ON THE FINITENESS OF PYTHAGORAS NUMBERS OF REAL MEROMORPHIC FUNCTIONS

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Dedicated to Eberhard Becker
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Abstract

We consider the 17th Hilbert Problem for global analytic functions in a modified form that involves infinite sums of squares. Then we prove a local-global principle for an analytic function to be a sum of squares. We deduce that an affirmative solution to the 17th Hilbert Problem for global analytic functions implies the finiteness of the Pythagoras number of the fields : (i) of global meromorphic functions, and (ii) of meromorphic function germs. This measures the difficulty of the problem in the analytic case.

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1 Introduction

Of all possible versions of the famous 17th Hilbert Problem, that for global analytic functions is the one standing apart from any substantial progress. As is well known, the problem is whether

Every positive semidefinite analytic function $f : \mathbb{R}^m \to \mathbb{R}$ *is a sum of squares.*

In this formulation, sums are *finite*. The best result we can state today goes back to the early '80s: *a positive semidefinite global analytic function whose zero set is discrete off a compact set is a sum of squares of meromorphic functions,* ([BKS] and [Rz],[Jw], see also [ABR]). Thus the non-compact case stands wide open, except for the case of surfaces ([ADR]). In this paper we explore some remarkable feature that makes the non-compact case very different from the compact one. Recall that the Pythagoras number of a ring is the smallest integer p (or $+\infty$) such that any sum of squares in the ring is a sum of p squares. We will prove:

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Theorem 1.1 *Suppose that every positive semidefinite global analytic function on $\mathbb{R}^m$ is a sum of squares of meromorphic functions. Then the field of global meromorphic functions on $\mathbb{R}^m$ has finite Pythagoras number.*

And we will prove the same conclusion for the field of meromorphic function germs. Thus, if we can represent every positive semidefinite function as a sum of squares (qualitative matter), we will not encounter sums of arbitrary length (quantitative matter). This kind of surprise will come out after the consideration of *infinite sums of squares*. Indeed, dealing with analytic functions, *infinite convergent* sums have a meaning, and they are a more subtle way to produce positive semidefinite functions. In this setting of infinite sums of squares, we localize the obstruction for a function to be a sum of squares at the germ of its zero set. After this sketchy preamble, let us now be more precise.

In what follows, we consider a real analytic manifold $M \subset \mathbb{R}^n$ (which we can suppose embedded as a closed set). This embedding dimension n will appear in various bounds in our results; the dimension of M will be denoted by m.

(1.2) Germs at a closed set $Z \subset M$. Germs at Z are defined exactly as germs at a point, through neighborhoods of Z in M; we will denote by f_Z the germ at Z of an analytic function f defined in some neighborhood of Z. We have the ring $\mathcal{O}(M_Z)$ of analytic function germs at Z, and its total ring of fractions $\mathcal{M}(M_Z)$, which is the ring of meromorphic function germs at Z. Note that for $Z = M$ we get nothing but global analytic and global meromorphic functions on M. If Z is connected, then $\mathcal{O}(M_Z)$ is a domain and $\mathcal{M}(M_Z)$ a field.

As usual, a germ f_Z is *positive semidefinite* when some representative f is positive semidefinite on some neighborhood of Z.

Next, we define infinite sums of squares. The first attempt to use convergent, even uniformly convergent, series of squares cannot work, as in the real case uniform convergence does not guarantee analyticity. As we must operate freely with these infinite sums, we must resort to complexification, which on the other hand is costumary in real analytic geometry. Thus, we are led to the following:

Definition 1.3 *Let $Z \subset M$ be closed. An* infinite sum of squares of analytic function germs *at Z is a series $\sum_{k\geq 1} f_k^2$ where all $f_k \in \mathcal{O}(M_Z)$, such that:*

(i) The f_k's have holomorphic extensions F_k's, all defined in the same neighborhood V of Z in some complexification of M, and

(ii) For every compact set $L \subset V$, $\sum_{k\geq 1} \sup_L |F_k|^2 < +\infty$.

The condition *(ii)* is the standard bound one uses to check that a function series is absolutely and uniformly convergent on compact sets. Accordingly, the infinite sum $\sum_{k\geq 1} f_k^2$ defines well an analytic function f on $\Omega = V \cap M$, which is a neighborhood of Z, and hence we have an analytic function germ f_Z: we write $f_Z = \sum_{k\geq 1} f_k^2 \in \mathcal{O}(M_Z)$. Hence, it makes sense to say that an element of the ring $\mathcal{O}(M_Z)$ is a sum of p squares in $\mathcal{O}(M_Z)$, even for $p = +\infty$.

Next, we consider meromorphic functions:

Definition 1.4 *Let $Z \subset M$ be closed. An analytic function germ f_Z is a* sum of $p \leq +\infty$ squares of meromorphic function germs at Z *if there is $g_Z \in \mathcal{O}(M_Z)$ such that $g_Z^2 f_Z$ is a sum of p squares of analytic function germs at Z. The zero set $\{g_Z = 0\}$ is called the* bad set *of the sum of squares.*

The above notion of bad set mimics the terminology introduced in [Dz], but notice that here we refer to each given sum of squares, not to the function it represents.

The fact that a germ f_Z may have different representations as a sum of squares, either finite or infinite, is a part of Hilbert's 17 Problem. The choice of a suitable sum of squares representation is often made to have a *controlled bad set,* that is, a bad set contained in the zero set germ $\{f_Z = 0\}$.

There is little need to remark here that the preceding definitions are restrictive in many ways. But this only means that any result on the representation of a function as a sum of squares will be stronger than one would have stated naively.

