On unitary representations of groups of isometries

Vladimir V. Uspenskij

Department of Mathematics
321 Morton Hall
Ohio University
Athens, Ohio 45701, USA
uspensk@math.ohiou.edu

Dedicated to Professor Enrique Outerelo.

ABSTRACT

Is the topological group of all motions (including translations) of an infinite dimensional Hilbert space H isomorphic to a subgroup of the unitary group $U(H)$? This question was asked by Su Gao. We answer the question in the affirmative.

2000 Mathematics Subject Classification: 43A65, 43A35, 22D10, 54E35.
Key words: Hilbert space, motion, representation, unitary groups, isometry.

1. Introduction

For a Hilbert space H we denote by $U_s(H)$ the group of all unitary operators on H, equipped with the strong operator topology. This topology is the same as the topology of pointwise convergence, it is induced on $U(H)$ by the product topology on H^H. The group $U_s(H)$ is a topological group, in other words, its topology is compatible with its group structure.

Given a topological group G, a *unitary representation* of G is a continuous homomorphism $f : G \to U_s(H)$. The representation f is *faithful* if it is injective, and *topologically faithful* if f is a homeomorphic embedding of G into $U_s(H)$. Thus a group admits a topologically faithful unitary representation if and only if it is isomorphic to a topological subgroup of a group of the form $U_s(H)$.

Every locally compact group G (in particular, every Lie group) admits a topologically faithful unitary representation. For example, the regular representation on $L^2(G)$ is such. Beyond the class of locally compact groups the situation is less clear. There are topological groups with a countable base which do not admit any faithful unitary representation or even any faithful representation by isometries on a reflexive Banach space (Megrelishvili). For example, the group $H(I)$ of all self-homeomorphisms of the interval $I = [0,1]$ has this property [2]. It follows that $\mathrm{Is}\,(U)$, the group of isometries of the Urysohn universal metric space U, also has this property, since it contains a topological copy of $H(I)$. See [4], [5], [6] for the proof and a discussion of the space U and its group of isometries.

The question arises: what are metric spaces M for which the group $\mathrm{Is}\,(M)$ of all isometries of M admits a (topologically) faithful unitary representation? We consider the topology of pointwise convergence on $\mathrm{Is}\,(M)$ (or the compact-open topology, which is the same on $\mathrm{Is}\,(M)$). If M is locally compact and connected, the group $\mathrm{Is}\,(M)$ is locally compact [1, Ch.1, Thm.4.7] and hence embeds in a unitary group. On the other hand, if M is the Urysohn space U (which topologically is the same as a Hilbert space [7]), then $\mathrm{Is}\,(M)$ has no faithful unitary representations, as we noted above.

Let H be a (real or complex) Hilbert space. Consider the topological group $\mathrm{Is}\,(H)$ of all (not necessarily linear) isometries of H. This group is a topological semidirect product of the subgroup of translations (which is isomorphic to the additive group of H) and the unitary group $U(H_{\mathbf{R}})$ of the real Hilbert space $H_{\mathbf{R}}$ underlying H. Su Gao asked the author whether the topological group $\mathrm{Is}\,(H)$ is isomorphic to a subgroup of a unitary group. The aim of the present note is to answer this question in the affirmative. It follows that the additive group of H admits a topologically faithful unitary representation. Even this corollary does not look self-evident.

Theorem 1.1 *Let H be a Hilbert space, and let $G = \mathrm{Is}\,(H)$ be the topological group of all (not necessarily linear) isometries of H. Then G is isomorphic to a subgroup of a unitary group.*

In Section 2 we remind some basic facts concerning positive–definite functions on groups, and prove (a generalization of) Theorem 1.1 in Section 3.

