An appropriate framework for handlebodies (an inverse function theorem without local convexity)

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Al profesor Don Enrique Outerelo Domínguez, con mucho cariño.

ABSTRACT

We prove an inverse function theorem for non-convex Euclidean quadrants and define handlebodies in the context of manifolds modelled on these quadrants.

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1. Introduction

Outerelo and Margalef have developed the necessary theory to deal with the product of manifolds with smooth boundary, considering manifolds with corners (see [3]). In this way, a simple manifold like the square $[0,1] \times [0,1]$ can be treated. Manifolds with corners are modelled on open sets of convex quadrants of real Banach spaces.

This note considers a wider category of manifolds with boundary in an intent to include handlebodies in a natural way. The idea is to include manifolds modelled on open sets of non-convex quadrants of Banach spaces. In the finite dimensional case, this seems the natural context to define the association of handles. The key step to define the suitable category is to establish the necessary inverse function theorem in this new context, which is made in section 3. In this section we also show that the category of manifolds with generalized corners, a concept that includes the manifolds in [3] and our generalization via non-convex quadrants, is itself included in the category of manifolds with generalized boundary as defined in [1].

Section 2 contains a brief description of the differential calculus on non-convex quadrants, including the appropriate notation. The index of the points in the boundary has to be necessarily revised. Finally section 4 describes the association of handles as a process that constructs manifolds with corners modelled on non-convex quadrants of Euclidean spaces.

We will use many results of the book [3], main reference in this paper.

2. Calculus on non-convex quadrants

Throughout this note E and F denote real Banach spaces and $\mathcal{L}(E,F)$ denotes the real Banach space of continuous linear maps from E to F. Let $\lambda \in \mathcal{L}(E,\mathbb{R})$ and let Λ be a finite linearly independent system of elements of $\mathcal{L}(E,\mathbb{R})$, possibly empty. We will use the notation

$$E_\lambda = \{x \in E \,/\, \lambda(x) \geq 0\}, \qquad E_\Lambda^\cap = \bigcap_{\lambda\in\Lambda} E_\lambda \qquad \text{and} \qquad E_\Lambda^\cup = \bigcup_{\lambda\in\Lambda} E_\lambda.$$

Of course, if $\Lambda = \{\lambda\}$ then $E_\Lambda^\cap = E_\lambda = E_\Lambda^\cup$. Both $E_\Lambda^\cap$ and $E_\Lambda^\cup$ are called quadrants of E defined by Λ. While $E_\Lambda^\cap$ is convex, $E_\Lambda^\cup$ is not if Λ has more than one element (see Figure 1). If we do not want to specify the type of quadrant, we will just write E_Λ.

Figure 1: Convex quadrant $(\mathbb{R}^2)_\Lambda^\cap$ and non-convex quadrant $(\mathbb{R}^2)_\Lambda^\cup$ with $\Lambda = \{\mathrm{pr}_1, \mathrm{pr}_2\}$.

In [3] $E_\Lambda^\cap$ is denoted by E_Λ^+. We keep the notation E_λ^0 for the kernel of λ and E_Λ^0 for $\bigcap_{\lambda\in\Lambda} E_\lambda^0$. If $\Lambda = \{\lambda_1, \ldots, \lambda_r\}$ is a linearly independent system of elements of $\mathcal{L}(E,\mathbb{R})$ there are vectors $x_1, \ldots, x_r \in E$ with $\lambda_i(x_j) = \delta_{ij}$, and then $E = E_\Lambda^0 \oplus_T \langle x_1, \ldots, x_r\rangle$.

Let U be an open set of a quadrant E_Λ. A map $f : U \to F$ is (Fréchet) differentiable at $x \in U$ if there exists a map $u \in \mathcal{L}(E,F)$ with $\lim_{y\to x} \frac{\|f(y)-f(x)-u(y-x)\|}{\|y-x\|} = 0$. In this case u is unique and we denote it by $Df(x)$. We say that the map f is of class 0 if it is continuous, f is of class p with $p \in \mathbb{N}$ if $Df(x)$ exists for every $x \in U$ and the map $Df : U \to \mathcal{L}(E,F)$ is of class $p-1$, and f is of class ∞ if f is of class p for every $p \in \mathbb{N}$. Let U and V be open sets of quadrants E_Λ y F_M respectively. A map $f : U \to V$ is a diffeomorphism of class p if f is bijective and of class p and the inverse map f^{-1} is of class p.

We now introduce the concept of index to distinguish different types of corner points. If U is an open set of E_Λ, we define the interior of U and write Int U to be the topological interior of U in E. The boundary of U is $\partial U = U - \text{Int } U$. Note that if $x \in \partial U$, there is at least one $\lambda \in \Lambda$ such that $\lambda(x) = 0$. If $x \in \text{Int } U$ we write $\text{ind}(x) = 0$. If $x \in \partial U$, then $|\text{ind}(x)| = \text{card}\{\lambda \in \Lambda \ / \ \lambda(x) = 0\} \geq 1$; $\text{ind}(x) > 0$ if x has a convex neighbourhood in U and $\text{ind}(x) < 0$ if x has not a convex neighbourhood in U. In the last case $E_\Lambda = E_\Lambda^\cup$ and $\lambda(x) \leq 0$ for any $\lambda \in \Lambda$. Note that there is no points with index -1. The proof of the following result is left to the reader:

Proposition 2.1 *Let U be an open set of a quadrant E_Λ with $\Lambda \neq \emptyset$. If $x \in U \cap E_\Lambda^0$ then $x \in \partial U$, $|\text{ind}(x)| = \text{card}\,\Lambda$ and $\text{ind}(x) > 0$ if and only if E_Λ is a convex quadrant.*

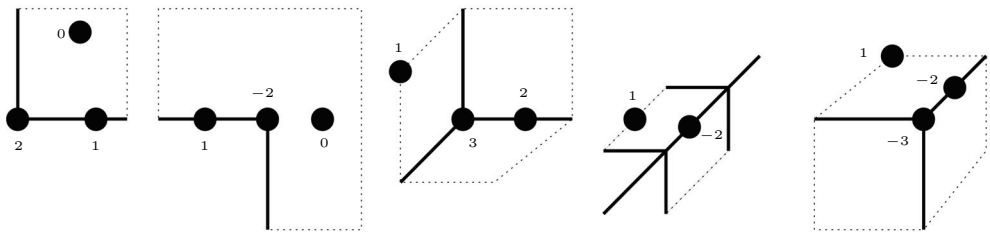

Figure 2: Different corner points with their indexes.