The central technical result in this paper is the following:

Theorem 1.5 *Let $f : M \to \mathbb{R}$ be a positive semidefinite analytic function, and $Z = \{f = 0\}$ its zero set. Suppose that f_Z is a sum of $p \leq +\infty$ squares. Then, f is a sum of $2^{n-1}p + 1$ squares with controlled bad set.*

Remark 1.6 To check that f_Z is a sum of $p \leq +\infty$ squares it is enough to check that *for every connected component Y of Z the germ f_Y is a sum of p squares.*

Indeed, those connected components Y form a locally finite family of disjoint closed sets of M, hence of any given complexification $\widetilde{M}$ of M. Thus, we can find a locally finite family of disjoint open sets $V \supset Y$, whose union $\bigcup V$ is an open neighborhood of Z. Then functions on that union are defined through their restrictions to the V's, and convergency on compact sets works fine. Indeed, *every compact set $L \subset \bigcup V$ is the union of the sets $L \cap V \neq \emptyset$, which are finitely many and compact; then we have the bound*

$$\sup_L |F|^2 \leq \sum_{L \cap V \neq \emptyset} \sup_{L \cap V} |F|^2$$

for any function F. □

Our central result Theorem 1.5 splits into two separated parts. Firstly, what concerns bad sets:

Proposition 1.7 *Let $Z \subset M$ be closed, and f_Z an analytic function germ which is a sum of $p \leq +\infty$ squares of meromorphic function germs. Then f_Z is a sum of $2^n p$ squares with controlled bad set. The number of squares can be lowered to $2^{n-1}p$ if f_Z vanishes on Z.*

Secondly, what refers specifically to sums of squares:

Proposition 1.8 *Let $f : M \to \mathbb{R}$ be a positive semidefinite analytic function, and $Z = \{f = 0\}$ its zero set. Suppose that the germ f_Z is a sum of $q \leq +\infty$ squares with controlled bad set. Then f is a sum of $q + 1$ squares with controlled bad set.*

Coming back to Theorem 1.5, the only general result we know so far is that if $Y \subset M$ is a compact set, the positive semidefinite germ f_Y is a finite sum of squares [ABR]. By Remark 1.6, we deduce:

Corollary 1.9 *Let $f : M \to \mathbb{R}$ be a positive semidefinite analytic function, such that all connected components of its zero set $\{f = 0\}$ are compact. Then f is a sum of squares.*

But notice that the sum here might well be infinite, since we have no common bound on the number of squares needed to represent the germs f_Y. In one case we do know such a bound: when Y is a singleton, f_Y is a sum of $p = 2^m + m$ squares ($m = \dim M$, because a suitable modification of the germ is algebraic, see [BKS])). In view of this, the result stated at the very beginning of this introduction follows readily:

Corollary 1.10 *Let $f : M \to \mathbb{R}$ be a positive semidefinite analytic function, such that the set $\{f = 0, \|x\| \geq \rho\}$ is discrete for some $\rho > 0$. Then f is a finite sum of squares.*

As was roughly explained at the beginning, from Theorem 1.5 we will deduce quantitative conclusions, namely:

Theorem 1.11 *Let $\mathcal{M}(\mathbb{R}^m)$ denote the field of global meromorphic functions on $\mathbb{R}^m$, and $\mathcal{M}_0(\mathbb{R}^m)$ that of meromorphic function germs at the origin. Suppose that every infinite sum of squares of meromorphic functions on $\mathbb{R}^m$ is a finite sum. Then the Pythagoras numbers of $\mathcal{M}(\mathbb{R}^m)$ and $\mathcal{M}_0(\mathbb{R}^m)$ are both finite.*

Since infinite sums of squares are positive semidefinite, this implies Theorem 1.1.

Note here that $\mathcal{M}_0(\mathbb{R}^n)$ is the field $\mathbb{R}(\{x_1, \ldots, x_m\})$ of meromorphic power series in m variables, and the computation of its Pythagoras number is an important old problem in the theory of quadratic forms: the only bounds known are $1, 2, 8$ for $n = 1, 2, 3$, and even finiteness remains open for larger m. This we consider a measure of the difficulty of the 17th Hilbert Problem.

The paper is organized as follows. In Section 2 we prove two key lemmas concerning the extension of holomorphic functions and sums of squares with fixed values on a given zero set.

Section 3 is devoted to the proof of Proposition 1.8. In Section 5 we prove Proposition 1.7, and Theorem 1.5 follows from this. Finally, Section 6 is devoted to the finiteness implications of the 17th Hilbert Problem (Theorem 1.11).

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2 Preliminaries on holomorphic functions

As said before, we will use some complex analysis. For holomorphic functions we refer the reader to the classical [GR], for real analytic functions and complexification, to [Ca].

(2.1) General terminology. In what follows we denote the coordinates in $\mathbb{C}^n$ by $z = (z_1, \ldots, z_n)$, with $z_i = x_i + \sqrt{-1}y_i$, where $x_i = \mathrm{Re}(z_i)$ and $y_i = \mathrm{Im}(z_i)$ are respectively the *real* and the *imaginary parts* of z_i. Also, we consider the usual conjugation $\sigma : \mathbb{C}^n \to \mathbb{C}^n, z \mapsto \overline{z} = (\overline{z_1}, \ldots, \overline{z_n})$, whose fixed points are $\mathbb{R}^n$. We say that a subset $Y \subset \mathbb{C}^n$ is *(σ-)invariant* if $\sigma(Y) = Y$; clearly, $Y \cap \sigma(Y)$ is the biggest invariant subset of Y. Thus, we see real spaces as subsets of complex spaces. We will use the notations Int and Cl to denote topological interiors and closures, respectively, with subscripts to specify the ambient space if necessary.

