On the Dirichlet boundary problem for quasi–degenerate elliptic linear equations

Gregorio Díaz*

Departamento de Matemática Aplicada
Universidad Complutense de Madrid
28040–Madrid, Spain
g_diaz@mat.ucm.es

Al profesor Enrique Outerelo.

ABSTRACT

Motivated by the structure of the Dynamics Programming Equations relative to Controlled Diffusions we study the Dirichlet value boundary problem governed by quasi–degenerate elliptic partial differential equations of second order. Interior properties of the solution involved to the Maximum Principle are also considered in the paper.

2000 Mathematics Subject Classification: 35B50, 35B37, 49L25
Key words: Maximum Principle, viscosity solutions, semi–convexity

1. Introduction

This paper is strongly motivated by some partial differential equations arising in the study of questions of the Stochastic Optimal Control theory (see [6], [11], [16], [19], [21]). In order to simplify, we present the ideas on the one-control case. On a probability space $(\mathcal{O}, \mathcal{F}, \mathbb{P})$ we consider a M-dimensional Brownian Motion $\{\mathcal{B}_t\}_{t\geq 0}$ and the relative *filtration*, $\{\mathcal{F}_t\}_{t\geq 0} \subset \mathcal{F}$, involved to $\{\mathcal{B}_t\}_{t\geq 0}$. So that for each couple of bounded and Lipschitz continuous functions $\mathbf{a} : \mathbb{R}^{\mathrm{N}} \to \mathbb{R}^{\mathrm{N}}$ and $\sigma : \mathbb{R}^{\mathrm{N}} \to \mathcal{M}(\mathrm{N}\times\mathrm{M};\mathbb{R})$ we consider the unique, in the probability sense, *stochastic trajectories* given by

$$\text{(SDS)} \quad \begin{cases} \mathrm{d}\mathcal{X}_t^x &= -\mathbf{a}(\mathcal{X}_t^x)\mathrm{d}t + \sqrt{2}\sigma(\mathcal{X}_t^x)\mathrm{d}\mathcal{B}_t, \quad t > 0, \\ \mathcal{X}_0^x &= x \in \mathbb{R}^{\mathrm{N}}. \end{cases}$$

*Partially supported by the project REN2003-0223 of the DGISGPI (Spain)

Involved to two continuous functions $f,\ a_0\ : \Omega \to \mathbb{R}$, one defines the *Feynman-Kac function*

$$\left\{ \begin{array}{ll} \mathbf{U}(x) = & \mathbb{E}_x \left[\displaystyle\int_0^{\tau_x} f(\mathcal{X}_t^x) \exp\left(- \int_0^t a_0(\mathcal{X}_s^x) ds \right) \mathrm{d}t \right. \\ & \left. + \varphi\left(\mathcal{X}_{\tau_x}^x\right) \exp\left(- \displaystyle\int_0^{\tau_x} a_0(\mathcal{X}_s^x) ds \right) \right], \quad x \in \overline{\Omega}, \end{array} \right. \tag{1.1}$$

where $\mathbb{E}_x$ stands for the conditional expectation with respect to the event $\{\mathcal{X}_0^x = x\}$ and the stochastic process

$$\tau_x \doteq \inf\{t \geq 0 : \mathcal{X}_t^x \notin \Omega\}$$

is the *first exit time* of the trajectory $\mathcal{X}_t^x$ from an open Ω of $\mathbb{R}^{\mathrm{N}}$. Classical arguments lead to think $\mathbf{U}$ as being a solution of a partial differential equation and a Dirichlet boundary condition

$$u = \varphi \quad \text{on } \Omega. \tag{1.2}$$

Indeed, it is proved that $\mathbf{U}$ verifies, in some sense to be precised, the equation

$$\mathcal{L}u + a_0 u = f \quad \text{in } \Omega \tag{1.3}$$

where

$$\mathcal{L}u \doteq -\text{trace } \left(\mathcal{A}(x) \cdot \mathrm{D}^2 u\right) + \langle \mathbf{a}(x), \mathrm{D}u \rangle.$$

Here $\mathcal{A}(\cdot) \doteq (\sigma\sigma^t)(\cdot)$ where σ^t is the transpose matrix of σ.

Certainly, some assumptions on the data must be required in order to prove that $\mathbf{U}$ satisfies (1.3)–(1.2) in a classical sense. An almost "unavoidable" hypothesis for that goal is the *non-degeneracy* of the diffusion: $\sigma(\cdot)$ must be a $\mathrm{N} \times \mathrm{N}$ matrix with $\sigma(\cdot) \geq \theta \mathrm{I}_{\mathrm{N}}$, for some $\theta > 0$ on $\overline{\Omega}$. Unfortunately, the condition does not hold in some important examples of the applications and consequently the regularity (even the continuity) the Feynman–Kac function can not be guaranteed. The viscosity solutions notion is adequate in order to remove the *non-degeneracy* hypothesis. We send to the monograph by M.G. Crandall, H. Ishii and P.L. Lions [9] to understand how semi-continuous functions can solve (1.3) in that framework. We note that linearity of the operator $\mathcal{L}$ can be lost for this kind of solutions. In Section 2 we recall this notion of viscosity solutions.

The eventual discontinuity interferes strongly with the fact that $\mathbf{U}$ does not satisfy (1.2). Furthermore, in [3] one constructs an example in which $\mathbf{U}$ is continuous on $\overline{\Omega}$ but (1.2) does not hold. This is the reason for which (1.2) is generalized to a boundary condition where from the control point of view the eventual behaviors of the dynamical system and the strategy of the controller must be considered. The *generalized Dirichlet boundary conditions* are

$$\min\{\mathcal{L}u + a_0 u - f, u - \varphi\} \leq 0 \quad \text{on } \partial\Omega \tag{1.4}$$

and

$$\max\{\mathcal{L}u + a_0 u - f, u - \varphi\} \geq 0 \quad \text{on } \partial\Omega \tag{1.5}$$

in the viscosity sense. Conditions (1.4) and (1.5) arise when we pass to limit smooth solutions of the classical Dirichlet boundary value problem in the *vanishing viscosity method* (see [4] and [9] for an introduction of the so–called *half–relaxed limits method*). If $\mathbf{U}$ is continuous on $\overline{\Omega}$ one proves that is the unique solution of the generalized Dirichlet boundary problem, even when $\mathbf{U} \neq \varphi$ on $\partial\Omega$. However we only know the results of G. Barles and E. Roury [3] and M. Katsoulakis [15] in order to prove the continuity of optimal value functions. We note that the eventual discontinuity of τ_x with respect to x is the main difficulty in proving the continuity of $\mathbf{U}$. Clearly, the continuity of $\mathcal{X}_t^x$ implies the upper semi-continuity of τ_x.

So that, our interest here is the study of the *generalized Dirichlet value boundary problem* involved to

$$\begin{cases} \mathcal{L}u + a_0 u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \Omega. \end{cases}$$

In Section 4 we obtain existence and uniqueness of solutions under the simple assumption $\mathcal{A}(\cdot) \doteq (\sigma\sigma^t)(\cdot)$ on the leading part of $\mathcal{L}$. Also this possibly degenerate elliptic condition is the main hypothesis used in Sections 2 and 3 where Weak and Strong Maximum Principles are derived. Certainly some results apply to a more general problems but that will be not considered in the paper.

Our contributions on the Maximum Principle are extensions of well known results for uniformly elliptic equations (see [13]). As in [18] we will study, in a future paper, the behavior of the *first exit time* τ_x function by means of the properties of the first eigenvalue involved. We send to [5] or [20] in order to obtain a characterization of the *principal* eigenvalue by the Strong Maximum Principle.

In the Appendix A we collect some results on the semi–convex and concave approach used in the paper.