To establish the boundary invariance we will use the following result:

Proposition 2.2 *Let U be an open set of E, $F_M^\cup$ a quadrant of F and $f : U \to F_M^\cup$ a differentiable map at $x \in U$ such that $f(x) \in \partial F_M^\cup$. Then $Df(x)(E) \subset F_M^\cup$.*

The key of the proof is the equality $Df(x)(v) = \lim_{t \to 0} \frac{f(x+tv)-f(x)}{t}$ for any $v \in E$. Indeed, 1.2.10 of [3] follows from this result since $Df(x)(E)$ is a linear subspace of F.

Theorem 2.3 (boundary invariance) *Let $\Lambda \subset \mathcal{L}(E, \mathbb{R})$ and $M \subset \mathcal{L}(F, \mathbb{R})$ be finite and linearly independent systems, U and V open sets of E_Λ and F_M respectively and $f : U \to V$ a diffeomorphism of class p with $p \in \mathbb{N} \cup \{\infty\}$. Then $\text{ind}(x) = \text{ind}(f(x))$ for every $x \in U$.*

Proof. By Proposition 2.2 we may assume $\Lambda \neq \emptyset \neq M$, $x \in \partial U$ and $f(x) \in \partial V$. By continuity we can also assume that $x \in E_\Lambda^0$ and $f(x) \in F_M^0$, hence $|\text{ind}(x)| = \text{card}\,\Lambda$ and $|\text{ind}(f(x))| = \text{card}\,M$. The proofs of the following steps are left to the reader:

1. For every $\lambda \in \Lambda$ and $v \in \text{Ker}\lambda \cap \partial E_\Lambda$ there is a $\mu = \mu(\lambda, v)$ such that $\mu(Df(x)(v)) = 0$. Moreover, for every $\mu' \in M$ we have that $\mu'(Df(x)(v)) \leq 0$ if $F_M = F_M^\cup$ and $\mu'(Df(x)(v)) \geq 0$ if $F_M = F_M^\cap$. Again, Proposition 2.2 is used to prove this point.

2. For every $\lambda \in \Lambda$ there is a $\mu \in M$ such that $\mu(Df(x)(v)) = 0$ if $v \in \mathrm{Ker}\lambda \cap \partial E_\Lambda$.

It follows that for every $\lambda \in \Lambda$ there is a $\mu \in M$ such that $Df(x)(\mathrm{Ker}\lambda) \subset \mathrm{Ker}\mu$, since $\mathrm{Ker}\lambda = \langle \mathrm{Ker}\lambda \cap \partial E_\Lambda \rangle$. The inequality $\mathrm{card}\,\Lambda \leq \mathrm{card}\,M$ can be then deduced.

The same argument applied to f^{-1} says that $\mathrm{card}\,M \leq \mathrm{card}\,\Lambda$, hence we have deduced the equality $|\mathrm{ind}(x)| = |\mathrm{ind}(f(x))|$. To see that $\mathrm{ind}(x) = \mathrm{ind}(f(x))$ we first note that it is not possible the situation $E_\Lambda = E_\Lambda^{\cup}$ non-convex and $F_M = F_M^{\cap}$. In this case for every $\lambda \in \Lambda$ we would have that $U \cap E_\lambda^0$ is an open set of E_λ^0 and by 1.2.10 of [3] it would follow that $Df(x)(E_\lambda^0) = D(f_{|U \cap E_\lambda^0})(x)(E_\lambda^0) \subset F_M^0$ (recall that $f(x) \in F_M^0$). Moreover, since $E = \langle \cup_{\lambda \in \Lambda} E_\lambda^0 \rangle$ when $E_\Lambda = E_\Lambda^{\cup}$ is not convex, it would happen that $Df(x)(E) \subset F_M^0$ and $Df(x)$ would not be an isomorphism. Working analogously with f^{-1} we deduce that it is not possible the situation $E_\Lambda = E_\Lambda^{\cap}$ and $F_M = F_M^{\cup}$ non-convex, hence or both quadrants are convex or both are not. Since $x \in E_\Lambda^0$ and $f(x) \in F_M^0$ the proof follows then from Proposition 2.1. □

3. The inverse function theorem for non-convex quadrants

We begin with two basic results about connectivity and quadrants of real Banach spaces. They constitute the necessary generalization of Lemma 2.2.3 of [3] for the non-convex context:

Proposition 3.1 *Suppose that $x \in W \cap F_M$ where W is an open set of F and F_M is a quadrant of F. Then there is an open neighbourhood U of x in F such that $U \subset W$ and $U \cap F_M$ is a connected set.*

Proof. Let V^x be a convex open neighbourhood of x in F with $V^x \subset W$. If F_M is convex we can take $U = V^x$. If F_M is not convex we take

$$U = V^x \cap \bigcap_{\mu \in M'} F - F_\mu$$

where $M' = \{\mu \in M \ / \ \mu(x) < 0\}$. Clearly U is an open neighbourhood of x in F and $U \subset W$. Since U is convex it follows that

$$U \cap F_M = U \cap \bigcup_{\mu \in M} F_\mu = \bigcup_{\mu \in M} U \cap F_\mu = \bigcup_{\mu \in M - M'} U \cap F_\mu$$

is a union of convex sets with a common point x, hence it is a connected set. □

Proposition 3.2 *Let $g : A \to B$ be a homeomorphism where A and B are open sets of E and F respectively. Let E_Λ and F_M be quadrants of E and F respectively, $A \cap E_\Lambda$ non-empty. Suppose that $B \cap F_M$ is a connected set, $g(A \cap \mathrm{Int}E_\Lambda) \subset F_M$ and $g(A \cap \partial E_\Lambda) \subset \partial F_M$. Then $g(A \cap (E - E_\Lambda)) \subset F - F_M$.*

Proof. The set $X = g^{-1}(B \cap \mathrm{Int}F_M)$ is also connected and is included in $\mathrm{Int}E_\Lambda \cup (E - E_\Lambda)$, which is a disjoint union of open sets. Since $X \cap \mathrm{Int}E_\Lambda \neq \emptyset$ we have the inclusion $X \subset \mathrm{Int}E_\Lambda$ hence $g(A \cap (E - E_\Lambda)) \subset F - \mathrm{Int}F_M$. Since $g(A \cap (E - E_\Lambda))$ is open it follows that $g(A \cap (E - E_\Lambda)) \subset F - F_M$. □

Let U be an open set of a quadrant of E and F_M a quadrant of F. Let $f : U \to F_M$ be a map, $x \in U$ and $p \in \mathbb{N} \cup \{\infty\}$. We say that f is a local diffeomorphism of class p at x if there is an open neighbourhood V of x in U such that $f(V)$ is an open set of F_M and $f_{|V} : V \to f(V)$ is a diffeomorphism of class p. If $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B_r^=(x)$ the closed ball $\{y \in \mathbb{R}^n \ / \ \|y - x\| \leq r\}$ of radius r and centre x.