Let $U \subset \mathbb{C}^n$ be an invariant open set and let $F : U \to \mathbb{C}$ be a holomorphic function. We say that F is *(σ-)invariant* if $F(z) = \overline{F(\overline{z})}$. This implies that F restricts to a real analytic function on $U \cap \mathbb{R}^n$. In general, we denote by:

$$\Re(F) : \begin{array}{ccc} U & \to & \mathbb{C} \\ z & \mapsto & \frac{F(z)+\overline{F(\overline{z})}}{2} \end{array} \qquad \Im(F) : \begin{array}{ccc} U & \to & \mathbb{C} \\ z & \mapsto & \frac{F(z)-\overline{F(\overline{z})}}{2\sqrt{-1}} \end{array}$$

the *real* and the *imaginary* parts of F, which satisfy $F = \Re(F) + \sqrt{-1}\,\Im(F)$. Note that both are invariant holomorphic functions.

We also recall that $\mathbb{R}^n$ has in $\mathbb{C}^n$ a neighborhood basis consisting of *invariant open Stein neighborhoods.* □

We next see how to extend an holomorphic function modulo another with some control on its behaviour.

Lemma 2.2 *Let $\mathcal{U}$ be an invariant open Stein neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$ and let $\Phi : \mathcal{U} \to \mathbb{C}$ be an invariant holomorphic function. Let V be an invariant open neighborhood of the connected components of $\Phi^{-1}(0)$ that meet $\mathbb{R}^n$, and suppose that V does not meet the other connected components. Let $K \subset \mathcal{U}$ be an invariant compact set. Then there exist a real constant $\mu > 0$ and an invariant compact set $L \subset V$ for which the following property holds:*

(*) *for every invariant holomorphic function* $C : V \to \mathbb{C}$ *there exists an invariant holomorphic function* $A : \mathcal{U} \to \mathbb{C}$ *such that* $\Phi|_V$ *divides* $A|_V - C$ *and*

$$\sup_K |A| < \mu \sup_L |C|.$$

Proof. First, consider the coherent sheaf of ideals $\mathcal{J} \subset \mathcal{O}_{\mathcal{U}}^{\mathbb{C}}$ generated by Φ, and the exact sequence of coherent sheafs

$$0 \to \mathcal{J} \to \mathcal{O}_{\mathcal{U}} \to \mathcal{O}_{\mathcal{U}}/\mathcal{J} \to 0.$$

Now, we have a corresponding diagram of cross sections:

$$\begin{array}{ccccc} \mathcal{J}(\mathcal{U}) & \longrightarrow & \mathcal{O}(\mathcal{U}) & \longrightarrow & \Gamma(\mathcal{U}, \mathcal{O}_{\mathcal{U}}/\mathcal{J}) \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{J}(V) & \longrightarrow & \mathcal{O}(V) & \longrightarrow & \Gamma(V, \mathcal{O}_{\mathcal{U}}/\mathcal{J}) \end{array}$$

Here, the upper right arrow is onto because $\mathcal{U}$ is Stein. Furthermore, the right vertical arrow is onto too. Indeed, each cross section of $\mathcal{O}_{\mathcal{U}}/\mathcal{J}$ on V can be extended by zero to $\mathcal{U}$, because the support of $\mathcal{O}_{\mathcal{U}}/\mathcal{J}$ in V is closed in $\mathcal{U}$. Hence we have a linear surjective homomorphism

$$\varphi : \mathcal{O}(\mathcal{U}) \longrightarrow \mathcal{O}(V)/\mathcal{J}(V) \equiv \Gamma(V, \mathcal{O}_{\mathcal{U}}/\mathcal{J}).$$

We equip these vector spaces with their natural topologies. As is well known $\mathcal{O}(\mathcal{U})$ and $\mathcal{O}(V)$ are Frechet spaces with the topology of the uniform convergence on compact sets. Also, by the closure of modules theorem, we know that $\mathcal{J}(V)$ is a closed subspace of $\mathcal{O}(V)$, and $\mathcal{O}(V)/\mathcal{J}(V)$ is also a Frechet space with the quotient topology. Summing up, φ is a continuous surjective homomorphism of Frechet spaces, consequently open [Sc, III.1.2]. In order to make use of this, we describe explicitly the topologies involved.

Let $\{K_i\}_i$ and $\{L_i\}_i$ be families of invariant compact sets in $\mathcal{U}$ and V, such that:

- $\mathrm{Int}_{\mathbb{C}^n}(L_1) \neq \emptyset$,
- $L_i \subset K_i$ for all i,
- $L_i \subset L_{i+1}$ and $K_i \subset K_{i+1}$ for all i, and
- $\bigcup_i \mathrm{Int}_{\mathbb{C}^n}(L_i) = V$ and $\bigcup_i \mathrm{Int}_{\mathbb{C}^n}(K_i) = \mathcal{U}$.

Then the topology of $\mathcal{O}(\mathcal{U})$ (resp. $\mathcal{O}(V)$) is defined by the pseudonorm:

$$\|F\| = \textstyle\sum_i \frac{1}{2^i} \frac{\sup_{K_i} |F|}{1 + \sup_{K_i} |F|} \qquad \text{for} \quad F \in \mathcal{O}(\mathcal{U})$$

$$(\text{resp.} \ \|G\|' = \textstyle\sum_i \frac{1}{2^\ell} \frac{\sup_{L_i} |G|}{1 + \sup_{L_i} |G|} \qquad \text{for} \quad G \in \mathcal{O}(V) \ ).$$

Moreover, by [Sc, I.6.3], the quotient topology of $\mathcal{O}(V)/\mathcal{J}(V)$ is given by the following third pseudonorm:

$$\|\xi\|^* = \inf_G\{\|G\|' : \xi = G + \mathcal{J}(V)\} \text{ for } \xi \in \mathcal{O}(V)/\mathcal{J}(V).$$