2. The Weak Maximum Principle

Due to the lack of the uniformly ellipticity, in this paper we consider viscosity solutions. Let $\mathbb{F} : \mathcal{O} \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}} \times \mathcal{S}^{\mathrm{N}} \to \mathbb{R}$ a continuous function, where $\mathcal{O}$ is an arbitrary set of $\mathbb{R}^{\mathrm{N}}$ and $\mathcal{S}^{\mathrm{N}}$ stands for the space of the real and symmetric matrices of $\mathrm{N} \times \mathrm{N}$. Then a function $u : \mathcal{O} \to \mathbb{R}$ is a viscosity solution of

$$\mathbb{F}(x, u, \mathrm{D}u, \mathrm{D}^2 u) \leq 0 \quad \text{on } \mathcal{O} \tag{2.1}$$

if

$$\mathbb{F}\left(x_0, u^*(x_0), \mathrm{D}\Phi(x_0), \mathrm{D}^2\Phi(x_0)\right) \leq 0$$

holds for all $x_0 \in \mathcal{O}$ and all $\Phi \in \mathcal{C}^2(\mathcal{O})$ such that $(u^* - \Phi)$ attains a maximum on $\mathcal{O}$ at x_0. Here u^* means the relative upper semi–continuous envelope

$$u^*(x) \doteq \lim_{r \to 0} \sup \ \{u(y) : \ |x - y| \leq r\}, \quad x \in \mathcal{O}.$$

Analogously, u is a viscosity solution of

$$\mathbb{F}(x, u, \mathrm{D}u, \mathrm{D}^2u) \geq 0 \quad \text{on } \mathcal{O} \tag{2.2}$$

if

$$\mathbb{F}\left(x_0, u_*(x_0), \mathrm{D}\Phi(x_0), \mathrm{D}^2\Phi(x_0)\right) \geq 0$$

holds for all $x_0 \in \mathcal{O}$ and all $\Phi \in \mathcal{C}^2(\mathcal{O})$ such that $(u_* - \Phi)$ attains a minimum on $\mathcal{O}$ at x_0. Now u_* is the relative lower semi–continuous envelope

$$u_*(x) \doteq \lim_{r \to 0} \inf \ \{u(y) : \ |x - y| \leq r\}, \quad x \in \mathcal{O}.$$

Certainly, u is a viscosity solution of

$$\mathbb{F}(x, u, \mathrm{D}u, \mathrm{D}^2u) = 0 \quad \text{on } \mathcal{O} \tag{2.3}$$

if it is a viscosity solution of (2.1) and (2.2) simultaneously. For this reason a viscosity solution of (2.1) (respectively (2.2)) is a subsolution (respectively supersolution) of (2.3).

Remark 2.1 In the above definitions we may equivalently consider local or global, strict or not strict extremes. We send to [9] for a general exposition on the theory of viscosity solutions. □

In the paper we focuss the interest on the choice

$$\mathbb{F}_{\mathcal{L},a_0,f}\left(x, r, p, \mathcal{Z}\right) \doteq -\text{trace } \left(\mathcal{A}(x) \cdot \mathcal{Z}\right) + \langle \mathbf{a}(x), p\rangle + a_0(x)r - f(x),$$

for $(x, r, p, \mathcal{Z}) \in \overline{\mathcal{O}} \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}} \times \mathcal{S}^{\mathrm{N}}$, where Ω is an open set of $\mathbb{R}^{\mathrm{N}}$. We note that for smooth functions, $u \in \mathcal{C}^2(\mathcal{O})$, one has

$$\mathbb{F}_{\mathcal{L},a_0,f}\left(x, u(x), \mathrm{D}u(x), \mathrm{D}^2u(x)\right) = \mathcal{L}u(x) + a_0(x)u(x) - f(x)$$

for

$$\mathcal{L}u(x) \doteq -\text{trace}\left(\mathcal{A}(x) \cdot \mathrm{D}^2u(x)\right) + \langle \mathbf{a}(x), \mathrm{D}u(x)\rangle,$$

where the matricial function $\mathcal{A}(\cdot) \in \mathcal{M}(\mathrm{N}; \mathbb{R})$ is given by

$$\mathcal{A}(\cdot) \doteq \left(\sigma\sigma^t(\cdot)\right).$$

As it was pointed out in the Introduction, this choice of the leading part arises in Controlled Diffusions.

Remark 2.2 Since

$$\sigma_{ik}\sigma_{jk}\xi_i\xi_j \geq -\frac{\sigma_{ik}^2\xi_i^2 + \sigma_{jk}^2\xi_j^2}{2} \quad \Rightarrow \quad \sum_{\substack{i,j=1 \\ i<j}}^{\mathrm{N}} \sigma_{ik}\sigma_{jk}\xi_i\xi_j \geq -\sum_{i=1}^{\mathrm{N}} \sigma_{ik}^2\xi_i^2,$$

the covariance matrix, $\mathcal{A}_{ij}(\cdot) = \sum_{k=1}^{\mathrm{M}} (\sigma_{ik}\sigma_{jk})(\cdot)$, is *elliptic possibly degenerate*, *i.e.*

$$\sum_{i,j=1}^{\mathrm{N}} \mathcal{A}_{ij}(\cdot)\xi_i\xi_j \geq 0, \quad \xi \in \mathbb{R}^{\mathrm{N}}.$$

Moreover, from $\sigma \not\equiv 0$ it verifies

$$\text{trace } \mathcal{A}(\cdot) > 0, \quad 1 \leq i \leq \mathrm{N},$$

we say then that $\mathcal{A}(\cdot)$ is *elliptic quasi non–degenerate.* On the other hand, the property

$$\mathcal{A}(\cdot) = \big(\sigma\sigma^t(\cdot)\big)$$

where $\sigma : \mathbb{R}^{\mathrm{N}} \to \mathcal{M}(\mathrm{N} \times \mathrm{M}; \mathbb{R})$ is a Lipschitz continuous functions, fails, in general, for any Lipschitz continuous positive semi–definite symmetric matricial function $\mathcal{A}$. However, we may consider the technicality: $\mathcal{A} \in \mathrm{W}^{2,\infty}$ *implies that* $\sqrt{\mathcal{A}}$ *is uniformly Lipschitz continuous* (see [12] for some results of the factorization of non–negative definite matrices). □

We will assume that the coefficients $\mathcal{A}(\cdot) = \{\mathcal{A}_{ij}(\cdot) : \ 1 \leq i, j \leq \mathrm{N}\}$ and $\mathbf{a}(\cdot) = (a_i(\cdot) : \ 1 \leq i \leq \mathrm{N})$, as well as the function a_0 are uniformly continuous.

The main result here is

Theorem 2.1 (A Weak Maximum Principle) *Let Ω be a bounded open set. Assume*

$$\begin{cases} \displaystyle\sum_{i,j=1}^{\mathrm{N}} \mathcal{A}_{ij}(\cdot)\xi_i\xi_j \geq 0, \quad \xi \in \mathbb{R}^{\mathrm{N}}, \\ \mathcal{A}_{kk}(\cdot) > 0, \quad \textit{for some } k \in \{1, \ldots, \mathrm{N}\} \end{cases} \tag{2.4}$$

and

$$\frac{a_k}{\mathcal{A}_{kk}} \textit{ has a positive upper bound in } \Omega. \tag{2.5}$$

Then if an upper semi–continuous function u is a viscosity solution of

$$\mathcal{L}u \leq 0 \quad \textit{in } \Omega$$

one has

$$\sup_{\Omega} u = \sup_{\partial\Omega} u. \tag{2.6}$$

Proof. From

$$a_k(x) \le M\mathcal{A}_{kk}(x), \quad x \in \Omega, \text{ for some } M > 0$$

(see (2.5)) one derives

$$\mathcal{L}e^{\gamma x_k} = \left(-\gamma^2 \mathcal{A}_{kk}(x) + \gamma a_k(x)\right) e^{\gamma x_k} \le \gamma \mathcal{A}_{kk}(x)(-\gamma + M)e^{\gamma x_k} < 0$$

for some $\gamma > 0$ sufficiently large. We claim that $u + \varepsilon\Phi$ can not attain a maximum in the interior of Ω, for $\Phi(x) = e^{\gamma x_k}$ and $\varepsilon > 0$. Indeed, otherwise at any interior maximum point $x_0 \in \Omega$ one derives the contradiction

$$\begin{cases} \mathcal{L}\Phi(x_0) \ge 0 & \text{(by definition of viscosity solution),} \\ 0 > \mathcal{L}\Phi(x_0) & \text{(by construction).} \end{cases}$$

Therefore

$$\sup_{\Omega}(u + \varepsilon\Phi) = \sup_{\partial\Omega}(u + \varepsilon\Phi).$$

Finally, letting $\varepsilon \to 0$ one concludes (2.6). □

Corollary 2.1 *Under assumptions of Theorem 2.1. If an upper semi–continuous function u is a viscosity solution of*

$$\mathcal{L}u + a_0 u \le 0 \quad \text{in } \Omega$$

with $a_0 \ge 0$ one has

$$\sup_{\Omega} u \le \sup_{\partial\Omega} u^+. \tag{2.7}$$

(*Here* $r^+ = \max\{r, 0\}$). □

Inequality (2.7) follows arguing in $\Omega^+ = \{x \in \Omega : \ u(x) > 0\}$, where

$$\mathcal{L}u + a_0 u \le 0 \quad \text{in } \Omega^+ \quad \Rightarrow \quad \mathcal{L}u \le 0 \quad \text{in } \Omega^+.$$

The above arguments, as well as the next result, are extensions to viscosity solutions of well known results for uniformly elliptic operators (see for instance [13]).

