CARLOS ANDRADAS
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Constructible Sets in Real Geometry

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Preface

The plan to write this book was laid out in April 1987 at Oberwolfach, during the Conference "Reelle Algebraische Geometrie". Afterwards we met at various conferences and seminars in Luminy, Madrid, Münster, Oberwolfach, Segovia, Soesterberg, Trento and La Turballe. We would like to thank the organizers and the institutions which supported these meetings. With pleasure we remember the special year on Real Algebraic Geometry and Quadratic Forms (Ragsquad) in Berkeley 1990/91 where an essential part of this book was written. Thanks to T.Y. Lam and R. Robson.

We are indepted to our Departments: Universidad Complutense de Madrid and Westfälische Wilhelms-Universität Münster, as well as the D.A.A.D. and the D.G.I.C.Y.T..

It is not possible to mention here all colleagues and friends who showed permanent interest in the project. They encouraged us to continue and complete this work. In particular, we are obliged to Jacek Bochnak, Mike Buchner, Michel Coste and Claus Scheiderer for proofreading and helpful suggestions. While the work was still in progress, Manfred Knebusch used parts of our manuscript for a course on Real Algebraic Geometry. His experience convinced us that it was worth pursuing a fully abstract approach. We were also in permanent contact with Murray Marshall, and the reader will recognize the mutual influence of ideas.

We are also indebted to Professor Reinhold Remmert and to the Springer-Verlag for publishing the book in this series.

Finally, we thank Erika Becker for her care in converting many pages of handwritten mathematics into beautiful TEX.

Madrid, Münster, Majadahonda *C. Andradas, L. Bröcker, J. M. Ruiz*
March 1996

Contents

Introduction

Let A be the ring of polynomials in n indeterminates over $\mathbb{R}$. Then any subset S of $\mathbb{R}^n$ which is the solution set of a polynomial system $f_1(x) > 0, \ldots, f_k(x) > 0$ is also the solution set of a system of n inequalities $g_1(x) > 0, \ldots, g_n(x) > 0$, no matter how big k is. This observation, made about twelve years ago for $n \leq 3$ and proved in full generality five years later is the starting point of the present book.

Many similar problems can be raised: Firstly one may generalize the type of sets S which appear as solution of polynomial systems, and ask questions of the following type. What happens if the strict inequalities are replaced by non-strict ones? What if we consider arbitrary semialgebraic sets S, that is, sets described by finitely many polynomial inequalities where unions are also involved? Is it still possible to bound by a function depending only on n the number of polynomials needed for the description? How can one recognize those semialgebraic sets which are given by pure intersections of strict or non-strict inequalities, called basic open and basic closed respectively? If S is basic open, is its closure basic closed?

Secondly, one may change the underlying space $\mathbb{R}^n$ and the ring A, and ask corresponding questions in the new situation. For instance, take instead of $\mathbb{R}^n$ an algebraic subset V of $\mathbb{R}^n$, or replace $\mathbb{R}$ by an arbitrary real closed field R or even by an ordered field K with real closure R. The corresponding rings are then $\mathbb{R}[V]$, $R[V]$ and $K[V]$ respectively. One can also take a real analytic set together with its ring of analytic functions, as well as an analytic set germ with its ring of analytic function germs. Or, more generally, one may take for A any commutative ring with unit and consider as underlying space the real spectrum $\mathrm{Spec}_r(A)$ of A.

As the reader may realize, all these questions can be posed at a very basic level, although it is not apparent how to handle them. Their solution seems to require, at least until the present moment, advanced methods from analytic and algebraic geometry, model theory, commutative algebra, quadratic forms and real algebra, including Marshall's spaces of orderings and Coste-Roy's real spectra, both highly important in this book.

Our purpose is to introduce a way of attacking all the above questions in a systematic manner. To that end, we exhibit a theory whose objects include all the different kinds of *constructible sets* which appear in real geometry. For instance, if the underlying space is $\mathbb{R}^n$ and A is the ring of polynomials, they are the semialgebraic sets, which, by Tarski's principle, form the smallest class of sets containing all real algebraic varieties and being closed under boolean operations and projections. So they coincide with the *definable sets*, that is, sets that can be defined by *formulae in the language of ordered fields*. This allows the use of methods from model theory to prove in a very efficient way many properties of semialgebraic sets. In other situations Tarski's principle is no longer available. That is the case for semianalytic sets and germs, which are the constructible sets in real analytic geometry. Furthermore, the notion of constructible set is well established in abstract settings as real spectra and spaces of orderings.

Now, what is the essence of constructible sets? Firstly, their definition involves the relation $>$ which appears naturally in real geometry. Secondly, more than the concrete values of the defining functions, what matters are their signs. This leads us to consider a space X together with a monoid G of functions taking the values -1, 0 and 1, which stand for being < 0, $= 0$ and > 0, respectively. Under minor extra conditions we call (X, G) a *real space*, and there are natural notions of *basic open*, *constructible* and *Zariski-closed* sets, as well as the corresponding topologies. In addition, there is also an abstract notion of quadratic *form*. Then, to every constructible set there is attached a unique form, and the complexity of the former can be studied in terms of the latter. This opens the door to apply the theory of quadratic forms to constructible sets.