2. Positive–definite functions

For a complex matrix $A = (a_{ij})$ we denote by A^* the matrix $(\bar{a}_{ji})$. A is *Hermitian* if $A = A^*$. A Hermitian matrix A is *positive* if all the eigenvalues of A are ≥ 0 or, equivalently, if $A = B^2$ for some Hermitian B. A complex function p on a group G is *positive–definite* if for every $g_1, \dots, g_n \in G$ the $n \times n$-matrix $p(g_i^{-1}g_j)$ is Hermitian and positive. If $f : G \to U(H)$ is a unitary representation of G, then for every vector $v \in H$ the function $p = p_v$ on G defined by $p(g) = (gv, v)$ is positive–definite. (We denote by (x, y) the scalar product of $x, y \in H$.) Conversely, let p be a positive–definite function on G. Then $p = p_v$ for some unitary representation $f : G \to U(H)$ and some $v \in H$. Indeed, consider the group algebra $\mathbf{C}[G]$, equip it with the scalar product defined by $(g, h) = p(h^{-1}g)$, quotient out the kernel, and take the completion to get H. The regular representation of G on $\mathbf{C}[G]$ gives rise to a unitary representation on H with the required property. If G is a topological group and the function p is continuous, the resulting unitary representation is continuous as well. These arguments yield the following criterion:

Proposition 2.1 *A topological group G admits a topologically faithful unitary representation (in other words, is isomorphic to a subgroup of $U_s(H)$ for some Hilbert space H) if and only if for every neighborhood U of the neutral element e of G there exist a continuous positive–definite function $p : G \to \mathbf{C}$ and $a > 0$ such that $p(e) = 1$ and $|1 - p(g)| > a$ for every $g \in G \setminus U$.*

Proof. The necessity is easy: consider convex linear combinations of the functions p_v defined above, where $v \in H$ is a unit vector. We prove that the condition is sufficient. According to the remarks preceding the proposition, for every neighborhood U of e there exists a unitary representation $f_U : G \to U_s(H)$ such that $f_U^{-1}(V) \subset U$ for some neighborhood V of 1 in $U_s(H)$. Indeed, pick a positive–definite function p on G such that $p(e) = 1$ and

$$\inf\{|1 - p(g)| : g \in G \setminus U\} = a > 0.$$

Let f_U be a representation such that $p(g) = (f_U(g)v, v)$ for every $g \in G$ and some unit vector $v \in H$. Then the neighborhood

$$V = \{A \in U_s(H) : |1 - (Av, v)| < a\}$$

of 1 in $U_s(H)$ is as required.

It follows that the Hilbert direct sum of the representations f_U, taken over all neighborhoods U of e, is topologically faithful. □

Let G be a locally compact Abelian group, $\hat{G}$ its Pontryagin dual. Recall that continuous positive–definite functions on $\hat{G}$ are precisely the Fourier transforms of positive measures on G (Bochner's theorem). In particular, every positive–definite function $p : \mathbf{R}^k \to \mathbf{C}$ has the form

$$p(x) = \int_{\mathbf{R}^k} \exp\left(-2\pi i(x, y)\right) d\mu(y)$$

for some positive measure μ on $\mathbf{R}^k$. Here $(x, y) = \sum_{i=1}^k x_i y_i$ for $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$. Let $p(x) = \exp\left(-\pi(x, x)\right)$. Then p is the Fourier transform of the Gaussian measure $p(y)dy$ and hence is positive–definite. Thus for any $x_1, \dots, x_n \in \mathbf{R}^k$ the matrix $(\exp(-\|x_i - x_j\|^2))$ is positive. Here $\|\cdot\|$ is the Hilbert space norm on $\mathbf{R}^k$ defined by $\|y\| = \sqrt{(y, y)}$. If H is an infinite-dimensional Hilbert space and $x_1, \dots, x_n \in H$, then the finite-dimensional linear subspace of H spanned by $x_1, \dots, x_n$ is isometric to $\mathbf{R}^k$ for some k. Thus we arrive at the following:

Lemma 2.1 *Let H be a Hilbert space, and let $x_1, \dots, x_n$ be points in H. Then the symmetric $n \times n$-matrix $(\exp(-\|x_i - x_j\|^2))$ is positive.*

3. Proof of the main theorem

We prove a theorem which is more general than Theorem 1.1.