Theorem 2.2 *Let Ω be a bounded open set. Assume* (2.4) *and*

$$a_k(x) \le M_k \mathcal{A}_{kk}(x), \quad x \in \Omega, \tag{2.8}$$

for some $M_k > 0$. Then if an upper semi–continuous function u is a viscosity solution of

$$\mathcal{L}u + a_0 u \le f \quad \text{in } \Omega$$

one has

$$u(x) \le \sup_{\partial\Omega} u^+ + \left(e^{(M_k+1)(D_k - d_k)} - 1\right) \frac{1}{\inf_{\Omega} \mathcal{A}_{kk}} \sup_{\Omega} f^+, \quad x \in \overline{\Omega}, \tag{2.9}$$

whenever $a_0(\cdot) \geq 0$, $f: \Omega \to \mathbb{R}$ *is a continuous function and*

$$d_k < x_k < D_k, \quad x \in \Omega.$$

Furthermore, inequality

$$u(x) \leq \frac{1}{C}\left(\sup_{\partial\Omega} u^+ + \left(e^{(M_k+1)(D_k-d_k)} - 1\right)\frac{1}{\inf_{\Omega} \mathcal{A}_{kk}} \sup_{\Omega} f^+\right), \quad x \in \overline{\Omega} \tag{2.10}$$

holds for $C = 1 - \left(e^{M_k(D_k-d_k)} - 1\right)\frac{1}{\inf_{\Omega} \mathcal{A}_{kk}} \sup_{\Omega} a_0^-$, *provided*

$$-a_0(x) < \frac{\inf_{\Omega} \mathcal{A}_{kk}}{\left(e^{(M_k+1)(D_k-d_k)} - 1\right)}, \quad x \in \Omega. \tag{2.11}$$

(*Here* $r^- = \max\{-r, 0\}$).

Proof. Let us consider the non–negative function

$$\Phi(x) \doteq \frac{1}{\inf_{\Omega} \mathcal{A}_{kk}} \sup_{\Omega} f^+ \left(e^{(M_k+1)(D_k-d_k)} - e^{(M_k+1)(x_k-d_k)}\right), \quad x \in \overline{\Omega}.$$

By straightforward computations as in the proof of Theorem 2.1, one verifies

$$\mathcal{L}\Phi(x) > f(x), \quad x \in \Omega.$$

Again, we claim that $u - \Phi$ can not attain a positive maximum in the interior of Ω. Indeed, otherwise at any interior positive maximum point $x_0 \in \Omega$ with $u(x_0) > \Phi(x_0) \geq 0$, the condition $a_0 \geq 0$ derives the contradiction

$$f(x_0) < \mathcal{L}\Phi(x_0) \leq \mathcal{L}\Phi(x_0) + a_0(x_0)u(x_0) \leq f(x_0).$$

Therefore,

$$u(x) \leq \sup_{\partial\Omega}(u - \Phi)^+ + \Phi(x) \leq \sup_{\partial\Omega} u^+ + \left(e^{(M_k+1)(D_k-d_k)} - 1\right)\frac{1}{\inf_{\Omega} \mathcal{A}_{kk}} \sup_{\Omega} f^+, \quad x \in \overline{\Omega}$$

concludes (2.9).
For general a_0 functions we have

$$\mathcal{L}u + a_0^+ u \leq f + (a_0^+ - a_0)u \quad \text{in } \overline{\Omega}.$$

Since $a_0 = a_0^+ - a_0^-$, (2.9) implies

$$\sup_{\Omega} u^+ \leq \sup_{\partial\Omega} u^+ + \left(e^{(M_k+1)(D_k-d_k)} - 1\right)\frac{1}{\inf_{\Omega} \mathcal{A}_{kk}}\left[\sup_{\Omega} f^+ + \sup_{\Omega} u^+ \sup_{\Omega} a_0^-\right]$$

whence (2.10) holds, provided (2.11). □

Remark 2.3 One deduces the estimate

$$\sup_{\Omega}|u| \leq \frac{1}{C}\left(\sup_{\partial\Omega}|u| + \left(e^{(M_k+1)(D_k-d_k)}-1\right)\frac{1}{\inf\limits_{\Omega}\mathcal{A}_{kk}}\sup_{\Omega}|f|\right) \tag{2.12}$$

if $u \in \mathcal{C}(\Omega)$ is a viscosity solution of

$$\mathcal{L}u + a_0 u = f \quad \text{in } \Omega$$

for

$$-a_0(x) < \frac{\inf\limits_{\Omega}\mathcal{A}_{kk}}{\left(e^{(M_k+1)(D_k-d_k)}-1\right)}, \quad x \in \Omega.$$

Therefore, one derives the estimate

$$\lambda_{\mathcal{L},\Omega} \geq \frac{\inf\limits_{\Omega}\mathcal{A}_{kk}}{\left(e^{(M_k+1)(D_k-d_k)}-1\right)} \tag{2.13}$$

for the eventual *first eigenvalue* of the operator $\mathcal{L}$ on Ω with Dirichlet boundary condition.

3. The Strong Maximum Principle

As usually we follow the classical argumental lines by E. Hopf. So, we begin with

Lemma 3.1 (The boundary point lemma) *Assume* (2.4) *and*

$$\frac{|a_k|}{\mathcal{A}_{kk}} \text{ has an upper bound in } \Omega. \tag{3.1}$$

Let u be a viscosity solution of

$$\mathcal{L}u \leq 0 \quad \text{in } \Omega.$$

Let $x_0 \in \partial\Omega$ be such that

$$\begin{cases} i) & u \text{ attains a strict maximum on } \Omega \cup \{x_0\}, \\ ii) & \exists\, \mathbf{B}_{\mathrm{R}}(x_0 - \mathrm{R}\vec{n}(x_0)) \subset \Omega, \quad (\ \partial\Omega \text{ satisfies an interior sphere condition at } x_0). \end{cases}$$

Then if we denote $u(x_0) \doteq \limsup\limits_{\substack{x\to x_0\\ x\in\Omega}} u(x)$ one has

$$\liminf_{\lambda\to 0}\frac{u(x_0)-u(x_0-\lambda\vec{n})}{\lambda} \geq C > 0, \tag{3.2}$$

where $\vec{n}$ stands for the outer normal unit vector of $\partial\Omega$ at x_0 and C is a positive constant depending only on the geometry at x_0.

Proof. From (3.1) we may assume

$$a_k(x) \geq -M\mathcal{A}_{kk}(x), \quad x \in \Omega, \text{ for some } M > 0.$$

Let us introduce the function

$$\Phi(x) = e^{-\alpha r_k^2} - e^{-\alpha \mathrm{R}^2}, \quad x \in \mathbf{B}_{\mathrm{R}} \doteq \mathbf{B}_{\mathrm{R}}(y), \; y = x_0 - \mathrm{R}\vec{n}(x_0), \tag{3.3}$$

where $r_k = x_k - y_k$ and α is a positive constant to be chosen later. Since

$$|r_k| \leq \mathrm{R} \quad \text{and} \quad a_k(x)r_k \geq -M\mathrm{R}\mathcal{A}_{kk}(x), \quad x \in \Omega,$$

straightforward computations lead to

$$\begin{aligned} \mathcal{L}\Phi(x) &= 2\alpha e^{-\alpha r_k^2}\left[-2\alpha\mathcal{A}_{kk}(x)r_k^2 + \mathcal{A}_{kk}(x) - a_k(x)r_k\right] \\ &\leq 2\alpha e^{-\alpha r_k^2}\left[-\inf_{\Omega}\mathcal{A}_{kk}\frac{\alpha\mathrm{R}^2}{2} + (1+M\mathrm{R})\sup_{\Omega}\mathcal{A}_{kk}\right] \\ &< 0, \qquad \text{if } \frac{\mathrm{R}}{2} < r < \mathrm{R}, \end{aligned} \tag{3.4}$$

for some $\alpha > 0$ large enough. Moreover, by construction

$$u(x) - u(x_0) < 0, \quad x \in \partial\mathbf{B}_{\frac{\mathrm{R}}{2}} \quad \Rightarrow \quad u(x) - u(x_0) + \varepsilon\Phi(x) \leq 0, \quad x \in \partial\mathbf{B}_{\frac{\mathrm{R}}{2}},$$

for $\varepsilon > 0$ small enough. Here we claim the inequality ,

$$(u - u(x_0) + \varepsilon\Phi)(x) \leq 0, \quad x \in \overline{\mathcal{O}} \tag{3.5}$$

whence for $\mathcal{O} = \mathbf{B}_{\mathrm{R}} \setminus \overline{\mathbf{B}}_{\frac{\mathrm{R}}{2}}$. Indeed, if $u - u(x_0) + \varepsilon\Phi$ attains a maximum at some point $\overline{x} \in \mathcal{O}$ one derives the contradiction

$$\begin{cases} \mathcal{L}\Phi(\overline{x}) \geq 0 & \text{(by definition of viscosity solution),} \\ 0 > \mathcal{L}\Phi(\overline{x}) & \text{(by construction).} \end{cases}$$

Therefore, (3.5) leads to

$$\frac{u(x_0) - u(x_0 - \lambda\vec{n})}{\lambda} \geq \varepsilon\frac{\Phi(x_0 - \lambda\vec{n})}{\lambda}, \qquad (\lambda \ll 1)$$

whence

$$\liminf_{\lambda\to 0}\frac{u(x_0) - u(x_0 - \lambda\vec{n})}{\lambda} \geq C > 0.$$

□

Remark 3.1 In fact, above result implies

$$\liminf_{\substack{x\to x_0 \\ x\in\Omega}}\frac{u(x_0) - u(x)}{|x - x_0|} \geq C > 0.$$

Then we get to the main result here

Theorem 3.1 (Hopf's Strong Maximum Principle) *Assume* (2.4) *and* (3.1). *Let an upper semi–continuous function u be a viscosity solution of*

$$\mathcal{L}u \leq 0 \quad \text{in } \Omega,$$

where Ω is not necessarily bounded. Then u cannot achieve a local maximum at some $x_0 \in \Omega$ unless u is constant in a neighborhood of x_0.