Clearly, not much can be done at this very general level. Thus, we define *spaces of signs* by imposing four axioms on real spaces. They are inspired by geometric properties of semialgebraic sets and algebraic properties of reduced quadratic forms. The theory of spaces of signs, which we develop in Chapters III, IV and V, is surprisingly rich: it combines the power of real spectra and spaces of orderings. In fact, any commutative ring with unit A gives rise to a space of signs (X, G), where $X = \operatorname{Spec}_r(A)$ and $G = \{\operatorname{sign}[f] \mid f \in A\}$. If the monoid G is a group, the field like situation, then (X, G) is just a space of orderings in the sense of Marshall.

Thus we need a good knowledge of both theories. A general outline of real spectra is given at the beginning of Chapter II and Marshall's spaces of orderings are studied in full detail in Chapter IV. One key point of the whole theory is that questions on spaces of signs can be reduced, via *local-global principles*, to finite *subspaces*, and then to combinatorics. Of high interest are certain subspaces called *fans*, which on the one hand admit quite a simple structure, and on the other govern the whole theory.

Even after we have developed spaces of signs, only part of the way is done. Coming back to the situation $X = \mathrm{Spec}_r(A)$, in order to get quantitative results one needs an algebraic description of the fans, or at least an estimation of their size. This information is hidden in the residue fields of A, and to read it one needs a good understanding of the real algebra of fields. The basic ideas of this topic are presented in Chapter VI. They grew during the 70's out of the study of quadratic forms and Witt rings over formally real fields, without any geometric application in mind. They are sufficient to give estimates for the fans occuring in real algebraic geometry, where the residue fields are algebraic function fields. The situation is more complicated in analytic geometry, where very little is known about the residue fields.

A natural class of rings which is still accessible is that of *excellent rings*, to which Chapter VII is devoted. It contains the algebras finitely generated over fields and also the rings of analytic functions and analytic function germs. Moving back and forth through henselizations and completions, where power series methods can be applied, a deep understanding of the real algebra of excellent rings is achieved. Along the way, we show constructibility of closures and characterize local constructibility of connected components.

Closing the circle, how do we come back to the geometric problems posed at the beginning? Let X be a real algebraic, a compact analytic set or an analytic set germ, and A the suitable ring of functions on X. Since A is excellent, we already know a lot about $\mathrm{Spec}_r(A)$, although in the analytic case some extra work is still required to estimate the size of fans in the space of signs associated to A. Finally, we need a transfer from $\mathrm{Spec}_r(A)$ to X, or, more precisely, a one-to-one correspondence between constructible sets in $\mathrm{Spec}_r(A)$ and X. In the algebraic case this transfer comes from the Artin-Lang homomorphism theorem. In the analytic case, this originated about ten years ago in the solution of Hilbert's 17th problem for meromorphic functions on X. All of this is done in Chapter VIII.

In the end, we can harvest the fruits of the theory: we answer the questions posed at the beginning, not only in the algebraic case but also in the analytic.

Look at the long path we have drawn from geometry to real spectra, spaces of signs, combinatorics and back to geometry. We have tried to display this path in Chapter I by examples and through heuristic considerations, before going into technicalities. This first chapter also contains a few remarks on how this work fits into the general stream of real geometry. Therefore it should be seen as an extended introduction.

Chapter I. A First Look at Semialgebraic Geometry

Summary. This chapter can be viewed as an introduction to the book and as motivation for the problems considered. It contains almost no proofs. In Section 1 we introduce the Tarski-Seidenberg theorem in several forms which are practical and sufficiently general, without entering too far into the terminology of model theory. In the next section we discuss some typical problems of semialgebraic geometry, trying to show how the topic of this book –the description of semialgebraic and more general sets by few generators– fits into the theory that we develop. For this kind of complexity problem we introduce in Section 3 the unifying terminology of real spaces, the spaces which occur in various contexts like semialgebraic geometry. semianalytic geometry, real spectra of rings and spaces of orderings of fields. This book deals with the relations between these. However, in Section 4 we first look at typical examples and illustrations in the semialgebraic situation. So this section is mostly recommended for motivation.

1. Real Closed Fields and Transfer Principles

The objects we have in mind at our first look at semialgebraic geometry are certain subsets of $\mathbb{R}^n$, which are defined by real polynomials. Here, one might first think of solutions of systems of polynomial equations, that is, affine algebraic varieties over the reals. But then, of course, one considers not merely the solutions in $\mathbb{R}^n$ but also in $\mathbb{C}^n$ or in $\mathbb{P}^n(\mathbb{C})$, where the variety shows its full beauty.

