Jesús M. Ruiz, Dept. Geometría y Topología
Fac. Ciencias Matemáticas, Univ. Complutense de Madrid, Spain.

Degree theory, with applications and history

This course was the offspring of the teaching of Degree Theory to many generations of Ph.D. students at the Universidad Complutense de Madrid. That experience showed possible to give a quite condensed course that exhibits the power of the theory to reach very important theorems that usually are obtained by quite other means, and we designed a course on Degree Theory with special emphasis on applications to topology.

By the invitation of Prof. Fabrizio Broglia, the course took place at the Pisa Ph.D. School in 2006, for three months at the pace of four hours per week.

I. The course. The final goal are several very important and deep results of General Topology by the use of Degree Theory. The main tool being the so-called *Euclidean Degree Theory.* Thus the course starts with the detailed presentation of this theory. This can require, on demand from the potential audience, a previous up-dating of the basics of Differential Topology:

1. *Manifolds:* Analysis on manifolds, diffeotopies and orientation.
2. *Immersions, submersions and complete intersections:* Sard-Brown theorem.
3. *Transversality:* Density theorem.
4. *Approximation:* Spaces of mappings, homotopy, approximation and trasversality.

Then a recall of the classical degree theory for smooth mappings of compact manifolds will be in order too, as motivation and preparation for later use:

5. *Degree of smooth mappings of compact manifolds.*

Afterwards, the more elaborated euclidean degree theory can be properly defined:

6. *Degree of mappings of open sets of the euclidean space:* Definition and homotopy invariance, addition formula.

Then, we have the first characteristic result:

7. **Borsuk-Ulam Theorem.** *Let $D \subset \mathbb{R}^n$ be a bounded symmetric open set, and $f : X \to \mathbb{R}^n$ a continuous function defined on its boundary X. Then:*
   (a) If f is odd, then so is its degree (hence, $\neq 0$).
   (b) If f is even, then so is its degree (and $\neq 0$ for n odd).

Right from the Borsuk-Ulam theorem, we will deduce a very important general topology theorem:

8. **Invariance of Domain.** *(1) Let $f : U \to \mathbb{R}^n$ be a continuous map on an open set $U \subset \mathbb{R}^n$. If f is* locally injective, *then f is* open.
   (2) Let $f : A \to B$ a homeomorphism of two arbitrary sets $A, B \subset \mathbb{R}^n$. Then f maps interior *points of A in $\mathbb{R}^n$ to* interior *points of B in $\mathbb{R}^n$.*
   (3) Two manifolds which are homeomorphic *have the same dimension.*

Back to degree theory, we obtain the second essential fact:

9. *Multiplication Formula:* This gives a way to compute the degree of a composition of continuous mappings. Is the key technical result of the theory, and gives way to many computations in the flavour of the standard degree theory of smooth mappings among compact manifolds.

Once degree theory is well established, we turn to:

10. *Applications:* Here we will mainly prove two famous theorems:

    Jordan-Brouwer Separation Theorem. *Let $K, L \subset \mathbb{R}^n$, $n \geq 2$, be two compact sets. If they are homeomorphic, then the number of connected components of their complements are the same.*
    (As a bonus, from this we get another proof of the invariance of domain.)

    Hopf Homotopy Theorem. *Let $X \subset \mathbb{R}^n$ be a compact set that separates $\mathbb{R}^n$ into two connected components. Then two continuous mappings $X \to \mathbb{R}^n \setminus \{0\}$ are homotopic if and only if both have the same degree.*

And all these ten sections include comments on the historical development development of the theory and its *who is who.*

II. Background and bibliography. As may be apparent from the above program, the prerequisites to follow such a course are analysis and topology on euclidean spaces, and, at least to some elementary extent, on embedded manifolds. However, one purpose of ours is to make everything as accesible as possible, and convince the audience these are not that difficult mathematics. In fact, for the initial steps there are two very good references to resort to:

[1] J. Milnor: *Topology from the differentiable viewpoint,* University Press of Virginia, Charlottesville 1965.

[2] V. Guillemin, S. Pollack: *Differential topology,* Prentice Hall, New Jersey 1974.

The first book is a jewel, whose reading should be compulsory for all mathematicians. The second one is also an extraordinary reference, for anyone interested in Differential Topology. Both (specially the second) go far beyond our needs, but were the inspiration to write the following humbler booklet:

[3] E. Outerelo, J.M. Ruiz: *Topología Diferencial,* Addison-Wesley, Madrid 1998.

This covers quite exactly what is to be reviewed in I.1-5. Then, for I.6 we resort to:

[4] N.G. Lloyd: *Degree Theory,* Cambridge University Press, Cambridge 1978.

[5] K. Deimling: *Nonlinear Functional Analysis,* Springer, Berlin 1985.

As additional bibliography, we cannot help mention:

[6] R. Abraham, J. Robbin: *Transversal mappings and flows*, Benjamin, New York 1967.

[7] E. Lima: *Introdução a topologia diferencial*, IMPA, Rio de Janeiro 1961.