Proof. By simplicity we only consider the case

$$\mathcal{A}(\cdot) \equiv \mathcal{A} \in \mathcal{M}(\mathrm{N};\mathbb{R}) \quad \text{and} \quad \mathbf{a}(\cdot) \equiv \mathbf{a} \in \mathbb{R}^{\mathrm{N}}.$$

Assume that u is non–constant and achieves a maximum value $u(x_0) = \mathrm{M}$ on some ball $\mathbf{B} \subset \Omega$. Then we consider the semi-convex approach of u, *i.e.*

$$u^\varepsilon(x) \doteq \sup_{y\in\Omega}\left\{u(y) - \frac{|x-y|^2}{2\varepsilon^2}\right\}, \quad x \in \mathbf{B}_\varepsilon \qquad (\varepsilon > 0),$$

where $\mathbf{B}_\varepsilon \doteq \{x \in \mathbf{B} :\ \mathrm{dist}(x, \partial\mathbf{B} > \varepsilon\sqrt{1 + 4\sup_{\mathbf{B}}|u|}\}$. For ε small enough we may assume $x_0 \in \mathbf{B}_\varepsilon$. Then u^ε attains M, as a maximum value, in $\mathbf{B}_\varepsilon$, with $u(x_0) = u^\varepsilon(x_0) = \mathrm{M}$. Moreover, u^ε satisfies

$$\mathcal{L}u^\varepsilon \leq 0 \quad \text{on } \mathbf{B}_\varepsilon.$$

A short exposition about the semi–convex and semi–concave approach can be found in the Appendix A below.

By classic arguments, if we denote

$$\mathbf{B}_\varepsilon^- \doteq \{x \in \mathbf{B}_\varepsilon :\ u^\varepsilon(x) < \mathrm{M}\},$$

there exists the largest ball $\mathbf{B}_\mathrm{R}(y) \subset \mathbf{B}_\varepsilon^-$ (see [13]). Certainly there exists some $z_0 \in \partial\mathbf{B}_\mathrm{R}(y) \cap \mathbf{B}_\varepsilon$ for which $u^\varepsilon(z_0) = \mathrm{M}$. Then, as in the proof of Lemma 3.1 the contradiction

$$\begin{cases} 0 = \mathrm{D}u^\varepsilon(z_0) & \text{by the DMP property (see Lemma A.2)} \\ 0 \neq \mathrm{D}u^\varepsilon(z_0) & \text{by Lemma 3.1} \end{cases}$$

appears. Therefore, u^ε is constant on $\mathbf{B} \subset \Omega$, *i.e.*

$$u^\varepsilon(y) = u^\varepsilon(x_0) = u(x_0), \quad y \in \mathbf{B}.$$

As in (A.3), for every $y \in \mathbf{B}$ there exists $\widehat{y}$ such that

$$u^\varepsilon(y) = u(\widehat{y}) - \frac{1}{2\varepsilon^2}|y - \widehat{y}|^2$$

whence

$$u(x_0) = u^\varepsilon(x_0) = u^\varepsilon(y) = u(y) - \frac{1}{2\varepsilon^2}|y-\widehat{y}|^2 \le u(x_0) - \frac{1}{2\varepsilon^2}|y-\widehat{y}|^2 \le u(x_0) \quad \Rightarrow \quad \widehat{y} = y.$$

So that, one concludes

$$u(y) = u^\varepsilon(y) = u^\varepsilon(x_0) = u(x_0), \quad y \in \mathbf{B}. \qquad \square$$

Finally, we may adapt [1, Theorem 1] and to obtain

Proposition 3.2 (Weak Comparison Principle) *Assume* (2.4) *and* (3.1). *Let an upper semi–continuous function v and a lower semi–continuous function V be viscosity solutions of*

$$\mathcal{L}v + a_0 v \le f \quad \text{in } \Omega$$

and

$$\mathcal{L}V + a_0 V \ge f \quad \text{in } \Omega,$$

where f is a uniformly continuous function and a_0 satisfies $a_0(\cdot) \ge 0$. Then

$$(u-v)(x) \le \sup_{\partial\Omega}(u-v)^+, \quad x \in \overline{\Omega},$$

provided that Ω is a bounded open set. $\square$

4. The boundary value problem

Now we focuss the attention on the Dirichlet boundary value problem

$$\begin{cases} \mathcal{L}u + a_0 u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \Omega. \end{cases}$$

For simplicity we will assume f and φ continuous. As it was pointed out above, by a subsolution we mean a function $u : \overline{\Omega} \to \mathbb{R}$ verifying

$$\begin{cases} \mathcal{L}u + a_0 u \le f & \text{in } \Omega, \\ \min\{\mathcal{L}u + a_0 u - f, u^* - \varphi\} \le 0 & \text{on } \partial\Omega, \end{cases}$$

in the viscosity sense. Analogously, $u : \overline{\Omega} \to \mathbb{R}$ is a supersolution if

$$\begin{cases} \mathcal{L}u + a_0 u \ge f & \text{in } \Omega, \\ \max\{\mathcal{L}u + a_0 u - f, u_* - \varphi\} \ge 0 & \text{on } \partial\Omega, \end{cases}$$

holds in the viscosity sense. Certainly, u is a viscosity solution if it is a subsolution and it is a supersolution simultaneously.

Remark 4.1 As it is well known, we note that u is a solution of the *classical Dirichlet value problem* if the above boundary conditions is replaced by

$$u^* \leq \varphi \quad \text{on } \partial\Omega$$

and

$$u_* \geq \varphi \quad \text{on } \partial\Omega$$

respectively. □

Here we use simple adaptations of relative results of [10]. In order to characterize the function $\mathbf{U}$ (see (1.1)) we use a classical argument. Indeed, as in [2] or [10] we may prove

Theorem 4.1 (Dynamics Programming Principle)
For every $t \geq 0$ one satisfies

$$\left\{ \begin{array}{ll} \mathbf{U}(x) = & \mathbb{E}_x \left[\displaystyle\int_0^{t\wedge\tau_x} f(\mathcal{X}_s^x) \exp\left(-\int_0^s a_0(\mathcal{X}_r^x) dr \right) \mathrm{d}s \right. \\ & + \mathbb{1}_{\{\tau_x > t\}} \mathbf{U}(\mathcal{X}_t^x) \exp\left(-\displaystyle\int_0^t a_0(\mathcal{X}_s^x) ds \right) \\ & \left. + \mathbb{1}_{\{\tau_x \leq t\}} \varphi(\mathcal{X}_{\tau_x}^x) \exp\left(-\displaystyle\int_0^{\tau_x} a_0(\mathcal{X}_s^x) ds \right) \right], \quad x \in \Omega. \square \end{array} \right. \tag{4.1}$$

So the arguments of [1] (see also [10]) lead to

Theorem 4.2 *Assume*

$$f,\ a_0\ : \mathbb{R}^{\mathrm{N}} \to \mathbb{R} \text{ are continuous functions.} \tag{4.2}$$

The function $\mathbf{U}$*, given in* (1.1)*, is a viscosity solution of*

$$\left\{ \begin{array}{ll} \mathcal{L}u + a_0 u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \Omega. \end{array} \right.$$

Remark 4.2 In our argument the distance function $\mathrm{d}(\cdot) \doteq \mathrm{dist}(\cdot, \partial\Omega)$ plays an important role. As it is well known, $\mathrm{d} \in \mathcal{C}^k$, with $|\mathrm{Dd}| = 1$, near $\partial\Omega$ whenever Ω, is a $\mathcal{C}^k$ open set of $\mathbb{R}^{\mathrm{N}}$. Moreover, $\vec{n}(x_0) = -\mathrm{Dd}(x_0)$, $x_0 \in \partial\Omega$, is the unit outward normal vector at x_0 (see [13, Lemma 14.16]). □

In order to understand generalized Dirichlet boundary conditions (see (1.4) and (1.5)) a first question arises. How the equation holds on the boundary? Here we adapt a result of [2, Proposition 1.1] involving the geometry.