Theorem 3.3 (inverse function theorem) *Let U be an open set of a quadrant of $\mathbb{R}^n$, $(\mathbb{R}^n)_M$ a quadrant of $\mathbb{R}^n$, $f : U \to (\mathbb{R}^n)_M$ a map of class p with $p \in \mathbb{N} \cup \{\infty\}$ and x a point of U. Assume that $f(\partial U) \subset \partial(\mathbb{R}^n)_M$. Then if $Df(x)$ is an isomorphism f is a local diffeomorphism of class p at x.*

Proof. The proof follows the steps of 2.2.4 of [3]. If $x \in \mathrm{Int}\ U$ the argument is exactly the same.

Suppose then that $x \in \partial U$ where U is an open set of $(\mathbb{R}^n)_\Lambda$. Assuming that the hypothesis of the Whitney Extension Theorem are satisfied, we use it to get a local extension $\bar{f} : V^x \to \mathbb{R}^n$ of class 1. Here V^x is an open neighbourhood of x in $\mathbb{R}^n$, $V^x \cap (\mathbb{R}^n)_\Lambda \subset U$ and $\bar{f}(x) = f(x)$ whenever $x \in V^x \cap U$.

Since $D\bar{f}(x) = Df(x)$ is still an isomorphism, we can apply the inverse function theorem when there is no boundary (2.2.1 of [3]) to obtain an open neighbourhood W^x of x in V^x such that $W^{f(x)} = \bar{f}(W^x)$ is an open neighbourhood of $f(x)$ in $\mathbb{R}^n$ and $\bar{f}_{|W^x} : W^x \to W^{f(x)}$ is a diffeomorphism of class 1.

By Proposition 3.1 we can assume that $W^{f(x)} \cap (\mathbb{R}^n)_M$ is a connected set. It follows by Proposition 3.2 that $\bar{f}(W^x \cap (\mathbb{R}^n - (\mathbb{R}^n)_\Lambda)) \subset W^{f(x)} \cap (\mathbb{R}^n - (\mathbb{R}^n)_M)$, hence $f_{|U \cap W^x} : U \cap W^x \to W^{f(x)} \cap (\mathbb{R}^n)_M$ is a diffeomorphism of class 1.

Since f is of class p we deduce that f is a local diffeomorphism of class p at x using induction and the formula $Df^{-1}(y) = (Df(f^{-1}(y)))^{-1}$, as in 2.2.2 of [3].

It remains to prove the existence of local extensions for a map of class 1 defined on an open set U of a quadrant $(\mathbb{R}^n)_\Lambda$ of $\mathbb{R}^n$. We choose a $\mu > 0$ such that the closed neighbourhood $A = B_\mu^=(x) \cap (\mathbb{R}^n)_\Lambda$ of x in $(\mathbb{R}^n)_\Lambda$ is included in U and we check for A the hypothesis of 2.1.10 of [3], which are the Whitney conditions for extending maps

of class 1 defined on closed sets. In other words, we have to prove that given any $a \in A$, $k \in \{0,1\}$ and $\epsilon > 0$ there is a $\delta > 0$ such that

$$\|R_k(x,y)\| \leq \epsilon \|x-y\|^{1-k}$$

for all $x, y \in A \cap B^{=}_{\delta}(a)$, where

$$R_0(x,y) = f(y) - f(x) - Df(x)(y-x)$$

and

$$R_1(x,y) = Df(y) - Df(x).$$

For $k = 1$ the existence of a δ is a consequence of the continuity of Df (at a) and the triangular inequality $\|Df(y) - Df(x)\| \leq \|Df(y) - Df(a)\| + \|Df(a) - Df(x)\|$. Finally the case $k = 0$ is treated in the following result:

Lemma 3.1 *Let U be an open set of a quadrant (convex or not) of $\mathbb{R}^n$, $f : U \to F$ a map of class 1 and a point $a \in U$. Then for all $\epsilon > 0$ there is a $\delta > 0$ such that $\|f(y) - f(x) - Df(x)(y-x)\| \leq \epsilon \|y-x\|$ whenever $x, y \in U \cap B^{=}_{\delta}(a)$.*

Proof. It is known if $a \in \operatorname{Int} U$ or $(\mathbb{R}^n)_\Lambda = (\mathbb{R}^n)^{\cap}_{\Lambda}$ (2.1.30 of [3]). Suppose then that $(\mathbb{R}^n)_\Lambda = (\mathbb{R}^n)^{\cup}_{\Lambda}$ and $a \in \partial U$. By continuity we can assume that $a \in E^0_\Lambda$.

We may also assume that Λ has only two elements. Indeed, if for every pair $\lambda_i, \lambda_j \in \Lambda$ we obtain a δ_{ij} for the restriction of f to $U \cap (\mathbb{R}^n)^{\cup}_{\{\lambda_i,\lambda_j\}}$, then the solution would be the minimum of all of them.

Finally we may assume that $\Lambda = \{\mathrm{pr}_1, \mathrm{pr}_2\}$ since all the norms in $\mathbb{R}^n$ are equivalent and there is a linear isomorphism $\phi : \mathbb{R}^n \to \mathbb{R}^n$ with $\phi((\mathbb{R}^n)^{\cup}_{\{\mathrm{pr}_1,\mathrm{pr}_2\}}) = (\mathbb{R}^n)^{\cup}_{\{\lambda_i,\lambda_j\}}$, which is a consequence of 1.1.4 c) of [3].

Consider the restriction $f_{|U\cap(\mathbb{R}^n)_{\mathrm{pr}_1}}$ of class 1. By 2.1.30 of [3] there is a $\delta_1 > 0$ such that $B^{=}_{\delta_1}(a) \cap (\mathbb{R}^n)_{\mathrm{pr}_1} \subset U$ and $\|f(y) - f(x) - Df(x)(y-x)\| \leq \frac{\epsilon}{4}\|y-x\|$ for all $x, y \in B^{=}_{\delta_1}(a) \cap (\mathbb{R}^n)_{\mathrm{pr}_1}$. Analogously, considering the restriction $f_{|U\cap(\mathbb{R}^n)_{\mathrm{pr}_2}}$, we obtain a $\delta_2 > 0$ such that $B^{=}_{\delta_2}(a) \cap (\mathbb{R}^n)_{\mathrm{pr}_2} \subset U$ and $\|f(y) - f(x) - Df(x)(y-x)\| \leq \frac{\epsilon}{4}\|y-x\|$ for all $x, y \in B^{=}_{\delta_2}(a) \cap (\mathbb{R}^n)_{\mathrm{pr}_2}$.