The Abraham-Robbin book is a classic (with an appendix on real semialgebraic sets, a remarkable feature!). The second is a wonderful book, by any measure a highly recommendable reading.

All in all, the course was largely selfcontained.

III. The actual contents of every single lesson given at Pisa were recorded and we include the *Libretto* in the following pages.

ISTITUTO NAZIONALE DI ALTA MATEMATICA

CITTÀ UNIVERSITARIA - ROMA

LIBRETTO DELLE LEZIONI

DI Degree theory, with Applications and History

DETTATE DAL PROF. Jesus M. Ruiz

NELL'ANNO ACCADEMICO 2005 - 2006

AVVERTENZE

1. - Per le lezioni non impartite si prega di non numerare la casella.
2. - Al termine del corso il professore deve restituire il libretto alla Segreteria dell'Istituto.

Argomento della lezione N. 1	Argomento della lezione N. 2
Presentation of the course. Prerequisites and goals. Main results and applications	Differentiable and smooth mappings on arbitrary sets of affine space by local extension. Differentiable partitions of unity. Bump and Urysohn's functions. Tietze extension
Addì, 11 - IV 2006 11:00 → 12:00 *Firma dell'Insegnante*	Addì, 11 - IV 2006 12:00 → 13:00 *Firma dell'Insegnante*

Argomento della lezione N. 3	Argomento della lezione N. 4
Local diffeomorphisms and manifolds. Parametrizations, local coordinate systems, dimension. Local equations. Manifolds with boundary. Boundary invariance	Tangent spaces. Moving frames of partial derivatives. Local equations and tangency. Derivatives of differentiable mappings of manifolds. Curve germs.
Addì, 13 - IV 2006 15:00 → 16:00 *Firma dell'Insegnante*	Addì, 13 - IV 2006 16:00 → 17:00 *Firma dell'Insegnante*

Argomento della lezione N. 5	Argomento della lezione N. 6
Critical points and regular values. Inverse images. The Sard-Brown theorem. Classification of differentiable curves.	Retractions. Globalization of local retractions. Characterization of manifolds by local retractibility. Tubes and retractions via normal spaces.
Addì, 27 - IV 2006 15:00 → 16:00 *Firma dell'Insegnante*	Addì, 27 - IV 2006 16:00 → 17:00 *Firma dell'Insegnante*

Argomento della lezione N. 7

Proper mappings. Homotopy and proper homotopy. Approximation theorem. Approximation guarantees homotopy. Smoothening of continuous homotopy

Addì, 2-V 2006

11:00 → 12:00

Firma dell'Insegnante

Argomento della lezione N. 8

Diffeotopies. Local existence. Globalization. Moving points around in manifolds. Moving around families of points.

Addì, 2-V 2006

12:00 → 13:00

Firma dell'Insegnante

Argomento della lezione N. 9

Orientation of differentiable manifolds. Preserving orientation by local diffeomorphisms. Diffeotopies and orientation. Orientation of hypersurfaces, inverse images and boundaries

Addì, 4-V 2006

15:00 → 16:00

Firma dell'Insegnante

Argomento della lezione N. 10

The degree of a smooth mapping at a regular value. Independence of the value. Boundary theorem and homotopy invariance. Composite mappings. Fundamental theorem of Algebra

Addì, 4-V 2006

16:00 → 17:00

Firma dell'Insegnante

Argomento della lezione N. 11

de Rham cohomology. Stokes theorem and highest order cohomology group. Degree revisited. The Gauss-Bonnet statement as a degree computation: Gauss mapping and curvature

Addì, 9-V 2006

11:00 → 12:00

Firma dell'Insegnante

Argomento della lezione N. 12

The degree of a continuous mapping by (approximation and) homotopy. Degrees of mappings of spheres. Counterexamples for toruses

Addì, 9-V 2006

12:00 → 13:00

Firma dell'Insegnante

Argomento della lezione N. 13

Mod 2 degree theory. Jordan Separation theorem for closed (not necessarily compact) hypersurfaces. Global equations and orientability

Addì, 11-V 2006

15:00 → 16:00

Firma dell'Insegnante

Argomento della lezione N. 14

Brouwer theorems: no boundary is a proper retraction; every continuous mapping of a closed ball has fixed points, the hedgehog theorem on vector tangent fields

Addì, 11-V 2006

16:00 → 17:00

Firma dell'Insegnante

Argomento della lezione N. 15

Euclidean degree of a smooth mapping at a regular value. Degree of a continuous mapping through polynomial approximation

Addì, 16-V 2006

11:00 → 12:00

Firma dell'Insegnante

Argomento della lezione N. 16

Homotopy invariance of the Euclidean degree. Boundary theorem. Axiomatization: existence of solutions and additivity

Addì, 16-V 2006

12:00 → 13:00

Firma dell'Insegnante

Argomento della lezione N. 17

Euclidean degree of a differentiable mapping. First applications: Gauss proof of the Fundamental theorem of Algebra, refinements of Brouwer's fixed point theorem.