Proposition 4.3 *Let Ω be a $\mathcal{C}^2$ open set of $\mathbb{R}^{\mathrm{N}}$. Let u a viscosity solution of*

$$\mathcal{L}u + a_0 u \leq f \quad \text{in } \mathbf{B}_{\mathrm{R}}(x_0) \cap \overline{\Omega}, \qquad x_0 \in \partial\Omega,\ \mathrm{R} > 0,$$

assumed that f and a_0 are Lipschitz continuous functions and

$$\sum_{i,j}^{\mathrm{N}} \mathcal{A}_{ij}(\cdot)\xi_i\xi_j \geq 0, \quad \forall \xi \in \mathbb{R}^{\mathrm{N}}, \quad \text{in } \mathbf{B}_{\mathrm{R}}(x_0) \cap \overline{\Omega}.$$

Then

$$\left.\begin{array}{c} \mathcal{A}(x_0) \cdot \vec{n}(x_0) \otimes \vec{n}(x_0) = 0 \\ \text{and} \\ \text{trace } \big(\mathcal{A}(x_0) \cdot \mathrm{D}^2\mathrm{d}(x_0)\big) + \langle \mathbf{a}(x_0), \vec{n}(x_0)\rangle \geq 0 \end{array}\right\} \tag{4.3}$$

holds. With the same properties on the coefficients if u is a viscosity solution of

$$\mathcal{L}u + a_0 u \geq f \quad \text{in } \mathbf{B}_{\mathrm{R}}(x_0) \cap \overline{\Omega}, \qquad x_0 \in \partial\Omega,\ \mathrm{R} > 0,$$

then (4.3) *also holds.*

Proof. Assume $\varepsilon > 0$ so small than $\mathrm{d} \in \mathcal{C}^2(\Omega_\varepsilon^{\mathrm{R}})$, for

$$\Omega_\varepsilon^{\mathrm{R}} = \{x \in \mathbf{B}_{\mathrm{R}}(x_0) \cap \overline{\Omega} : \mathrm{d}(x) \leq \varepsilon\} \neq \emptyset.$$

Let us consider the upper semi–continuous function

$$\phi_\varepsilon(x) = u^*(x) - \frac{|x - x_0|^4}{\varepsilon^4} - \frac{\mathrm{d}(x)}{\varepsilon^2}\left[1 - \frac{\mathrm{d}(x)}{2\varepsilon}\right], \quad x \in \Omega_\varepsilon^{\mathrm{R}},$$

that attains on $\Omega_\varepsilon^{\mathrm{R}}$ a maximum at some x_ε for which

$$u^*(x_0) = \phi_\varepsilon(x_0) \leq \phi_\varepsilon(x_\varepsilon) = u^*(x_\varepsilon) - \frac{|x_\varepsilon - x_0|^4}{\varepsilon^4} - \frac{\mathrm{d}(x_\varepsilon)}{\varepsilon^2}\left[1 - \frac{\mathrm{d}(x_\varepsilon)}{2\varepsilon}\right]. \tag{4.4}$$

Then

$$\frac{|x_\varepsilon - x_0|^4}{\varepsilon^4} + \frac{\mathrm{d}(x_\varepsilon)}{\varepsilon^2} - \frac{(\mathrm{d}(x_\varepsilon))^2}{2\varepsilon^3} \leq u^*(x_\varepsilon) - u^*(x_0).$$

By $\mathrm{d}(x_\varepsilon) \leq \varepsilon$ inequality

$$0 \leq \max\left\{\frac{|x_\varepsilon - x_0|^4}{\varepsilon^4}, \frac{\mathrm{d}(x_\varepsilon)}{\varepsilon^2}\left[1 - \frac{\mathrm{d}(x_\varepsilon)}{2\varepsilon}\right]\right\} \leq u^*(x_\varepsilon) - u^*(x_0) < +\infty, \tag{4.5}$$

holds, therefore

$$\left\{\begin{array}{c} \{x_\varepsilon\}_\varepsilon \to x_0 \quad \text{as } \varepsilon \to 0 \\ u^*(x_0) \leq \limsup\limits_{\varepsilon \to 0} u^*(x_\varepsilon) \leq u^*(x_0) \quad \text{(see (4.4))}. \end{array}\right.$$

Moreover, as $\varepsilon \to 0$ one has

$$\begin{cases} |x_\varepsilon - x_0| = o(\varepsilon) & \left(\text{by} \quad \dfrac{|x_\varepsilon - x_0|^4}{\varepsilon^4} \to 0\right) \\ \mathrm{d}(x_\varepsilon) = o(\varepsilon^2) & \left(\text{by} \quad \mathrm{d}(x_\varepsilon) \le 2\varepsilon \Rightarrow \dfrac{\mathrm{d}(x_\varepsilon)}{\varepsilon^2}\left[1 - \dfrac{\mathrm{d}(x_\varepsilon)}{2\varepsilon}\right] \to 0\right). \end{cases} \tag{4.6}$$

From

$$\phi_\varepsilon(x_\varepsilon) \ge \phi_\varepsilon(x), \quad x \in \Omega_\varepsilon^{\mathrm{R}},$$

function

$$\Phi_\varepsilon(x) = \frac{|x - x_0|^4}{\varepsilon^4} + \frac{\mathrm{d}(x)}{\varepsilon^2}\left[1 - \frac{\mathrm{d}(x)}{2\varepsilon}\right], \quad x \in \Omega_\varepsilon^{\mathrm{R}},$$

verifies

$$(u^* - \Phi_\varepsilon)(x_\varepsilon) \ge (u^* - \Phi_\varepsilon)(x), \quad x \in \Omega_\varepsilon^{\mathrm{R}}.$$

Replacing Φ by

$$\widehat{\Phi}(x) = \Phi(x) + (u^* - \Phi)(x_0) + |x - x_0|^2$$

if necessary, we may assume

$$0 = (u^* - \Phi_\varepsilon)(x_\varepsilon) \ge (u^* - \Phi_\varepsilon)(x), \quad x \in \Omega_\varepsilon^{\mathrm{R}}.$$

Since $\mathrm{d}(x_\varepsilon) = o(\varepsilon^2)$, x_ε is also a local maximum on $\overline{\Omega}$, consequently

$$-\text{trace}\ \big(\mathcal{A}(x_\varepsilon) \cdot \mathrm{D}^2\Phi_\varepsilon(x_\varepsilon)\big) + \langle \mathbf{a}(x_\varepsilon), \mathrm{D}\Phi_\varepsilon(x_\varepsilon)\rangle + a_0(x_\varepsilon)v^*(x_\varepsilon) \le f(x_\varepsilon), \tag{4.7}$$

From

$$\mathrm{D}\Phi_\varepsilon(x_\varepsilon) = \frac{4}{\varepsilon^4}|x_\varepsilon - x_0|^2(x_\varepsilon - x_0) + \frac{\mathrm{Dd}(x_\varepsilon)}{\varepsilon^2} - \frac{\mathrm{d}(x_\varepsilon)}{\varepsilon^3}\mathrm{Dd}(x_\varepsilon)$$

property (4.6) implies

$$\varepsilon^2 \mathrm{D}\Phi_\varepsilon(x_\varepsilon) = \mathrm{Dd}(x_\varepsilon) + o(1)$$

and

$$\begin{cases} \varepsilon^2 \left|\langle \mathbf{a}(x_\varepsilon), \mathrm{D}\Phi_\varepsilon(x_\varepsilon)\rangle - \left\langle \mathbf{a}(x_0), \dfrac{\mathrm{Dd}(x_0)}{\varepsilon^2}\right\rangle\right| & \le & \varepsilon^2 \left|\langle \mathbf{a}(x_\varepsilon) - \mathbf{a}(x_0), \mathrm{D}\Phi_\varepsilon(x_\varepsilon)\rangle\right| \\ & + & \varepsilon^2 \left|\left\langle \mathbf{a}(x_0), \mathrm{D}\Phi_\varepsilon(x_\varepsilon) - \dfrac{\mathrm{Dd}(x_0)}{\varepsilon^2}\right\rangle\right| \\ & = & |\mathbf{a}(x_0)| \cdot |\mathrm{Dd}(x_\varepsilon) - \mathrm{Dd}(x_0)| + o(1) \end{cases} \tag{4.8}$$