There is also a $\delta_3 > 0$ such that $B^{=}_{\delta_3}(a) \cap (\mathbb{R}^n)^{\cup}_{\Lambda} \subset U$ and $\|Df(y) - Df(x)\| \leq \frac{\epsilon}{4}$ if $x, y \in B^{=}_{\delta_3}(a) \cap (\mathbb{R}^n)^{\cup}_{\Lambda}$. The proof of this fact is that of case $k = 1$.

We choose $\delta = \min\{\delta_1, \delta_2, \delta_3\} > 0$ and $x, y \in B^{=}_{\delta}(a) \cap U$. We want to prove that $\|f(y) - f(x) - Df(x)(y-x)\| \leq \epsilon\|y-x\|$. If both $x, y \in (\mathbb{R}^n)_{\mathrm{pr}_1}$ the inequality follows since $\delta \leq \delta_1$. If both $x, y \in (\mathbb{R}^n)_{\mathrm{pr}_2}$ the inequality follows since $\delta \leq \delta_2$. Suppose then that $x \in (\mathbb{R}^n)_{\mathrm{pr}_1} - (\mathbb{R}^n)_{\mathrm{pr}_2}$ and $y \in (\mathbb{R}^n)_{\mathrm{pr}_2} - (\mathbb{R}^n)_{\mathrm{pr}_1}$. We consider (see Figure 3 for case $n = 3$) $x_0 \in \mathbb{R}^n$ defined by the conditions

$$\mathrm{pr}_1(x_0) = 0 = \mathrm{pr}_2(x_0),\ \mathrm{pr}_i(x_0) = \mathrm{pr}_i(x),\ i \in \{1, \ldots, n-2\},$$

and $y_0 \in \mathbb{R}^n$ defined by the conditions

$$\mathrm{pr}_1(y_0) = 0 = \mathrm{pr}_2(y_0),\ \mathrm{pr}_i(y_0) = \mathrm{pr}_i(y),\ i \in \{1, \ldots, n-2\}.$$

Then

$$\begin{aligned}
\|f(y) - f(x) - Df(x)(y-x)\| &= \|f(y) - f(y_0) - Df(y_0)(y-y_0) \\
&\quad -f(x_0)+f(y_0)+Df(y_0)(x_0-y_0)+Df(y_0)(y-x_0) \\
&\quad +f(x_0)-f(x)-Df(x)(x_0-x)-Df(x)(y-x_0)\| \\
&\leq \|f(y) - f(y_0) - Df(y_0)(y-y_0)\| \\
&\quad +\|f(x_0) - f(y_0) - Df(y_0)(x_0-y_0)\| \\
&\quad +\|f(x_0) - f(x) - Df(x)(x_0-x)\| \\
&\quad +\|Df(y_0) - Df(x)\|\ \|y-x_0\| \\
&\leq \frac{\epsilon}{4}\|y-y_0\| + \frac{\epsilon}{4}\|x_0-y_0\| + \frac{\epsilon}{4}\|x_0-x\| + \frac{\epsilon}{4}\|y-x_0\| \\
&\leq \epsilon\|y-x\|.
\end{aligned}$$

The last inequality uses that $\|y-y_0\|$, $\|x_0-y_0\|$, $\|x_0-x\|$ and $\|y-x_0\|$ are less than $\|y-x\|$. The second inequality is possible since $\|y_0-a\| \leq \|y-a\| \leq \delta$ and $\|x_0-a\| \leq \|x-a\| \leq \delta$. □

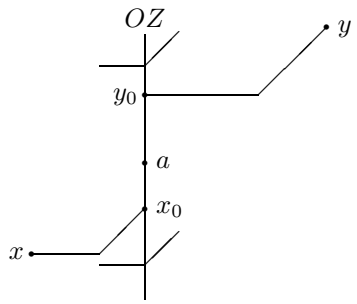

Figure 3: The inequality in the non-convex case.

Remark 3.1 The inverse function theorem given in 2.2.4 of [3] is for arbitrary real Banach spaces but with convex quadrants. Its proof uses the extension of maps of class 1 given in Proposition 2.1.35 of [3]. I do not know if a modification of this proposition could provide a Whitney Extension Theorem for non-convex quadrants in arbitrary real Banach spaces. This, apart from giving another proof of Theorem 3.3, would generalize it to the case of arbitrary real Banach spaces. One should take some care since, for example, there is no extension for a map defined on the union of the cones of the example of J. C. Wells (see [4], page 151). It is worth to mention also that Proposition 2.1.35 does not provide extensions of class ∞; because of this the proof by Margalef and Outerelo (as ours) uses only an extension of class 1.

Manifolds modelled on non-convex (and convex) quadrants of real Banach spaces will be called manifolds with generalized corners. By Theorem 2.3 and via charts we can define the index for these manifolds. We write B_kX for the set of points of the manifold X with index k, $k \in \mathbb{Z}$. We also write $\partial_k^+ X = \{x \in X \ / \ \mathrm{ind}(x) \geq k\}$ and $\partial_k^- X = \{x \in X \ / \ \mathrm{ind}(x) \leq k\}$. The boundary of X is $\partial X = \partial_1^+ X \bigcup \partial_{-1}^- X$. Manifolds without boundary ($\partial X = \emptyset$) or with smooth boundary ($\partial X = B_1 X$) have been traditionally studied. Margalef and Outerelo have carefully studied the manifolds with convex corners ($\partial X = \partial_1^+ X$) in [3]. In the following section we will define handlebodies as examples of manifolds X with $B_{-2}X \neq \emptyset$.

We finish this section showing that the category of manifolds with generalized corners is included in the category of manifolds with generalized boundary as defined by Graham in [1]. In his paper Graham considers maps with strong derivatives (a definition that includes itself a form of mean value theorem) defined on admissible sets (sets of real Banach spaces in which any point can be approximated by points of its interior). In this context the author proves an inverse function theorem (Theorem 2.9 of [1]).

Obviously any open set of a quadrant (convex or not) of a real Banach space is an admissible set of a real Banach space. On the other hand, with respect to the morphisms, the strong derivative and the concept of C_s^p used by Graham (see [1] for definitions) are apparently more restrictive that our Fréchet derivative and maps of class p. What we have found is that on open sets of quadrants, even non-convex quadrants, both concepts are equivalent. If $x \in \mathbb{R}^n$ and $r > 0$ we denote by $B_r(x)$ the open ball $\{y \in \mathbb{R}^n \ / \ \|y - x\| < r\}$ of radius r and centre x.