Addì, 18-V 2006

15:00 → 16:00

Firma dell'Insegnante

Argomento della lezione N. 18

Winding numbers. Homotopy invariance and boundary theorems. Representation of the Brouwer-Kronecker degree as a winding number for compact hypersurfaces.

Addì, 18-V 2006

16:00 → 17:00

Firma dell'Insegnante

Argomento della lezione N. 19	Argomento della lezione N. 20
Odd and even functions. Extension lemmas with non-vanishing conditions. The Borsuk-Ulam theorem. The Hirsch converse.	Fixed point theorems improved. The Invariance of Domain. Invariance of interiors. Topological invariance of dimension
Addì, 23-V 2006 11:00 → 12:00 *Firma dell'Insegnante*	Addì, 23-V 2006 12:00 → 13:00 *Firma dell'Insegnante*

Argomento della lezione N. 21	Argomento della lezione N. 22
Euclidean degree of composite mappings. Necessary conditions and the corresponding multiplication formula. Consistency of the formula.	Proof of the multiplication formula. Approximation data with preservation of conditions. Additivity computations and conclusion
Addì, 25-V 2006 15:00 → 16:00 *Firma dell'Insegnante*	Addì, 25-V 2006 16:00 → 17:00 *Firma dell'Insegnante*

Argomento della lezione N. 23	Argomento della lezione N. 24
The Jordan separation theorem: compact homeomorphic subsets disconnect affine space in the same way. Proof by the multiplication formula.	Topological consequences of the Jordan theorem. Formulation and proof for compact subsets of a sphere. Generalization for closed subsets of affine space. A new proof of Invariance of Domain.
Addì, 30-V 2006 11:00 → 12:00 *Firma dell'Insegnante*	Addì, 30-V 2006 12:00 → 13:00 *Firma dell'Insegnante*

Argomento della lezione N. 25

Construction of mappings into spheres. Antipodal extension, linearization and normalization. The Hopf theorem for the Brouwer-Kronecker degree.

Addì, 1-VI 2006
15:00 → 16:00

Firma dell'Insegnante

Argomento della lezione N. 26

Normal homotopic form of a mapping of degree d into a sphere: orientable boundaryless compact case. The other cases: boundary, non-compact, non-orientable.

Addì, 1-VI 2006
16:00 → 17:00

Firma dell'Insegnante

Argomento della lezione N. 27

Homotopy classification of continuous mappings by their winding numbers. Proof by reduction to the computation of suitable Brouwer-Kronecker degrees on spheres.

Addì, 6-VI 2006
11:00 → 12:00

Firma dell'Insegnante

Argomento della lezione N. 28

Cohomotopy groups of compact subsets that disconnect tamely affine spaces. The Borsuk homotopy treatment of the continuous extension problem. The converse boundary theorem.

Addì, 6-VI 2006
12:00 → 13:00

Firma dell'Insegnante

Argomento della lezione N. 29

Continuous mappings between spheres of different dimensions. The Hopf invariant as a link number. Consistency and homotopy invariance

Addì, 7-VI 2006
11:00 → 12:00

Firma dell'Insegnante

Argomento della lezione N. 30

Hopf invariant of a composite mapping: multiplication times degree. The Hopf fibration of a sphere of low dimension $m = 2, 4, 8$. Computation of its Hopf invariant for $m = 2$.

Addì, 7-VI 2006
12:00 → 13:00

Firma dell'Insegnante

Argomento della lezione N. 31	Argomento della lezione N. 32
Smooth tangent fields. Zeros and non degenerated zeros. Index of a tangent field at a non-degenerated zero. Impulsors, attractors and saddles.	Splitting of a degenerated zero of a tangent field. Index of a tangent field at a degenerated zero: splittings versus Brouwer-Kronecker degree. Examples.
Addì, 8-VI 2006. 15:00→16:00 Firma dell'Insegnante	Addì, 8-VI 2006. 16:00→17:00 Firma dell'Insegnante
Argomento della lezione N. 33	Argomento della lezione N. 34
Gradient tangent fields on manifolds. Critical points of a function and singularities of its gradient. Non-degeneracy: the Hessian. Index of a non-degenerated critical point.	Morse functions. Total index of the associated gradient. Linear forms that define Morse functions (i) on hypersurfaces, a residual set (ii) on arbitrary manifolds, a dense set.
Addì, 13-VI 2006 11:00→12:00 Firma dell'Insegnante	Addì, 13-VI 2006 12:00→13:00 Firma dell'Insegnante
Argomento della lezione N. 35	Argomento della lezione N. 36
The Poincaré-Hopf theorem. Computation of the total index of a tangent field through the degree of a normal field on a tube. Vanishing in odd dimension	Computation of the Euler characteristic: (i) via triangulations (ii) via flows, case of compact orientable surfaces. Morse inequalities. The Gauss-Bonnet formula.
Addì, 15-VI 2006 15:00→16:00 Firma dell'Insegnante	Addì, 15-VI 2006 16:00→17:00 Firma dell'Insegnante