(by the regularity of function $\mathbf{a}$). On the other hand

$$\begin{aligned} \mathrm{D}^2\Phi_\varepsilon(x_\varepsilon) \quad = \quad & \left[\frac{8}{\varepsilon^4}(x_\varepsilon - x_0) \otimes (x_\varepsilon - x_0) + \frac{4}{\varepsilon^4}|x_\varepsilon - x_0|^2 \mathrm{I}_{\mathrm{N}}\right] \\ & -\frac{1}{\varepsilon^3}\big(\mathrm{Dd}(x_\varepsilon) \otimes \mathrm{Dd}(x_\varepsilon) + (\mathrm{d}(x_\varepsilon) - \varepsilon)\mathrm{D}^2\mathrm{d}(x_\varepsilon)\big), \end{aligned}$$

and the property (4.6) implies

$$\varepsilon^2 \mathrm{D}^2\Phi_\varepsilon(x_\varepsilon) = -\frac{1}{\varepsilon}\mathrm{Dd}(x_\varepsilon)\otimes \mathrm{Dd}(x_\varepsilon) + \mathrm{D}^2\mathrm{d}(x_\varepsilon) + o(1),$$

and

$$\begin{cases} \varepsilon^2 \left|\text{trace } \left(\mathcal{A}(x_\varepsilon)\cdot \mathrm{D}^2\Phi_\varepsilon(x_\varepsilon)\right) - \text{trace } \left(\mathcal{A}(x_0)\cdot\left(-\dfrac{\mathrm{Dd}(x_0)\otimes\mathrm{Dd}(x_0)}{\varepsilon^3} + \dfrac{\mathrm{D}^2\mathrm{d}(x_0)}{\varepsilon^2}\right)\right)\right| \\ \leq \varepsilon^2 \left|\text{trace } \left((\mathcal{A}(x_\varepsilon) - \mathcal{A}(x_0))\cdot \mathrm{D}^2\Phi_\varepsilon(x_\varepsilon)\right)\right| \\ \qquad +\varepsilon^2 \left|\text{trace } \left(\mathcal{A}(x_0)\cdot\left(\mathrm{D}^2\Phi_\varepsilon(x_\varepsilon) + \dfrac{\mathrm{Dd}(x_0)\otimes\mathrm{Dd}(x_0)}{\varepsilon^3} - \dfrac{\mathrm{D}^2\mathrm{d}(x_0)}{\varepsilon^2}\right)\right)\right| \\ = \left|\text{trace } \mathcal{A}(x_0)\cdot\left(\dfrac{\mathrm{Dd}(x_\varepsilon)\otimes\mathrm{Dd}(x_\varepsilon) - \mathrm{Dd}(x_0)\otimes\mathrm{Dd}(x_0)}{\varepsilon} + \mathrm{D}^2\mathrm{d}(x_\varepsilon) - \mathrm{D}^2\mathrm{d}(x_0)\right)\right| \\ \quad +o(1) \end{cases} \tag{4.9}$$

(by the regularity of function σ and $\partial\Omega$). So that, if we multiply inequality (4.7) by ε^2, (4.8) and (4.9) imply

$$\text{trace } \left(\mathcal{A}(x_0)\cdot\left(\frac{\mathrm{Dd}(x_0)\otimes\mathrm{Dd}(x_0)}{\varepsilon} - \mathrm{D}^2\mathrm{d}(x_0)\right)\right) + \langle \mathbf{a}(x_0), \mathrm{Dd}(x_0)\rangle \leq o(1).$$

Finally, if we multiply this inequality by ε, and then we send ε to 0 we obtain

$$\begin{cases} \text{trace } \left(\mathcal{A}(x_0)\cdot \mathrm{Dd}(x_0)\otimes\mathrm{Dd}(x_0)\right) = 0 \\ -\text{trace } \left(\mathcal{A}(x_0)\cdot \mathrm{D}^2\mathrm{d}(x_0)\right) + \langle \mathbf{a}(x_0), \mathrm{Dd}(x_0)\rangle \leq 0. \end{cases}$$

For supersolutions one replaces ϕ_ε by the lower semi–continuous function

$$\psi_\varepsilon(x) = u_*(x) + \frac{|x-x_0|^4}{\varepsilon^4} + \frac{\mathrm{d}(x)}{\varepsilon^2}\left[1 - \frac{\mathrm{d}(x)}{2\varepsilon}\right], \quad x\in\Omega_\varepsilon^{\mathrm{R}}.$$

Then analogous arguments end the proof. □

Remark 4.3 Obviously, (4.3) is a necessary conditions for the existence of equations on the boundary. We note that they are independent of the zeroth term of the equation. Therefore, denoting

$$\partial_{gbc}\Omega \doteq \{x_0\in\partial\Omega : \ (4.3) \text{ holds}\},$$

the generalized boundary conditions like

$$\min\{\mathcal{L}u + a_0 u - f, \mathrm{B}(x,u,\mathrm{D}u)\} \leq 0 \quad \text{on } \partial\Omega$$

and

$$\max\{\mathcal{L}u + a_0 u - f, \mathrm{B}(x, u, \mathrm{D}u)\} \geq 0 \quad \text{on } \partial\Omega$$

are, in fact, *classical boundary conditions*

$$\mathrm{B}(x, u, \mathrm{D}u) = 0 \quad \text{on } \partial\Omega,$$

whenever $\partial_{gbc}\Omega = \emptyset$. Finally, we also note that (4.3) is used to characterize the *regular points* from the probabilistic point of view (see [12]). □

Theorem 4.4 (Existence and uniqueness) *Assume* (2.4), (3.1) *and*

$$a_0(\cdot) \geq 0 \text{ in } \Omega. \tag{4.10}$$

Then if $\partial_{gbc}\Omega = \emptyset$, whenever f and a_0 are Lipschitz continuous functions, there exists unique viscosity solution in $\mathcal{C}(\overline{\Omega})$ of the classical Dirichlet problem involved to

$$\begin{cases} \mathcal{L}u + a_0 u = f & \text{in } \Omega, \\ u = \varphi & \text{on } \Omega, \end{cases}$$

provided Ω is a bounded open and $\varphi \in \mathcal{C}(\partial\Omega)$.

Proof. Since

$$\mathcal{G} \doteq \{w : \ w \text{ is a subsolution of the relative generalized Dirichlet boundary}\} \neq \emptyset$$

(see Theorem 4.2), inequality (2.10) enables us to define the *Perron function*

$$u_{\mathbf{P}}(x) = \sup \ \{w(x) : \ w \in \mathcal{G}\}, \quad x \in \Omega. \tag{4.11}$$

Obviously, by definitions of semi–continuous envelopes one has

$$(u_{\mathbf{P}})_*(x) \leq u_{\mathbf{P}}(x) \leq (u_{\mathbf{P}})^*(x), \quad x \in \Omega.$$

Arguing as in [9] one proves that $(u_{\mathbf{P}})^*$ and $(u_{\mathbf{P}})_*$ are sub and supersolutions, respectively. Moreover, by $\partial_{gbc}\Omega = \emptyset$ one deduces

$$(u_{\mathbf{P}})^*(x) \leq \varphi(x) \leq (u_{\mathbf{P}})_*(x), \quad x \in \partial\Omega$$

(see Proposition 4.3). By comparison results, this inequality implies

$$(u_{\mathbf{P}})^*(x) \leq (u_{\mathbf{P}})_*(x), \quad x \in \overline{\Omega},$$

whence $u_{\mathbf{P}} \in \mathcal{C}(\overline{\Omega})$ and $u_{\mathbf{P}} = \varphi$ on $\partial\Omega$.
Uniqueness relies again on comparison results. Indeed, by $\partial_{gbc}\Omega = \emptyset$ one satisfies

$$v(x)^*(x) \leq \varphi(x) \leq V_*(x), \quad x \in \partial\Omega$$

if v and V are sub and supersolutions, respectively. Then comparison results imply

$$v(x)^*(x) \leq V_*(x), \quad x \in \overline{\Omega}$$

and uniqueness follows. □

Remark 4.4 The Lipschitz continuity of f and a_0 can be relaxed to the mere uniformly continuity. □

Remark 4.5 Complementary regularity can be obtained for the non–degenerate case

$$\sum_{i,j=1}^{\mathrm{N}} \mathcal{A}_{ij}(\cdot)\xi_i\xi_j \geq \theta|\xi|^2 > 0, \quad \xi \in \mathbb{R}^{\mathrm{N}} \setminus \{0\}$$

as in [8]. □

Appendix

A. The semi–convex and concave approach.

As it is well known the classical maximum principle is an usual key in obtaining comparison and uniqueness results. There are several versions. The Jensen's maximum principle for semi–convex functions is the main step in obtaining uniqueness results for viscosity solutions.

We recall that a function $\psi : \mathcal{O} \to \mathbb{R}$ is semi–convex if $x \mapsto \psi(x) + K|x|^2$ is convex for some positive constant K. Consequently, by straightforward computations ψ is convex if and only if

$$\psi(\mu x + (1-\mu)y) - \mu\psi(x) - (1-\mu)\psi(y) \leq K\mu(1-\mu)|x-y|^2, \quad x, y \in \mathcal{O},\ 0 < \mu < 1.$$

On the other hand, $\psi : \mathcal{O} \to \mathbb{R}$ is semi–concave if $-\psi$ is semi–convex. Next we collect some properties of these functions (for the details we send to [1], [9], [11], [14] or [17]).

We begin with a classical result.

