cong \mathbb{D}_n \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Set $\delta = \frac{g-1}{n} + 4$ and let Δ be an NEC group with signature $(0; +; [-]; \{(2, \overset{\delta}{\ldots}, 2)\}$. Let $\left\langle c_0, ..., c_\delta, e : c_j^2 = (c_i c_{i+1})^2 = e c_0 e^{-1} c_\delta = 1 \right\rangle$ be a canonical presentation of Δ.

We define the epimorphism

$$\theta : \Delta \to \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{D}_n = \left\langle x, y, z, t : \begin{array}{l} x^2 = y^2 = z^2 = t^2 = (zt)^n = 1 \\ xy = yx,\ xt = tx,\ xz = zx \\ yt = ty,\ yz = zy \end{array} \right\rangle,$$

by:

$\theta(c_0) = x$, $\theta(c_1) = y$, $\theta(c_2) = x$, $\theta(c_3) = z$, $\theta(c_4) = x$, $\theta(c_5) = t$, $\theta(c_{2i}) = x$, $\theta(c_{2i+1}) = y$, $3 \leq i \leq \delta/2$, $\theta(e) = 1$.

The surface $X_g = D/\ker\theta$ has four symmetries $\alpha_1, \alpha_2, \alpha_3, \alpha_4$, such that $X_g/\langle\alpha_1\rangle$ is uniformized by $\theta^{-1}(\langle x \rangle)$, $X_g/\langle\alpha_2\rangle$ is uniformized by $\theta^{-1}(\langle y \rangle)$, $X_g/\langle\alpha_3\rangle$ is uniformized by $\theta^{-1}(\langle z \rangle)$, $X_g/\langle\alpha_4\rangle$ is uniformized by $\theta^{-1}(\langle t \rangle)$.

Since $\theta^{-1}(\langle x \rangle)$, $\theta^{-1}(\langle y \rangle)$ and $\theta^{-1}(\langle z, t \rangle)$ are normal subgroups of Δ then, by Section 2.3 in [BEGG], the NEC group $\theta^{-1}(\langle x \rangle)$ has exactly $g+1$ empty period cycles in his signature, $\theta^{-1}(\langle y \rangle)$ has $g-1$ empty period cycles and $\theta^{-1}(\langle z, t \rangle)$ has four. The hyperbolic generators associate to each period cycle of $\theta^{-1}(\langle z, t \rangle)$ is sent to the identity by θ, thus by Theorem 2 in [BCS], the number of empty period cycles in $\theta^{-1}(\langle z \rangle)$ plus the number of period cycles in $\theta^{-1}(\langle t \rangle)$ is 8. Hence $|\alpha_1| + |\alpha_2| + |\alpha_3| + |\alpha_4| = 2g+8$. □

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