Next, given the compact set $K \subset \mathcal{U}$, we have the open subset of $\mathcal{O}(\mathcal{U})$ given by

$$W = \{H \in \mathcal{O}(\mathcal{U}) : \sup_K |H| < 1\}.$$

Since φ is open, $\varphi(W)$ is an open neigborhood of 0 in $\mathcal{O}(V)/\mathcal{J}(V)$, and, there exists $\varepsilon > 0$ such that

$$W^* = \{\xi : \|\xi\|^* < \varepsilon\} \subset \varphi(W).$$

Then, we pick $\mu > \frac{2}{\varepsilon}$, and $L = L_i$ with i such that $\sum_{j>i} \frac{1}{2^j} < \frac{\varepsilon}{2}$. We will prove the condition $(*)$ in the statement for such $\mu > 0$ and $L \subset V$, which by construction depend only on K.

Let $C \in \mathcal{O}(V)$ a non-zero holomorphic function. Since the interior of L in $\mathbb{C}^n$ is not empty, $a = \sup_L |C| > 0$, and we denote $G = \frac{1}{a\mu} C \in \mathcal{O}(V)$. Then $\sup_L |G| = \frac{1}{\mu} < \frac{\varepsilon}{2}$, and we have:

$$\|G\|' = \sum_j \frac{1}{2^j} \frac{\sup_{L_j} |G|}{1 + \sup_{L_j} |G|} = \sum_{j=1}^{i} \frac{1}{2^j} \frac{\sup_{L_j} |G|}{1 + \sup_{L_j} |G|} + \sum_{j>i} \frac{1}{2^j} \frac{\sup_{L_j} |G|}{1 + \sup_{L_j} |G|}$$

$$< \sup_L |G| \sum_{j=1}^{i} \frac{1}{2^j} + \sum_{j>i} \frac{1}{2^j} < \sup_L |G| \sum_{j=1}^{i} \frac{1}{2^j} + \frac{\varepsilon}{2} < \varepsilon.$$

Whence, setting $\xi = G + \mathcal{J}(V)$, we get

$$\|\xi\|^* \leq \|G\|' < \varepsilon,$$

and $\xi \in W^* \subset \varphi(W)$. Consequently, there exists $H \in W$ such that $\varphi(H) = \xi$, that is $H|_V - G \in \mathcal{J}(V)$, and the holomorphic function $F = a\mu H \in \mathcal{O}(\mathcal{U})$ verifies the required conditions. For, $F|_V - C = a\mu(H|_V - G) \in \mathcal{J}(V)$ and since $\sup_K |H| < 1$:

$$\sup_K |F| = a\mu \sup_K |H| < a\mu = \mu \sup_L |C|$$

Finally, if C invariant we take $A = \Re(F)$, and A satisfies the same conditions. First,

$$A|_V - C = \Re(F|_V) - C = \Re(F|_V - C) = \Re(\Lambda\Phi|_V) = \Re(\Lambda)\Phi|_V \in \mathcal{J}(V)$$

for some $\Lambda \in \mathcal{O}(V)$. Secondly, as K is invariant:

$$\sup_K |A| = \sup_K |\Re(F)| = \sup_K \left| \frac{F + \overline{F \circ \sigma}}{2} \right| \leq \sup_K \frac{|F| + |\overline{F \circ \sigma}|}{2} \leq \sup_K |F| < \mu \sup_L |C|,$$

and the proof is complete. □

Now we apply this to infinite sums of squares:

Proposition 2.3 *Let $\mathcal{U}$ be an invariant open Stein neighborhood of $\mathbb{R}^n$ in $\mathbb{C}^n$ and let $\Phi : \mathcal{U} \to \mathbb{C}$ be an invariant holomorphic function. Let V be an open invariant neighborhood of the connected components of $\Phi^{-1}(0)$ that meet $\mathbb{R}^n$, and suppose that V does not meet the other connected components. Let $C_k : V \to \mathbb{C}$ be invariant holomorphic functions such that $\sum_k \sup_L |C_k|^2 < +\infty$ for every compact set $L \subset V$. Then there exist invariant holomorphic functions $A_k : \mathcal{U} \to \mathbb{C}$, such that $\sum_k \sup_K |A_k|^2 < +\infty$ for every compact set $K \subset \mathcal{U}$ and $\Phi|_V$ divides all the differences $A_k|_V - C_k$.*

Furthermore, if C_1 is divisible on V by some holomorphic function $H : \mathcal{U} \to \mathbb{C}$, the function A_1 can be chosen divisible on $\mathcal{U}$ by H.

Proof. Let $\{K_i\}$ be a family of invariant compact sets such that

- $K_i \subset K_{i+1}$ for all i, and
- $\bigcup_i \mathrm{Int}_{\mathbb{C}^n}(K_i) = \mathcal{U}$.

By 2.2, for each i there exists $\mu_i > 0$ and a compact set $L_i \subset V$ such that if $C \in \mathcal{O}(V)$ is invariant there exists $A \in \mathcal{O}(\mathcal{U})$ invariant such that $A|_V - C = \Phi|_V B$ for some $B \in \mathcal{O}(V)$ and $\sup_{K_i} |A| < \mu_i \sup_{L_i} |C|$. We may assume that $L_i \subset L_{i+1}$ for all i.