Theorem A.1 (Aleksandrov Theorem (see [9])) *If $\psi : \mathcal{O} \to \mathbb{R}$ is semi–convex then ψ is twice differentiable almost everywhere on $\mathcal{O}$.* □

The next properties concern with the differentiability of the semi–convex functions

Lemma A.1 (The partial continuity of the gradient (PCG) property) *Let $\psi : \mathcal{O} \to \mathbb{R}$ be a semi–convex or a semi–concave function $\psi : \mathcal{O} \to \mathbb{R}$. If $x_n \to x$, as $n \to \infty$, then $\mathrm{D}\psi(x_n) \to \mathrm{D}\psi(x)$, as $n \to \infty$, provided that these gradients exist.* □

Lemma A.2 (The differentiability at maximum points (DMP) property) *If a semi–convex (respectively semi–concave) function $\psi : \mathcal{O} \to \mathbb{R}$ achieves an local maximum (respectively minimum) at some $x_0 \in \mathcal{O}$ then ψ is differentiable at x_0 and $\mathrm{D}\psi(x_0) = 0$.*

Proof. Obviously, we only consider the semi–convex case. By simplicity we may take $x_0 = 0 \in \mathcal{O}$. Then by applying the separation theorem for the convex function

$$\widehat{\psi}(x) \doteq \psi(x) + K|x|^2$$

there exists $p \in \mathbb{R}^{\mathrm{N}}$ such that

$$\widehat{\psi}(x) \geq \widehat{\psi}(0) + \langle p, x\rangle = \psi(0) + \langle p, x\rangle, \quad x \in \mathcal{O}$$

(see [7]). Since 0 is a maximum point of ψ on a small ball $\mathbf{B} \subset \mathcal{O}$ centered at 0 we have

$$\begin{aligned} \psi(x) &= \widehat{\psi}(x) - K|x|^2 \geq \psi(0) + \langle p, x\rangle - K|x|^2 \\ &\geq \psi(x) + \langle p, x\rangle - K|x|^2, \quad x \in \mathbf{B}, \end{aligned} \tag{A.1}$$

whence

$$\langle p, x\rangle \leq K|x|^2, \quad x \in \mathbf{B}.$$

For $\varepsilon > 0$ sufficiently small we may choose $x = \varepsilon p \in \mathbf{B}$ for which

$$\varepsilon|p|^2 \leq K\varepsilon^2|p|^2,$$

therefore it implies $p = 0$. Finally, (A.1) leads to

$$0 \leq \psi(0) - \psi(x) \leq K|x|^2, \quad x \in \mathbf{B}$$

and the result follows. □

Remark A.1 It is cleat that the *DMP property* also holds for $\psi - \Psi$ if $\psi : \mathcal{O} \to \mathbb{R}$ is semi–convex and $\Psi : \mathcal{O} \to \mathbb{R}$ is semi–concave. □

Lemma A.3 (Jensen Lemma) *If $\psi : \mathcal{O} \to \mathbb{R}$ is semi–convex and x_0 is a strict local maximum of ψ then for $r,\ \varrho > 0$ the set*

$$\mathcal{M} \doteq \{x \in \mathbf{B}_r(x_0) : \ \exists p \in \mathbf{B}_\varrho(0) \textit{ for which } \psi_p \textit{ has a local maximum at } x\}$$

has positive measure, where

$$\psi_p(x) = \psi(x) + \langle p, x\rangle, \quad x \in \mathcal{O}.$$

Sketch of the Proof. We may assume that x_0 is the unique strict maximum point of ψ on $\mathbf{B}_r(x_0)$, if $r > 0$ small. So, for $\varrho > 0$ small all maximum points of ψ_p on $\overline{\mathbf{B}}_r(x_0)$ are interior points. Then from

$$\mathrm{D}\psi(x) = -p \quad \text{at all these maximum points of } \psi_p$$

we deduce $\mathbf{B}_\varrho(0) \subset \mathrm{D}\psi(\mathcal{M})$.

Assume $\psi \in \mathcal{C}^2$ for the moment. Since $\psi(x) + K|x|^2$ is convex, the construction of $\mathcal{M}$ implies

$$-2K\mathrm{I} \leq \mathrm{D}^2\psi(x) \leq 0 \quad \text{in } \mathcal{S}^{\mathrm{N}}, \quad x \in \mathcal{M},$$

whence $|\det \mathrm{D}^2\varphi(x)| \leq (2K)^{\mathrm{N}},\ x \in \mathcal{M}$, and

$$\text{meas } (\mathbf{B}_\varrho(0)) \leq \text{meas } (\mathrm{D}\psi(\mathcal{M})) \leq \int_{\mathcal{M}} |\det \mathrm{D}^2\varphi(x)|dx \leq \text{meas } (\mathcal{M})(2K)^{\mathrm{N}}.$$

The general case of ψ the result follows from the fact that any mollification to a smooth function ψ_ε is also semi–convex with the same constant K and converge uniformly to ψ on $\mathbf{B}_r(x_0)$. Then if $\mathcal{M}_\varepsilon$ is the relative set to ψ_ε, the inclusion

$$\bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} \mathcal{M}_{\frac{1}{m}} \subset \mathcal{M}$$

concludes the proof. □

Next we present the semi–convex, concave approach. Let $v:\ \mathcal{O} \to \mathbb{R}$ be a bounded upper semi–continuous function on an arbitrary open set $\mathcal{O} \subset \mathbb{R}^{\mathrm{N}}$. For $\varepsilon > 0$ we define at $x \in \overline{\mathcal{O}}$

$$v^\varepsilon(x) \doteq \sup\left\{v(y) - \frac{1}{2\varepsilon^2}|x-y|^2 :\ y \in \overline{\mathcal{O}}\right\},$$

Analogously, if $v:\ \mathcal{O} \to \mathbb{R}$ is a bounded lower semi–continuous function we define

$$v_\varepsilon(x) \doteq \inf\left\{v(y) + \frac{1}{2\varepsilon^2}|x-y|^2 :\ y \in \overline{\mathcal{O}}\right\}.$$

We also introduce a slight extension to unbounded Lipschitz continuous functions.

Proposition A.4 *Let $v:\ \mathcal{O} \to \mathbb{R}$ be a bounded upper semi–continuous function or a unbounded Lipschitz continuous function. Then any maximum point, x_0, of v is also a maximum point of v^ε and $v(x_0) = v^\varepsilon(x_0)$. Moreover v^ε is semi–convex on $\overline{\mathcal{O}}$ and*

$$\sup_{\overline{\mathcal{O}}}(v^\varepsilon - v) \searrow 0, \quad \text{as } \varepsilon \to 0$$

(*for v unbounded Lipschitz continuous this convergence is uniform*).

Proof. For all $\eta > 0$ there exists $x_\eta \in \overline{\mathcal{O}}$ such that

$$v(x) \leq v^\varepsilon(x) \leq v(x_\eta) - \frac{1}{2\varepsilon^2}|x - x_\eta|^2 + \eta. \tag{A.2}$$

If v is bounded upper semi–continuous one has then

$$|x - x_\eta|^2 \leq 2\eta\varepsilon^2 + 4\varepsilon^2 \sup_{\mathcal{O}} |v|$$

Since

$$|x_\eta| \le |x| + \sqrt{2\eta\varepsilon^2 + 4\varepsilon^2 \sup_{\mathcal{O}} |v|}$$

there exists $\widehat{x} = \lim_{\eta\to 0} x_\eta$ for which

$$v^\varepsilon(x) = v(\widehat{x}) - \frac{1}{2\varepsilon^2}|x-\widehat{x}|^2 \quad \text{and} \quad |x-\widehat{x}| \le 2\varepsilon\sqrt{\sup_{\mathcal{O}}|v|}. \tag{A.3}$$

For the unbounded case. Let $L > 0$ be a positive constant such that

$$|v(x) - v(y)| \le L|x-y|, \quad x, y \in \overline{\mathcal{O}}.$$

Then (A.2) implies

$$|x - x_\eta|^2 \le +2\eta\varepsilon^2 + 2\varepsilon^2$$

i.e.

$$|x - x_\eta| \le L\varepsilon^2 + \sqrt{L^2\varepsilon^4 + 2\eta\varepsilon^2}.$$

Since

$$|x_\eta| \le |x| + L\varepsilon^2 + \sqrt{L^2\varepsilon^4 + 2\eta\varepsilon^2}$$

there exists $\widehat{x} = \lim_{\eta\to 0} x_\eta$ for which

$$v^\varepsilon(x) = v(\widehat{x}) - \frac{1}{2\varepsilon^2}|x-\widehat{x}|^2 \quad \text{and} \quad |x-\widehat{x}| \le 2L\varepsilon^2. \tag{A.4}$$

Now, both (A.3) and (A.4) lead to

$$0 \le v^\varepsilon(x) - v(x) \le v(\widehat{x}) - v(x),$$

hence

$$\sup_{\overline{\mathcal{O}}}(v^\varepsilon - v) \to 0, \quad \text{as } \varepsilon \to 0.$$