Proposition 3.4 *Let U be an open set of a quadrant (convex or not) of a real Banach space and $f : U \to F$ a map of class p with $p \in \mathbb{N} \cup \{\infty\}$. Then f is C_s^p and $d^p f = D^p f$.*

Proof. By induction on p. We first prove the case $p = 1$. Recall that C_s^1 is equivalent to be strongly differentiable. Let $a \in U$ and $T = Df(a)$ be the differential of f at a. We have to see that for every $\epsilon > 0$ there is a $\delta > 0$ such that $\|f(y) - f(x) - T(y - x)\| \leq \epsilon \|y - x\|$ whenever $x, y \in B_\delta(a) \cap U$.

If the quadrant is convex the argument is that on page 48 in [1]. By the mean value theorem (1.1.12 of [3]) we have that

$$\|f(y) - f(x) - T(y - x)\| \leq \sup\{\|Df(c) - Df(a)\| \ / \ c \in [x, y]\} \ \|y - x\|,$$

where $[x, y] = \{x + t(y - x) \ / \ t \in [0, 1]\}$. Continuity of Df at a and the local convexity of U yield the result that $Df(a)$ is a strong derivative of f at each point a of U.

Suppose now that the quadrant is not convex. As in Lemma 3.1 we can assume that $E_\Lambda^\cup = E_{\lambda_1} \bigcup E_{\lambda_2}$ and $a \in E_0^\Lambda$.

Considering the restriction of f to E_{λ_1} we obtain a $\delta_1 > 0$ such that $B_{\delta_1}(a) \cap E_{\lambda_1} \subset U$ and $\|f(y) - f(x) - T(y - x)\| \leq \frac{\epsilon}{2} \|y - x\|$ whenever $x, y \in B_{\delta_1}(a) \cap E_{\lambda_1}$ (convex case

already proved). Analogously, considering the restriction of f to E_{λ_2}, we obtain a $\delta_2 > 0$ such that $B_{\delta_2}(a) \cap E_{\lambda_2} \subset U$ and $\|f(y) - f(x) - T(y-x)\| \leq \frac{\epsilon}{2}\|y-x\|$ whenever $x, y \in B_{\delta_2}(a) \cap E_{\lambda_2}$.

We choose $\delta = \min\{\delta_1, \delta_2\}$ and $x, y \in B_\delta(a) \cap U$. If both x, y are in the quadrant E_{λ_1} the wanted inequality follows since $\delta \leq \delta_1$. If both x, y are in the quadrant E_{λ_2} the inequality follows since $\delta \leq \delta_2$. Suppose then that $x \in E_{\lambda_1} - E_{\lambda_2}$ and $y \in E_{\lambda_2} - E_{\lambda_1}$. Recall that $E = E^0_\Lambda \oplus_T \langle x_1, x_2 \rangle$ where $\lambda_i(x_j) = \delta_{ij}$, hence there is a vector $y_0 \in E^0_\Lambda$ such that $y - y_0 \in \langle x_1, x_2 \rangle$. It follows that there is an equivalent norm on E such that

$$\begin{aligned}
\|f(y)-f(x)-T(y-x)\| &= \|f(y) - f(y_0) - T(y - y_0) + f(y_0) - f(x) - T(y_0 - x)\| \\
&\leq \|f(y) - f(y_0) - T(y-y_0)\| + \|f(y_0) - f(x) - T(y_0 - x)\| \\
&\leq \frac{\epsilon}{2}\|y - y_0\| + \frac{\epsilon}{2}\|y_0 - x\| \\
&\leq \epsilon\|y - x\|.
\end{aligned}$$

Indeed, let $\| \ \|_0$ be the restriction to the closed subspace E^0_Λ of the norm of E. The norm of E is equivalent to the norm $\| \ \|$ defined by the formula $\|e_0 + \alpha x_1 + \beta x_2\| = \|e_0\|_0 + |\alpha| + |\beta|$ for every $e_0 \in E^0_\Lambda$ and $\alpha, \beta \in \mathbb{R}$.

We have that $y = y_0 - \alpha x_1 + \beta x_2$ where $y_0 \in E^0_\Lambda$, $\alpha > 0$ and $\beta \geq 0$ since $y \in E_{\lambda_2} - E_{\lambda_1}$, and $x = x_0 + \theta x_1 - \tau x_2$ where $x_0 \in E^0_\Lambda$, $\theta \geq 0$ and $\tau > 0$ since $x \in E_{\lambda_1} - E_{\lambda_2}$. Also $a \in E^0_\Lambda$, hence $\|y - y_0\| = \alpha + \beta$, $\|y_0 - x\| = \|y_0 - x_0\|_0 + \theta + \tau$, $\|y - x\| = \|y_0 - x_0\|_0 + \alpha + \theta + \beta + \tau$ and $\|y - a\| = \|y_0 - a\|_0 + \alpha + \beta > \|y_0 - a\|$. In particular $\|y - y_0\|$ and $\|y_0 - x\|$ are less than $\|y - x\|$ and the last inequality is proved.

To check the second inequality note that $y, y_0 \in B_{\delta_2}(a) \cap E_{\lambda_2}$ since $\|y_0 - a\| < \|y - a\| < \delta \leq \delta_2$, hence $\|f(y) - f(y_0) - T(y - y_0)\| \leq \frac{\epsilon}{2}\|y - y_0\|$. Analogously, $y_0, x \in B_{\delta_1}(a) \cap E_{\lambda_1}$ since $\|x - a\| \leq \delta \leq \delta_1$ and $\|y_0 - a\| < \|y - a\| < \delta \leq \delta_1$, hence $\|f(y_0) - f(x) - T(y_0 - x)\| \leq \frac{\epsilon}{2}\|y_0 - x\|$. This proves that f is C^1_s and that $df = Df$ at each point of U.