Since $\sum_k \sup_{L_i} |C_k|^2 < +\infty$ for all i, there exist a strictly increasing sequence (k_i) of positive integers such that

$$\sum_{k \geq k_i} \sup_{L_i} |C_k|^2 < \frac{1}{2^i \mu_i^2}$$

For each k such that $k_i \leq k < k_{i+1}$ there exists a holomophic function $A_k : \mathcal{U} \to \mathbb{C}$ such that $\sup_{K_i} |A_k| < \mu_i \sup_{L_i} |C_k|$ and $\Phi|_V$ divides $A_k|_V - C_k$. Let us see that for every compact set $K \subset \mathcal{U}$ the series $\sum_k \sup_K |A_k|^2 < +\infty$. Since $\bigcup_i \mathrm{Int}_{\mathbb{C}^n}(K_i) = \mathcal{U}$, it is enough to check that $\sum_k \sup_{K_i} |A_k|^2 < +\infty$ for all i. But,

$$\begin{aligned}
\sum_k \sup_{K_i} |A_k|^2 &= \sum_{1 \leq k < k_i} \sup_{K_i} |A_k|^2 + \sum_{j \geq i} \Big(\sum_{k_j \leq k < k_{j+1}} \sup_{K_i} |A_k|^2 \Big) \\
&\leq \sum_{1 \leq k < k_i} \sup_{K_i} |A_k|^2 + \sum_{j \geq i} \Big(\sum_{k_j \leq k < k_{j+1}} \sup_{K_j} |A_k|^2 \Big) \\
&\leq \sum_{1 \leq k < k_i} \sup_{K_i} |A_k|^2 + \sum_{j \geq i} \Big(\sum_{k \geq k_j} \mu_j^2 \sup_{L_j} |C_k|^2 \Big) \\
&< \sum_{1 \leq k < k_i} \sup_{K_i} |A_k|^2 + \sum_{j \geq i} \mu_j^2 \frac{1}{2^j \mu_j^2} \leq \sum_{1 \leq k < k_{i+1}} \sup_{K_i} |A_k|^2 + 1 < +\infty
\end{aligned}$$

This concludes the proof of the statement, except for its last assertion. To see that, notice that the convergency bound does not depend on the choice of a single term of the series. Then, we write $C_1 = C_1^* H$, where C_1^* is holomorphic on V, and by 2.2 there is an holomorphic function A_1^* such that $\Phi|_V$ divides $A_1^*|_V - C_1^*$. Whence, we conclude by taking $A_1 = A_1^* H$. □

3 Gobalization of sums of squares

The purpose of this section is to prove Proposition 1.8. We start with the following:

Proposition 3.1 *Let $f : \mathbb{R}^n \to \mathbb{R}$ be an analytic function, and $Z = \{f = 0\}$ its zero set. Suppose that the germ f_Z is a sum of $q \leq +\infty$ squares of analytic function germs. Then f divides a sum $\sum_{k=1}^q a_k^2$ of q squares of analytic functions on $\mathbb{R}^n$ and the analytic function $\sum_{k=1}^q a_k^2/f$ is strictly positive on Z.*

Furthermore, if a global analytic function $h : \mathbb{R}^n \to \mathbb{R}$ divides the first addend of the representation of f_Z as a sum of squares, then a_1 can be chosen divisible by h.

Proof. Consider an open Stein neighborhood $\mathcal{U}$ of $\mathbb{R}^n$ in $\mathbb{C}^n$ on which f has an invariant holomorphic extension F. By hypothesis, there are invariant holomorphic functions $C_k : V \to \mathbb{C}$, defined on an open neighborhood $V \subset \mathcal{U}$ of Z in $\mathbb{C}^n$, such that $F|_V = \sum_k C_k^2$, where the series converges in the strong sense of 1.3*(ii)*.

Up to shrinking V, we may assume that it is invariant and does not intersect the connected components of $F^{-1}(0)$ that do not meet $\mathbb{R}^n$. By 2.3, applied to $\Phi = F^2$, there exist invariant holomorphic functions $A_k : \mathcal{U} \to \mathbb{C}$, such that $\sum_k \sup_K |A_k|^2 < +\infty$ for all compact set $K \subset \mathcal{U}$ and F^2 divides $A_k - C_k$ on V (and if h is given, its complexification H is defined on $\mathcal{U}$ and divides A_1).

On V we have:

$$\sum_k A_k^2 - F = \sum_k A_k^2 - \sum_k C_k^2 = \sum_k \left(A_k^2 - C_k^2\right),$$

and this series is convergent on compact sets, as $\sum_k A_k^2$ and $\sum_k C_k^2$ are so. By construction, F^2 divides on V each term $A_k^2 - C_k^2 = (A_k + C_k)(A_k - C_k)$, hence it divides their sum $\sum_k A_k^2 - F$. Thus there is an holomorphic function $\Psi : V \to \mathbb{C}$ such that on V we have:

$$\sum_k A_k^2 = F + \Psi F^2 = uF, \quad \text{where } u = 1 + \Psi F.$$

Clearly, u has no zeros in the zero set of F, hence, u is a holomorphic unit in a perhaps smaller V. To concude take $a_k = A_k|_{\mathbb{R}^n}$. □

We are ready for the:

Proof of Proposition 1.8. Recall that M is embedded as a closed set in $\mathbb{R}^n$, hence there is an analytic function $h : \mathbb{R}^n \to \mathbb{R}$ with zero set $\{h = 0\} = M$. Consider an open tubular neighborhood Ω of M in $\mathbb{R}^n$, endowed with the corresponding analytic retraction $\pi : \Omega \to M$. By composition with π, all functions extend from M to Ω; henceforth, we denote those extensions with bars.

By hypothesis, on an open neighborhood V of $Z = \{f = 0\}$ in M we have a representation as a sum of q squares:

$$g^2 f = \sum_k b_k^2,$$

with $\{g = 0\} \subset Z$; multiplying by g^2, we can assume $g \geq 0$.