For the both cases, we claim that

$$x \mapsto v^\varepsilon(x) + \frac{1}{2\varepsilon^2}|x|^2$$

is convex in every convex subset of $\overline{\mathcal{O}}$. Let us consider $x+z,\ x-z,\ x \in \overline{\mathcal{O}}$, for which

$$v^\varepsilon(x \pm z) \ge v(\widehat{x}) - \frac{1}{2\varepsilon^2}|x \pm z - \widehat{x}|^2$$

holds. Then the function $w^\varepsilon(x) \doteq v^\varepsilon(x) + \frac{1}{2\varepsilon^2}|x|^2$ satisfies

$$\begin{aligned} w^\varepsilon(x+z) + w^\varepsilon(x-z) - 2w^\varepsilon(x) &\ge -\frac{1}{2\varepsilon^2}\left[|x+z-\widehat{x}|^2 + |x-z-\widehat{x}|^2 - 2|x-\widehat{x}|^2\right] \\ &\quad + \frac{1}{2\varepsilon^2}\left[|x+z|^2 + |x-z|^2 - 2|x|^2\right] \\ &= 0. \end{aligned}$$

Finally, if $v(x) \leq v(x_0),\ x \in \overline{\mathcal{O}}$, one has

$$\begin{cases} v(x_0) \leq v^\varepsilon(x_0) \leq v(\widehat{x}_0) - \dfrac{1}{2\varepsilon^2}|x_0 - \widehat{x}_0|^2 \leq v(x_0), \\ v^\varepsilon(x) \leq v(x_\eta) - \dfrac{1}{2\varepsilon^2}|x - x_\eta|^2 + \eta \leq v(x_0) + \eta \end{cases}$$

whence one concludes the proof. □

Remark A.2 Analogously, one proves that if $v:\ \mathcal{O} \to \mathbb{R}$ is a bounded lower semi–continuous function or a unbounded Lipschitz continuous function. Then any minimum point, x_0, of v is also a minimum point of v_ε and $v(x_0) = v_\varepsilon(x_0)$. Moreover v_ε is semi–concave on $\overline{\mathcal{O}}$ and

$$\sup_{\overline{\mathcal{O}}}(v_\varepsilon - v) \nearrow 0, \quad \text{as } \varepsilon \to 0$$

(for v unbounded Lipschitz continuous this convergence is uniform). □

Proposition A.5 *Fix $\varepsilon > 0$. Let $x \in \mathcal{O}$ with $\operatorname{dist}(x, \partial\mathcal{O}) > \varepsilon\sqrt{1 + 4\sup_{\mathcal{O}}|v|}$ if v is bounded upper semi–continuous or $\operatorname{dist}(x, \partial\mathcal{O}) > 2L\varepsilon^2$ if v is unbounded Lipschitz continuous. Let $\Phi \in \mathcal{C}^2(\overline{\mathcal{O}})$ such that for which $v^\varepsilon - \Phi$ attains a local maximum at x. Then $w - \Phi$ also attains a local maximum at x for*

$$w(z) \doteq v(\widehat{x} + z - x),$$

where $\widehat{x}$ is given by (A.3) *or* (A.4).

Proof. We only consider the unbounded Lipschitz continuous case. The point $\widehat{x}$ satisfies $|x - \widehat{x}| \leq 2L\varepsilon^2$ (see (A.4) above).

Next, by construction if z is near x, the point $\widehat{x} + z - x$ belongs $\mathcal{O}$, whence

$$v^\varepsilon(z) \geq v(\widehat{x} + z - x) - \frac{1}{2\varepsilon^2}|\widehat{x} - x|^2.$$

Then, if for $\Phi \in \mathcal{C}^2(\overline{\mathcal{O}})$ the function $v^\varepsilon - \Phi$ attains a local maximum at x, *i.e.*

$$(v^\varepsilon - \Phi)(x) \geq (v^\varepsilon - \Phi)(z) \quad \text{for } z \text{ near } x$$

one has

$$v(\widehat{x}) - \frac{1}{2\varepsilon^2}|x - \widehat{x}|^2 - \Phi(x) \geq v(\widehat{x} + z - x) - \frac{1}{2\varepsilon^2}|\widehat{x} - x|^2 - \Phi(z)$$

for z near x. Therefore, the function

$$w(z) \doteq v(\widehat{x} + z - x)$$

satisfies

$$(w - \Phi)(x) \geq (w - \Phi)(z) \quad \text{for } z \text{ near } x.$$ □

Remark A.3 Proposition A.5 also holds whenever we replace upper semi–continuous by lower semi–continuous, local maximum by local minimum and v^ε by v_ε, respectively. □

Relative to nonlinear partial differential equations governed by general operators $\mathbb{F} \in \mathcal{C}\left(\mathcal{O} \times \mathbb{R} \times \mathbb{R}^{\mathrm{N}} \times \mathcal{S}^{\mathrm{N}}\right)$, we have

Theorem A.2 *Assume that*

$$r \mapsto \mathbb{F}(y, r, p, \mathcal{Z})$$

is non–decreasing for every $(x, p, \mathcal{Z})$ *and* v *is a viscosity solution of*

$$\mathbb{F}(x, u, \mathrm{D}u, \mathrm{D}^2 u) \leq (\textit{respectively} \geq)\ 0 \quad \textit{on } \mathcal{O}.$$

Then v^ε *(respectively* v_ε*) is a viscosity solution of*

$$\mathbb{F}^+_\varepsilon(x, u, \mathrm{D}u, \mathrm{D}^2 u) \leq (\textit{respectively } \mathbb{F}^-_\varepsilon(x, u, \mathrm{D}u, \mathrm{D}^2 u) \geq)\ 0 \quad \textit{on } \mathcal{O}_\varepsilon,$$

where $\mathcal{O}_\varepsilon = \{z \in \mathcal{O} :\ \mathrm{dist}(z, \partial\mathcal{O}) > \varepsilon\sqrt{1 + 4\sup_{\mathcal{O}} |v|}\}$ *and*

$$\begin{cases} \mathbb{F}^+_\varepsilon(x, r, p, \mathcal{Z}) \doteq \inf\left\{\mathbb{F}(y, r, p, \mathcal{Z}) :\ |x - y| \leq \varepsilon\sqrt{1 + 4\sup_{\mathcal{O}} |v|}\right\}, \\ \mathbb{F}^-_\varepsilon(x, r, p, \mathcal{Z}) \doteq \sup\left\{\mathbb{F}(y, r, p, \mathcal{Z}) :\ |x - y| \leq \varepsilon\sqrt{1 + 4\sup_{\mathcal{O}} |v|}\right\}, \end{cases}$$

if v *is a bounded upper semi–continuous function and* $\mathcal{O}_\varepsilon = \{z \in \mathcal{O} :\ \mathrm{dist}(z, \partial\mathcal{O}) > 2L\varepsilon^2\}$ *and*

$$\begin{cases} \mathbb{F}^+_\varepsilon(x, r, p, \mathcal{Z}) \doteq \inf\left\{\mathbb{F}(y, r, p, \mathcal{Z}) :\ |x - y| \leq 2L\varepsilon^2\right\}, \\ \mathbb{F}^-_\varepsilon(x, r, p, \mathcal{Z}) \doteq \sup\left\{\mathbb{F}(y, r, p, \mathcal{Z}) :\ |x - y| \leq 2L\varepsilon^2\right\}, \end{cases}$$

if v *is an unbounded Lipschitz continuous.*

Proof. Again, we only consider the unbounded Lipschitz continuous case. Assume that v is a viscosity solution of

$$\mathbb{F}(x, u, \mathrm{D}u, \mathrm{D}^2 u) \leq 0 \quad \text{on } \mathcal{O},$$

then if $x \in \mathcal{O}_\varepsilon$, the function $w(z) \doteq v(\widehat{x} + z - x)$ is also a viscosity solution of

$$\mathbb{F}(x, u, \mathrm{D}u, \mathrm{D}^2 u) \leq 0 \quad \text{on } \mathcal{O}_\varepsilon,$$

for $|x - \widehat{x}| \leq 2L\varepsilon^2$, therefore

$$\mathbb{F}(\widehat{x}, v(\widehat{x}), \mathrm{D}\Phi(x), \mathrm{D}^2\Phi(x)) \leq 0$$

if $v^\varepsilon - \Phi$ attains a local maximum at x for $\Phi \in \mathcal{C}^2(\overline{\mathcal{O}})$ (see Proposition A.5). Then by construction of $\widehat{x}$ and $\mathbb{F}^+_\varepsilon(x, r, p, \mathcal{Z})$ we obtain that v^ε is a viscosity solution of

$$\mathbb{F}^+_\varepsilon(x, u, \mathrm{D}u, \mathrm{D}^2 u) \leq 0 \quad \text{on } \mathcal{O}_\varepsilon.$$

The property of v_ε one proves by an analogous way. □

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