Suppose now $p > 1$ and assume the result for $1, \ldots, p-1$. Since f is of class p we have that f is of class 1 and Df is of class $p-1$. By induction f is C^1_s, $df = Df$, Df is C^{p-1}_s and $d^{p-1}(Df) = D^{p-1}(Df)$. Since f is C^1_s and $df = Df$ is C^{p-1}_s, f is C^p_s and

$$\begin{aligned}
d^p f &= d(d^{p-1} f) && \text{(by definition, page 49 of [1])} \\
&= d(D^{p-1} f) && \text{(by induction } d^{p-1} f = D^{p-1} f) \\
&= D(D^{p-1} f) && \text{(by induction since } D^{p-1} f \text{ is of class 1)} \\
&= D^p f.
\end{aligned}$$

The case $p = \infty$ is obvious since f is C^∞_s if and only if f is C^p_s for all $p \in \mathbb{N}$. $\square$

4. Handlebodies

The aim of this section is to define handlebodies as manifolds with generalized corners. The term smooth will be synonymous of differentiable of class ∞, and we will use the notation $D^s = \{(x_1, \ldots, x_s) \in \mathbb{R}^s \ / \ \sum_{i=1}^s x_i^2 \leq 1\}$, $\mathbb{R}_+ = \{t \in \mathbb{R} \ / \ t \geq 0\}$, $\mathbb{R}_- = \{t \in \mathbb{R} \ / \ t \leq 0\}$ and Q for the non-convex quadrant $(\mathbb{R}^2)^{\cup}_{\{\mathrm{pr}_1, \mathrm{pr}_2\}}$ of $\mathbb{R}^2$, that is

$$Q = \mathbb{R}_+ \times \mathbb{R} \bigcup \mathbb{R} \times \mathbb{R}_+.$$

A collar (neighbourhood) of a manifold M with smooth boundary is a smooth embedding $c : \partial M \times \mathbb{R}_+ \to M$ such that $c(x, 0) = x$ for every $x \in \partial M$.

Let M be a manifold of dimension n with smooth boundary $\partial M = B_1 M \neq \emptyset$. Let $p, q \in \mathbb{N}$ with $n = p + q$. The handle $D^p \times D^q$ will be denoted by H and its corners set $\partial D^p \times \partial D^q$ by C.

Let $f : \partial D^p \times D^q \to \partial M$ be a smooth embedding and let $\sim$ be the smallest equivalence relation on the disjoint union $M + H$ such that $x \sim f(x)$ if $x \in \partial D^p \times D^q$ (and $f(x) \in \partial M$). We write

$$\pi : M + H \longrightarrow X = \frac{M + H}{\sim}$$

for the corresponding quotient map. In the following diagram $\bar{i}$ and $\bar{j}$ are the natural inclusions and by definition $i = \pi \circ \bar{i}$ and $j = \pi \circ \bar{j}$, hence it is a commutative diagram:

$$\begin{tikzcd}
M \arrow[rr, "i"] \arrow[d, phantom, "\bar{i}\bigcap"] & & X \\
M + H \arrow[rr, "\pi"] \arrow[d, phantom, "\bar{j}\bigcup"] & & X \\
H \arrow[rr, "j"'] & & X
\end{tikzcd}$$

Let T_X be the quotient topology on X and let $A = X - j(C) = X - i(f(C))$.

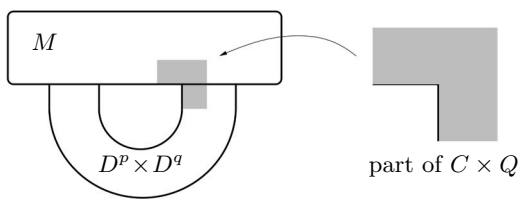

Figure 4: The natural context for handlebodies.

First step: construction of a smooth structure on X.

1. - The maps i and j are injective.
   - T_X is the final topology associated to $i : M \to X$ and $j : H \to X$.
   - The maps $i : M \to (X, T_X)$ and $j : H \to (X, T_X)$ are homeomorphisms onto the image and closed maps.
   - (X, T_X) is Haussdorf and paracompact.
   - (X, T_X) is compact if and only if M is compact.

2. Given collars $c_1 : \partial D^p \times \mathbb{R}_+ \to D^p$ and $c : \partial M \times \mathbb{R}_+ \to M$ of D^p and M respectively, the map $g : \partial D^p \times \mathrm{Int} D^q \times \mathbb{R} \to A$ defined by

$$g(x, y, s) = \begin{cases} i(c(f(x, y), s)) \text{ if } s \geq 0 \\ \\ j(c_1(x, -s), y) \text{ if } s \leq 0 \end{cases}$$

   is well defined and there is a unique smooth structure $\mathcal{A}$ on A such that the maps $i_{|M-f(C)} : M - f(C) \to (A, \mathcal{A})$ and $j_{|H-C} : H - C \to (A, \mathcal{A})$ are embeddings and $g : \partial D^p \times \mathrm{Int} D^q \times \mathbb{R} \to (A, \mathcal{A})$ is an open embedding.

   Moreover, $T_{\mathcal{A}} = (T_X)_{|A}$ and $\mathcal{A}$ is unique in the following sense: if $\mathcal{A}'$ is other smooth structure on A such that $i_{|M-f(C)} : M - f(C) \to (A, \mathcal{A}')$ and $j_{|H-C} : H - C \to (A, \mathcal{A}')$ are embeddings, then $(A, \mathcal{A})$ and $(A, \mathcal{A}')$ are diffeomorphic. Moreover, the diffeomorphism can be chosen to be the identity on $i(M - f(C))$ or on $j(H - C)$. For a proof of this item see [2], page 56.

3. Given a collar $c_2 : \partial D^q \times \mathbb{R}_+ \to D^q$ of the disk D^q there is a smooth embedding $T : f(C) \times \mathbb{R} \to \partial M$ such that

$$T(f(x, y), t) = f(x, c_2(y, t))$$

   whenever $x \in \partial D^p$, $y \in \partial D^q$ and $t \in \mathbb{R}_+$.

4. The map $G : C \times Q \to X$ defined by

$$G(x, y, t, s) = \begin{cases} i(c(T(f(x, y), t), s)) \text{ if } s \geq 0 \\ \\ j(c_1(x, -s), c_2(y, t)) \text{ if } s \leq 0 \end{cases}$$

   is well defined (by 3.) and there exists a smooth structure $\mathcal{X}$ on X such that the inclusion map $(A, \mathcal{A}) \hookrightarrow (X, \mathcal{X})$ and the map $G : C \times Q \to (X, \mathcal{X})$ are open embeddings. In particular $\mathcal{X}_{|A} = \mathcal{A}$.

   Moreover, $T_{\mathcal{X}} = T_X$ and $(X, \mathcal{X})$ is a manifold with generalized corners such that $B_{-2} = j(C)$, $\partial^-_{-3} = \emptyset$ and $\partial^+_2 = \emptyset$.

To prove this item we note that $C \times (\mathbb{R} \times \mathbb{R}_+ - \mathbb{R}_+ \times \{0\})$, $C \times \mathbb{R}_+ \times (\mathbb{R}_- - \{0\})$ and $C \times (\mathbb{R}_+ - \{0\}) \times \mathbb{R}$ are three open sets of $C \times Q$ whose union is $G^{-1}(A) = C \times (Q - \{(0,0)\})$, the restriction of G to $C \times (\mathbb{R} \times \mathbb{R}_+ - \mathbb{R}_+ \times \{0\})$ is an open embedding since $i_{|M-f(C)}$ is, the restriction of G to $C \times \mathbb{R}_+ \times (\mathbb{R}_- - \{0\})$ is an open embedding since $j_{|H-C}$ is, and the restriction of G to $C \times (\mathbb{R}_+ - \{0\}) \times \mathbb{R}$ is an open embeding since

$$G(x, y, t, s) = g(x, c_2(y, t), s)$$

whenever $(x, y, t, s) \in C \times (\mathbb{R}_+ - \{0\}) \times \mathbb{R}$.