Step I: Global denominator. Consider the function $g' = h^2 + \bar{g}$. It is analytic on the neighborhood $U = \pi^{-1}(V)$ of Z in $\mathbb{R}^n$ and $\{g' = 0\} = \{g = 0\} \subset Z$ by our choice of g. Thus, g' is an analytic function on U and its zero set is closed in $\mathbb{R}^n$. Consequently we can consider the locally principal coherent analytic sheaf of ideals defined by:

$$\mathcal{J}_x = \begin{cases} g'_x \, \mathcal{O}_{\mathbb{R}^n,x} & \text{if} \quad x \in U \\ \mathcal{O}_{\mathbb{R}^n,x} & \text{if} \quad x \notin Z. \end{cases}$$

As $H^1(\mathbb{R}^n, \mathcal{O}^*_{\mathbb{R}^n}) = H^1(\mathbb{R}^n, \mathbb{Z}_2) = 0$, this sheaf is globally principal, say generated by g''. The zero set of g'' is that of g' and contained in Z, and $v = g''/g'$ is an analytic unit on U. On $V = U \cap M$ we can write:

$$g''^2 f = (vg')^2 f = \sum_k (vb_k)^2 = \sum_k c_k^2.$$

Thus the denominator g'' is a global analytic function whose zero set is contained in Z.

Step II. Global sum of squares. After Step I, we only care for the sum of squares $\gamma = \sum_k c_k^2$. By composition with π, this sum of squares extends to $\sum_k \bar{c}_k^2$ on U (π respects convergency, as one easily sees by complexification), and we consider

$$\gamma' = h^2 + \bar{\gamma} = h^2 + \sum_{k=1}^{q} \bar{c}_k^2.$$

This analytic function is defined on U, but as done before for g and g', we find a global analytic function γ'' with the same zero set Z as γ and γ', and such that γ''/γ', is a positive analytic function on U. Now γ'' is a sum of $q+1$ squares on U, which is a neigborhood of its zero set, and the first square is h^2. Hence by Proposition 3.1, γ'' divides a sum $\sum_k a_k^2$ of $q+1$ squares of global analytic functions, and the quotient w is strictly positive on Z; furthermore we can choose a_1 divisible by h.

Step III. Additional square. The function

$$\alpha = \frac{\gamma''^2 + \sum_k a_k^2}{\gamma''} = \gamma'' + w$$

is a well defined strictly positive analytic function on $\mathbb{R}^n$: both addends in the right hand side are ≥ 0, the first one does not vanish off Z, and the second one does not vanish on Z. Thus, let β stand for the positive square root of α^{-1}, and we have the following sum of $q+2$ squares of global analytic functions on $\mathbb{R}^n$:

$$\gamma'' = (\beta\gamma'')^2 + \sum_k (\beta a_k)^2.$$

As h divides a_1, when we restrict this to M, the square $(\beta a_1)^2$ disappears, and $\gamma''|_M$ is a sum of $q+1$ squares of analytic functions.

Step IV. Conclusion. By construction, $g''^2 f$ and $\gamma''|_M$ have the same zero set Z, and $g''^2 f/\gamma''|_M$ is a positive unit on a neighborhood of it. Hence that quotient is a positive unit on M, hence has an analytic square root u. We conclude that $g''^2 f = u^2\gamma''|_M$ is a sum of $q+1$ squares of global analytic functions, as $\gamma''|_M$ is. □

4 Bad sets

The purpose of this section is to show how to control the bad set of a sum of squares of meromorphic functions, which is the content of Proposition 1.7. This control is essential to apply Proposition 1.8. First of all, we can always reduce to the case when $M \subset \mathbb{R}^n$ is an open set $\Omega \subset \mathbb{R}^n$ using a tubular neighborhood Ω of M.

After this remark, it is clear that the following statement implies Proposition 1.7:

Lemma 4.1 *Let $f : \Omega \to \mathbb{R}$ be an analytic function defined on an open set $\Omega \subset \mathbb{R}^n$. Let $h : \Omega \to \mathbb{R}$ be an analytic function such that $h^2 f$ is a sum of $p \leq +\infty$ squares of analytic functions. Set* $\dim\{h = 0, f \neq 0\} = d$. *Then, there exist an analytic function $g : \Omega \to \mathbb{R}$ such that $g^2 f$ is a sum of $q \leq 2^{d+1}p$ squares, and $\{g = 0\} \subset \{f = 0\}$. Moreover, on a smaller neighborhood of $\{f = 0\}$ we can assume $r \leq 2^d p$.*

Proof. Consider the global analytic set $Y = \{h = 0\}$. We pick a point y_i in each irreducible component Y_i of Y that is not contained in $\{f = 0\}$. Clearly, we can suppose $f(y_i) \neq 0$ and that the y_i's form a discrete set. By a small diffeotopy around each y_i we can move y_i off Y, to obtain a smooth diffeomorphism $\psi : \Omega \to \Omega$ which maps each y_i to $y_i' \notin Y$ and is the identity on a neighborhood of $\{f = 0\}$. By the latter condition, f^2 divides the map $\psi - \mathrm{Id}$, hence $\psi = \mathrm{Id} + f^2\mu$ for a smooth map $\mu : \Omega \to \mathbb{R}^n$. Now, let $\eta : \Omega \to \mathbb{R}^n$ be an analytic mapping close to μ. Then $\varphi = \mathrm{Id} + f^2\eta$ is close to ψ, and consequently φ is an analytic diffeomorphism of Ω. Note that φ is the identity on $\{f = 0\}$, and so f *and* $f \circ \varphi$ *have the same zeros*. Also, by looking at Taylor expansions, one sees that $f \circ \varphi = f + f^2 h$ for some analytic map $h : \Omega \to \mathbb{R}^n$. Thus we can write $f \circ \varphi = vf$, where $v = 1 + fh$ has no zero: a zero x of v would be a zero of $f \circ \varphi$, hence one of f, and $v(x) = 1 + f(x)h(x) = 1$! Moreover, as f is positive semidefinite, so is v, and $u = \sqrt{v}$ is a well defined strictly positive analytic function such that $f \circ \varphi = u^2 f$. By hypothesis $h^2 f = \sum_j h_j^2$, which gives:

$$u^2(h \circ \varphi)^2 f = (h \circ \varphi)^2 (f \circ \varphi) = \sum_j (h_j \circ \varphi)^2$$