Theorem 3.1 *Let (M, d) be a metric space, and let $G = \mathrm{Is}\,(M)$ be its group of isometries. Suppose that there exists a real–valued positive–definite function $p : \mathbf{R} \to \mathbf{R}$ such that:*

1. *$p(0) = 1$, and for every $\epsilon > 0$ we have $\sup\{p(x) : |x| \geq \epsilon\} < 1$;*
2. *for every points $a_1, \dots, a_n \in M$ the symmetric real $n \times n$-matrix $(p(d(a_i, a_j)))$ is positive.*

Then the topological group G is isomorphic to a subgroup of a unitary group.

In virtue of Lemma 2.1, this theorem is indeed stronger than Theorem 1.1: if $M = H$, we can take for p the function defined by $p(x) = \exp(-x^2)$.
Proof. According to Proposition 2.1, we must show that there are sufficiently many positive–definite functions on G. For every $a \in M$ consider the function $p_a : G \to \mathbf{R}$ defined by $p_a(g) = p(d(ga, a))$. This function is positive–definite. Indeed, let $g_1, \dots, g_n \in G$. Since each g_i is an isometry of M, we have $d(g_i^{-1}g_j a, a) = d(g_i a, g_j a)$, and hence the matrix $(p_a(g_i^{-1}g_j)) = (p(d(g_i^{-1}g_j a, a))) = (p(d(g_i a, g_j a)))$ is positive by our assumption.

Let U be a neighborhood of 1 (= the identity map of M) in G. Without loss of generality we may assume that

$$U = \{g \in G : d(ga_i, a_i) < \epsilon,\ i = 1, \dots, n\}$$

for some $a_1, \dots, a_n \in M$ and some $\epsilon > 0$. Let t be the average of the functions $p_{a_1}, \dots, p_{a_n}$. Then t is a positive–definite function on G such that

$$\sup\{t(g) : g \in G \setminus U\} < 1.$$

Indeed, let $\delta > 0$ be such that $p(x) \leq 1 - \delta$ for every x such that $|x| \geq \epsilon$. If $g \in G \setminus U$, then $d(ga_i, a_i) \geq \epsilon$ for some i, hence $p_{a_i}(g) = p(d(ga_i, a_i)) \leq 1 - \delta$ and

$$t(g) = \frac{1}{n} \sum_{k=1}^{n} p_{a_k}(g) \leq 1 - \frac{\delta}{n}.$$

We invoke Proposition 2.1 to conclude that G has a topologically faithful representation. $\square$

References

[1] Sh. Kobayashi, K. Nomizu: *Foundations of differential geometry, Vol. 1.* Interscience Publishers, New York-London 1963.

[2] M.G. Megrelishvili: *Every semitopological semigroup compactification of the group $H_+[0,1]$ is trivial.* Semigroup Forum **63** (2001), no.3, 357–370.

[3] V. Pestov: *Topological groups: Where to from here?* Topology Proc. **24** (1999), 421–502. E-print: math.GN/9910144.

[4] V. Uspenskij: *On the group of isometries of the Urysohn universal metric space.* Comment. Math. Univ. Carolinae **31** (1990), no.1, 181–182.

[5] ________: *On subgroups of minimal topological groups.* E-print: math.GN/0004119.

[6] ________: Compactifications of topological groups. Proceedings of the Ninth Prague Topological Symposium (2001), 331–346 (electronic), Topol. Atlas, North Bay, ON, 2002. E-print: math.GN/0204144.

[7] ________: *The Urysohn universal metric space is homeomorphic to a Hilbert space.* Topology Appl. **139** (2004), 145–149. E-print: math.GN/0309101.