First step is analogous to Lema 1.4 of the second chapter of [2], but without the map D used there for "straightening the corners".

Second step: the maps $i : M \to (X, \mathcal{X})$ and $j : H \to (X, \mathcal{X})$ are embeddings.

The maps i and j are smooth near $f(C)$ and C respectively since the following diagram is commutative and its "horizontal" maps are open embeddings:

$$\begin{tikzcd}
((x,y),(t,s)) \arrow[rrr] & & & c(T(f(x,y),t),s) \\
C \times (\mathbb{R} \times \mathbb{R}_+) \arrow[rr] & & M \arrow[d, "i"] & \\
C \times Q \arrow[u, phantom, "\bigcap"] \arrow[rr, "G"] & & (X, \mathcal{X}) & \\
C \times (\mathbb{R}_+ \times \mathbb{R}_-) \arrow[u, phantom, "\bigcup"] \arrow[rr] & & H \arrow[u, "j"] & \\
((x,y),(t,s)) \arrow[rrr] & & & (c_1(x,-s), c_2(y,t))
\end{tikzcd}$$

Remark 4.1 With the smooth structure on X constructed in Lema 1.4 of the second chapter of [2], the map $i : M \to X$ is not differentiable.

Third step: the construction made in first step determines the smooth structure on X. Precisely:

Theorem 4.1 *Let $\mathcal{X}$ and $\mathcal{X}'$ be two smooth structure on X constructed as in the first step, $\mathcal{X}$ via g and G depending on c_1, c_2, c and T, $\mathcal{X}'$ via g' and G' depending on c_1', c_2', c' and T'. Then there is a diffeomorphism $\psi : (X, \mathcal{X}) \to (X, \mathcal{X}')$ such that $\psi(i(M)) = i(M)$ and $\psi(j(H)) = j(H)$.*

Proof. There exist an open set $\Omega \subset \partial M \times \mathbb{R}_+$ such that $\partial M \times \{0\} \subset \Omega$, a $\epsilon > 0$ and homeomorphisms τ and α such that the following diagrams are commutative:

$$\begin{tikzcd}
 & \Omega \arrow[dl, "c_|"'] \arrow[dr, "c'_|"] & \\
M \arrow[rr, "\tau"] & & M
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \partial D^p \times [0,\epsilon) \arrow[dl, "c_{1|}"'] \arrow[dr, "c'_{1|}"] & \\
D^p \arrow[rr, "\alpha"] & & D^p
\end{tikzcd}$$

The existence of these homeomorphisms is given by the theorem of ambient isotopy for germs of collars ([2], Teorema C on page 55). We define $\psi : (X, \mathcal{X}) \to (X, \mathcal{X}')$ as the unique map that makes commutative the following diagram:

$$\begin{tikzcd}
M \arrow[r, "\tau"] \arrow[d, "i"'] & M \arrow[d, "i"] \\
(X, \mathcal{X}) \arrow[r, "\psi"] & (X, \mathcal{X}') \\
H \arrow[r, "\alpha \times 1_{D^q}"] \arrow[u, "j"] & H \arrow[u, "j"']
\end{tikzcd}$$

Precisely,

$$\psi(p) = \begin{cases} i(\tau(x)) \text{ if } p = i(x) \\ \\ j(\alpha \times 1_{D^q}(y)) \text{ if } p = j(y). \end{cases}$$

• The map ψ is well defined. We have to show that $\alpha \times 1_{D^q}(y) \in \partial D^p \times D^q$ and $f(\alpha \times 1_{D^q}(y)) = \tau(x)$ whenever $i(x) = p = j(y)$. In this case $y = (a, b) \in \partial D^p \times D^q$, $f(y) = x$ and $x \in \partial M$. Then $\alpha \times 1_{D^q}(y) = (\alpha(a), b) = (a, b) = y$ since $\alpha(a) = \alpha(c_1(a, 0)) = c_1'(a, 0) = a$, and

$$f(\alpha \times 1_{D^q}(y)) = f(y) = x = c'(x, 0) = \tau(c(x, 0)) = \tau(x),$$

the third equality since $x \in \partial M$.

• The map ψ is a homeomorphism. The maps τ and $\alpha \times 1_{D^q}$ are homeomorphisms and i and j are closed embeddings (1. of the first step).

• $\psi(A) = A$ and $\psi_{|A} : (A, \mathcal{X}_{|A}) \to (A, \mathcal{X}'_{|A})$ is a diffeomorphism. It is easy to see that $\psi(A) = A$. To prove that the restriction of ψ to A is smooth we first note that $U = \{(x, y, s) \in \partial D^p \times \mathrm{Int} D^q \times \mathbb{R} \ / s \in (-\epsilon, 0] \text{ or } s \geq 0 \text{ and } (f(x, y), s) \in \Omega\}$ is an open set of $\partial D^p \times \mathrm{Int} D^q \times \mathbb{R}$ and $\partial D^p \times \mathrm{Int} D^q \times \{0\} \subset U$, since Ω is a open set of $\partial M \times \mathbb{R}_+$ and $\partial M \times \{0\} \subset \Omega$. Having in mind how the smooth structure on A was constructed in 2. of the first step, it is then enough to show that the following diagram is commutative:

$$\begin{tikzcd}
& U \arrow[dl, "g"'] \arrow[dr, "g'"] & \\
(A, \mathcal{X}_{|A}) \arrow[rr, "\psi_{|A}"] & & (A, \mathcal{X}'_{|A})
\end{tikzcd}$$

Let $(x, y, s) \in U \subset \partial D^p \times \mathrm{Int} D^q \times \mathbb{R}$. If $s \geq 0$ then $\psi(g(x, y, s)) = \psi(i(c(f(x, y), s))) = i(\tau(c(f(x, y), s)))$ and $g'(x, y, s) = i(c'(f(x, y), s))$. Both things are the same since $(f(x, y), s) \in \Omega$ and $\tau \circ c = c'$ on Ω.

And if $s \leq 0$ then $\psi(g(x,y,s)) = \psi(j(c_1(x,-s),y)) = j(\alpha(c_1(x,-s)),y)$ and $g'(x,y,s) = j(c_1'(x,-s),y)$. Both things are the same since $s \in (-\epsilon, 0]$ and $\alpha \circ c_1 = c_1'$ on $\partial D^p \times [0,\epsilon)$.