(note that if the sum is infinite, it is well defined in the sense of 1.3). Hence,

$$(h^2 + u^2(h \circ \varphi)^2) f = \sum_j (h_j^2 + (h_j \circ \varphi)^2).$$

Now, we multiply both sides times $h^2 + u^2(h \circ \varphi)^2$ to get

$$\delta^2 f = \sum_j (h_j^2 + (h_j \circ \varphi)^2)(h^2 + u^2(h \circ \varphi)^2),$$

with $\delta = h^2 + u^2(h \circ \varphi)^2$. If the sum is infinite, we have another infinite sum. In case the sum is finite, then we recall that the product of two sums of two squares is again a sum of two squares, and we get twice the number of squares. Finally, the bad set now is:

$$\{\delta = 0\} = \{h = 0\} \cap \{h \circ \varphi = 0\} = \{h = 0\} \cap \varphi^{-1}(Y),$$

so that,

$$\{\delta = 0\} \setminus \{f = 0\} \subset \bigcup_i Y_i \cap \varphi^{-1}(Y).$$

But no irreducible component Y_i is contained in $\varphi^{-1}(Y)$, because $\varphi(y_i) \notin Y$, hence $\dim(Y_i \cap \varphi^{-1}(Y)) < \dim Y_i \leq d$.

Thus we drop the dimension of the bad set off $\{f = 0\}$, and after $d+1$ repetitions we get the first assertion of the statement. Instead, we can stop after d times, and then

$$\dim\{g = 0, f \neq 0\} \leq 0.$$

This means that $D = \{g = 0, f \neq 0\}$ is a discrete closed subset of Ω, and this latter can be replaced by $\Omega \setminus D$ to get the second assertion. □

Consequently, Propositions 1.7 and 1.8 are proved, and together they imply Theorem 1.5. In the next section we use the latter to expose the quantitative content of the 17th Hilbert problem in the non-compact case.

5 The finiteness implications

To start with, we prove the following reformulation of the first half of Theorem 1.11, which concerned global meromorphic functions on $\mathbb{R}^m$.

Proposition 5.1 *Suppose that the Pythagoras number of $\mathcal{M}(\mathbb{R}^m)$ is $+\infty$. Then, there is a positive semidefinite analytic function $f : \mathbb{R}^m \to \mathbb{R}$ that is a sum of squares of meromorphic functions (hence positive semidefinite), but not a finite sum of squares.*

Proof. By hypothesis, for each $p \geq 1$ there is an analytic function $f_p : \mathbb{R}^m \to \mathbb{R}$ which is a sum of squares of meromorphic functions, but not of p squares. We may suppose that the zero set Z_p of f_p has codimension 2.

Indeed, set $g = f_p$. At each zero x of f, we have a unique factorization of analytic germs $g_x = \zeta_x^2 \eta_x$, η_x without multiple factors. The germ $\{\eta_x = 0\}$ has codimension ≥ 2, since otherwise some irreducible factor ξ_x of η_x would be real, and g_x would change sign at x. Now, the ζ_x's generate a locally principal coherent analytic sheaf of ideals, $\mathcal{J}$, which is globally principal (once again we use that on $\mathbb{R}^m$ locally principal sheaves are all globally principal). Let h be a global generator of $\mathcal{J}$, so that $g = h^2 g'$. Each germ g'_x coincides with η_x up to a unit, hence its zero set has codimension ≥ 2 and g'_x does not change sign. Replacing g by g' we may suppose its zero set has codimension 2 as claimed.

Back to our f_p's, assume for a moment that Z_p can be moved into the open cylinder

$$V_p = \{x = (x', x_m) \in \mathbb{R}^m : \|x' - a'_p\| < \tfrac{1}{4}\},$$

where $a'_p = (p, 0, \ldots, 0) \in \mathbb{R}^{m-1}$. Then the Z_p's form a locally finite family, and $Z = \bigcup_p Z_p$ is a closed analytic subset of $\mathbb{R}^m$. Consequently, we can define the following locally principal sheaf :

$$\mathcal{J}_x = \begin{cases} f_p \mathcal{O}_{\mathbb{R}^m,x} & \text{if} \quad x \in Z_p \\ \mathcal{O}_{\mathbb{R}^m,x} & \text{if} \quad x \notin Z. \end{cases}$$

Again, we know that $\mathcal{J}$ has a global generator f. Thus, on each V_p there is an analytic unit v_p such that $f = v_p f_p$. Note also that the zero set of f is Z, which does not disconnect $\mathbb{R}^m$, because its codimension is ≥ 2. Hence, f has constant sign on $\mathbb{R}^m$ and we may assume $f \geq 0$, and $v_p > 0$. In particular, $+\sqrt{v_p}$ is a well defined analytic function on V_p, so that f and f_p behave the same concerning sums of squares. Since, by construction, the connected components Y of Z are the ones of the Z_p's, we deduce that each germ f_Y is a sum of squares, and by Remark 1.6, so is the germ f_Z. Thus, by Theorem 1.5, f is a sum of squares of meromorphic functions. However, this sum cannot be finite, say of q squares, because f_q is not a sum of q squares.