That the inverse map is smooth has a proof completely analogous.

• The map $\psi : (X,\mathcal{X}) \to (X,\mathcal{X}')$ is a diffeomorphism. We prove that ψ is smooth (the same proof is valid for ψ^{-1}). By 4. of first step and the previous point, it is enough to show that the following composition map is smooth:

$$\phi = \psi \circ G : C \times Q \xrightarrow{G} (X,\mathcal{X}) \xrightarrow{\psi} (X,\mathcal{X}').$$

Of course ϕ is a topological embedding and by the previous point is smooth at $C \times Q - C \times \{(0,0)\}$.

Moreover, $\phi_{|C\times(\mathbb{R}\times\mathbb{R}_+)}$ is smooth since $i : M \to (X,\mathcal{X}')$ is smooth and the following diagram is commutative:

$$\begin{tikzcd}
(x,y,t,s \geq 0) \arrow[r] \arrow[dddd] & c(T(f(x,y),t),s) \arrow[r] & \tau(c(T(f(x,y),t),s)) \arrow[dddd] \\
C \times (\mathbb{R} \times \mathbb{R}_+) \arrow[r] \arrow[d, "G_{|}"'] & M \arrow[r, "\tau"] & M \arrow[d, "i"] \\
(X,\mathcal{X}) \arrow[rr, "\psi"] & & (X,\mathcal{X}') \\
i(c(T(f(x,y),t),s)) \arrow[rr] & & \psi(i(c(T(f(x,y),t),s))) = i(\tau(c(T(f(x,y),t),s)))
\end{tikzcd}$$

Also, $\phi_{|C\times(\mathbb{R}_+\times\mathbb{R}_-)}$ is smooth since $j : H \to (X,\mathcal{X}')$ is smooth and the following diagram is commutative:

$$\begin{tikzcd}
(x,y,t \geq 0, s \leq 0) \arrow[r] \arrow[dddd] & (c_1(x,-s), c_2(y,t)) \arrow[r] & (\alpha(c_1(x,-s)), c_2(y,t)) \arrow[dddd] \\
C \times (\mathbb{R}_+ \times \mathbb{R}_-) \arrow[r] \arrow[d, "G_{|}"'] & H \arrow[r, "\alpha \times 1_{D^q}"] & H \arrow[d, "j"] \\
(X,\mathcal{X}) \arrow[rr, "\psi"] & & (X,\mathcal{X}') \\
j(c_1(x,-s), c_2(y,t)) \arrow[rr] & & \psi(j(c_1(x,-s), c_2(y,t))) = j(\alpha(c_1(x,-s)), c_2(y,t))
\end{tikzcd}$$

That ϕ is smooth follows then from the next result, applied to $Y = C$ and $Z = (X,\mathcal{X}')$.

Lemma 4.1 *Let Y, Z be two manifolds and $\phi : Y \times Q \to Z$ a continuous map. Suppose that the restrictions of ϕ to the sets*

$$Y \times (Q - \{(0,0)\}), Y \times (\mathbb{R} \times \mathbb{R}_+) \text{ and } Y \times (\mathbb{R}_+ \times \mathbb{R}_-)$$

are smooth maps. Then ϕ is a smooth map.

When Y is a point this lemma says that a continuous map defined on $Q = (\mathbb{R}^2)^{\cup}_{\{\mathrm{pr}_1,\mathrm{pr}_2\}}$ is smooth if its restrictions to $Q - \{(0,0)\}$, $\mathbb{R} \times \mathbb{R}+$ and $\mathbb{R}_+ \times \mathbb{R}_-$ are.

Proof. We may assume that Y and Z are quadrants of real Banach spaces. For $y \in Y$ write $u_y = D(\phi_{|Y\times(\mathbb{R}\times\mathbb{R}_+)})(y,0,0)$ and $v_y = D(\phi_{|Y\times(\mathbb{R}_+\times\mathbb{R}_-)})(y,0,0)$. Note that

$$\begin{aligned} u_y &= \lim_{n\to\infty} D\left(\phi_{|Y\times(\mathbb{R}\times\mathbb{R}_+)}\right)(y,1/n,0) \\ &= \lim_{n\to\infty} D\left(\phi_{|Y\times(Q-\{(0,0)\})}\right)(y,1/n,0) \\ &= \lim_{n\to\infty} D\left(\phi_{|Y\times(\mathbb{R}_+\times\mathbb{R}_-)}\right)(y,1/n,0) \\ &= v_y. \end{aligned}$$

We have that $u_y = v_y$ is the differential $D\phi(y,0,0)$ of ϕ at $(y,0,0)$. Indeed,

$$\lim_{(x,t,s\geq 0)\to(y,0,0)} \frac{\|\phi(x,t,s) - \phi(y,0,0) - D\phi(y,0,0)(x-y,t,s)\|}{\|(x-y,t,s)\|} = 0$$

since $D\phi(y,0,0) = u_y$ and

$$\lim_{(x,t,s\leq 0)\to(y,0,0)} \frac{\|\phi(x,t,s) - \phi(y,0,0) - D\phi(y,0,0)(x-y,t,s)\|}{\|(x-y,t,s)\|} = 0$$

since $D\phi(y,0,0) = v_y$.

Hence ϕ is differentiable and $D\phi$ is obviously continuous. In the same way we prove that $D\phi$ is a map of class 1 and so on, completing the proof of lemma and theorem. □

Remark 4.2 In the proof of Theorem 4.1 we have used that i and j are smooth maps. Due to this we do not have an analogous theorem for the construction given in Lemma 1.4 of the second chapter of [2] (recall Remark 4.1).

Fourth step (open question): suppose that $\mathcal{X}$ is a smooth structure on X such that $i : M \to (X,\mathcal{X})$ and $j : H \to (X,\mathcal{X})$ are embeddings (embedding meaning at least topological closed embedding plus injective tangent maps and possibly the situation of the second step too). Can we obtain $\mathcal{X}$ via g and G depending on c_1, c_2, c and T as in the first step? An affirmative answer together with Theorem 4.1 would prove the following conjecture: if $\mathcal{X}$ and $\mathcal{X}'$ are two smooth structures on X such that $i : M \to (X,\mathcal{X})$, $j : H \to (X,\mathcal{X})$, $i : M \to (X,\mathcal{X}')$ and $j : H \to (X,\mathcal{X}')$ are embeddings, then there is a diffeomorphism $\psi : (X,\mathcal{X}) \to (X,\mathcal{X}')$, with probably $\psi(i(M)) = i(M)$ and $\psi(j(H)) = j(H)$.

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