To complete the proof it only remains to move each Z_p by a suitable analytic diffeomorphism of $\mathbb{R}^m$. This we do now.

Since Z_p has codimension ≥ 2, many lines do not meet Z_p, and after a linear change of coordinates, we may assume this is the case for the x_m-axis. Then, we pick an analytic function $\delta(x_m)$ such that $0 < \delta(x_m) < \operatorname{dist}\big(Z_p, (0, \ldots, 0, x_m)\big)$, and the analytic diffeomorphism

$$(x', x_m) \mapsto \left(\frac{\sqrt{1 + x_m^2}}{\delta(x_m)} \, x', x_m \right)$$

moves Z_p off $\{\|x'\|^2 < 1 + x_m^2\}$. Thus, we henceforth assume $Z_p \subset \{\|x'\|^2 \geq 1 + x_m^2\}$. Then, we consider the analytic diffeomorphism: $\varphi(x', x_m) = (y', y_m)$ defined by the equations:

$$y' - a'_p = \frac{x'}{4(1 + y_m^2)}, \qquad y_m = 2\|x'\|^2 - x_m.$$

The conclusion is that $\varphi(Z_p)$ is contained in $\|y' - a'_p\| < \frac{1}{4}$, and we are done.

Indeed, if $(x', x_m) \in Z_p$, then $\|x'\|^2 \geq 1 + x_m^2$, so that:

$$y_m = 2\|x'\|^2 - x_m \geq \|x'\|^2 + 1 + x_m^2 - x_m > \|x'\|.$$

Consequently:

$$\|y' - a'_p\| = \frac{\|x'\|}{4(1 + y_m^2)} < \frac{\|x'\|}{4(1 + \|x'\|^2)} < \frac{1}{4}.$$

□

Next, we look at the Pythagoras number of the field $\mathcal{M}_0(\mathbb{R}^m) = \mathbb{R}(\{x_1, \dots, x_m\})$ of meromorphic power series. The second assertion of Theorem 1.11 can be written as follows:

Proposition 5.2 *Suppose that the Pythagoras number of the field $\mathcal{M}_0(\mathbb{R}^m)$ is $+\infty$. Then, there is a positive semidefinite analytic function $f : \mathbb{R}^m \to \mathbb{R}$ that is a sum of squares of meromorphic functions (hence positive semidefinite), but not a finite sum of squares.*

Proof. Fix for each integer $p \geq 1$ a germ g_p which is a sum of squares of meromorphic function germs, but not of p squares. After a change of coordinates, in a suitable neighborhood $W \times I \subset \mathbb{R}^{m-1} \times \mathbb{R} = \mathbb{R}^m$ of the origin we have analytic functions $\delta, h_k : W \times I \to \mathbb{R}$ and $a_i : W \to \mathbb{R}$ with $a_i(0) = 0$, such that

$$g_p = x_m^d + a_1 x_m^{d-1} + \cdots + a_d = \sum_k h_k^2/\delta^2.$$

We can suppose $I = (-2\rho, 2\rho) \subset \mathbb{R}$, with $\rho > 0$, and we choose $\varepsilon > 0$ small enough so that

$$|a_1(x')\rho^{d-1} + \cdots + a_{d-1}(x')\rho + a_d(x')| < \rho^d \quad \text{for} \quad \|x'\| < \varepsilon.$$

We shrink W to $W = \{\|x'\| < \varepsilon\}$, for g_p to have not zeros in $W \times \{|x_m| = \rho\}$. Next, pick an analytic diffeomorphism

$$\varphi_p : U_p = \mathbb{R}^{m-1} \times (p - \tfrac{1}{2}, p + \tfrac{1}{2}) \to W \times I$$

which maps each level $x_m = p + t$ to the level $x_m = 4\rho t$. Set $f_p = g_p \circ \varphi_p$. The construction guarantees that alike g_p, the analytic function f_p is a sum of squares of meromorphic functions on U_p, but not of p squares.

Next we consider the open set $V_p = U_p \cap \{|x_m - p| < \frac{1}{4}\}$, and claim that $Z_p = \{f_p = 0\} \cap V_p$ is closed in $\mathbb{R}^m$. For, suppose there is $x \notin Z_p$ adherent to Z_p: then $x \in U_p \cap \{|x_m - p| = \frac{1}{4}\}$, and $\varphi_p(x) \in W \times \{|x_m| = \rho\}$ is a zero of g_p, which is impossible.

By the claim, the union $Z = \bigcup_p Z_p$ is a closed analytic subset of $\mathbb{R}^m$, and we can define a coherent locally principal sheaf on $\mathbb{R}^m$ by

$$\mathcal{J}_x = \begin{cases} f_p \mathcal{O}_{\mathbb{R}^m,x} & \text{if} \quad x \in Z_p \\ \mathcal{O}_{\mathbb{R}^m,x} & \text{if} \quad x \notin Z \end{cases}$$

As usual, we know that $\mathcal{J}$ is globally principal, say generated by f. This is the function we sought.

Indeed, on each V_p there is an analytic unit v_p such that $f = v_p f_p$. Thus, the sign of f is locally constant, hence constant, and we can suppose $f \geq 0$, so that $v_p > 0$. Recall here that f_p is a sum of squares of meromorphic functions, hence f is also a sum of squares on V_p. On the other hand, the zero set of f is Z, and its connected components Y are the connected components of the Z_p's. Summing up, f verifies all conditions to apply Theorem 1.5, and we conclude that f is a sum of squares of meromorphic functions on $\mathbb{R}^m$. Finally, this sum cannot be finite, say of q squares, because then f_q would be a sum of q squares of meromorphic function germs, which we know is not the case. $\square$

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