However, the situation changes if we are concerned with polynomial relations instead of polynomial equations: For instance, the set

$$S = \{x \in \mathbb{R} \mid \exists\, y \in \mathbb{R} : x^2 + y^2 = 1\}$$

is not algebraic and not even a boolean combination of algebraic subsets of $\mathbb{R}$. On the other hand, S can be described by inequalities:

$$S = \{x \in \mathbb{R} \mid -1 \leq x \leq 1\}.$$

Chapter II. Real Algebra

Summary. We consider general properties of the real spectrum of a commutative ring with unit. In Sections 1 and 2 we collect the basic facts, and for more information we refer to [B-C-R] and [Kn-Schd]. In Section 3 valuation theory enters the scene. It is of fundamental importance for the whole work. As an application we obtain in Section 4 the first results on going-up and going-down in the real spectrum. In Section 5 we present the notion and basic properties of abstract semialgebraic functions on constructible sets of the real spectrum; this is done in two equivalent ways which have both their advantages. These functions are used in Section 6 to construct cylindrical decompositions with respect to systems of polynomials. Finally, in Section 7 we introduce real strict localizations, which are the real analogues of the strict localizations used in etale cohomology.

1. The Real Spectrum of a Ring

Let A be a commutative ring with unit.

Definition 1.1 *The* real spectrum of A, *denoted by* $\mathrm{Spec}_r(A)$, *is the set*

$$\{\alpha : A \to R_\alpha\}/\sim,$$

where α runs over all non-trivial homomorphisms from A into some real closed field R_α and "$\sim$" is the equivalence relation generated by all commutative triangles

$$\begin{array}{ccc} A & \xrightarrow{\ \alpha\ } & R_\alpha \\ & {\scriptstyle\beta}\searrow \quad \nearrow{\scriptstyle i} & \\ & R_\beta & \end{array}$$

Let us present two alternative descriptions of $\mathrm{Spec}_r(A)$. First we need:

Proposition and Definition 1.2 *A subset $P \subset A$ is called a* precone of A *if the following conditions hold:*

We close this section with a result concerning connected components which will be of use in Chapters VII and VIII. Given a topological space X, a subset $Y \subset X$ and a point $z \in X$ we shall denote by $cc(Y)$ the number $\leq \infty$ of connected components of Y and by $cc_z(Y)$ the number $\leq \infty$ of connected components of Y adherent to $z \in X$. Then:

Proposition 7.13 *Let C be a constructible set of* $\operatorname{Spec}_r(A)$ *and* $\alpha \in \operatorname{Adh}(C)$. *Consider the real strict localization $\varphi : A \to A_\alpha$ and let $C_\alpha = (\varphi^*)^{-1}(C) \subset$* $\operatorname{Spec}_r(A_\alpha)$. *Then:*

a) For every integer $p \leq cc(C_\alpha)$ there are arbitrarily small open constructible neighbourhoods U of α such that $U \cap C = C_1 \cup \cdots \cup C_p$, where the C_i's are disjoint open subsets of $U \cap C$ adherent to α.

b) $cc(C_\alpha) = cc(C \cap U_\alpha) \geq cc_\alpha(C)$.

Proof. *a)* If $p \leq cc_\alpha(C)$, $C \cap U_\alpha$ must be the union of p disjoint non-empty open subsets $T_1, \ldots, T_p$. Then they are also closed and, as $C \cap U_\alpha$ is proconstructible, by Proposition 1.12 we have $T_i = U_i \cap C \cap U_\alpha$ for some open constructible set U_i of $\operatorname{Spec}_r(A)$. Furthermore, we have $U_\alpha = \bigcap U$, for any basis $\{U\}$ of open constructible neighbourhoods of α, and we can write

$$C \cap \bigcap U \setminus (U_1 \cup \cdots \cup U_p) = \emptyset \quad \text{and} \quad C \cap \bigcap U \cap U_i \cap U_j = \emptyset \text{ for } i \neq j.$$

By compactness of the constructible topology there is some U such that

$$C \cap U \setminus (U_1 \cup \cdots \cup U_p) = \emptyset \quad \text{and} \quad C \cap U \cap U_i \cap U_j = \emptyset \text{ for } i \neq j.$$

Finally we put $C_i = C \cap U \cap U_i$ for $i = 1, \ldots, p$, and it remains to see that each C_i is adherent to α. But $T_i \subset C_i$, and $\emptyset \neq T_i \subset U_\alpha$; hence there is $\beta \in C_i$ with $\beta \to \alpha$, which means that $\alpha \in \operatorname{Adh}(C_i)$.

b) Since φ^* is a homeomorphism from $\operatorname{Spec}_r(A_\alpha)$ onto U_α (Proposition 7.10), it induces a bijection between the connected components of C_α and the ones of $C \cap U_\alpha$, which gives the first equality. On the other hand, if D is a connected component of C, it is closed in C, hence proconstructible. Then by Proposition 2.3, $\alpha \in \operatorname{Adh}(D)$ if and only if $D \cap U_\alpha \neq \emptyset$. Now choose a connected component T of $C \cap U_\alpha$ meeting $D \cap U_\alpha$. Necessarily $T \subset D$, and $D \mapsto T$ defines an injection from the connected components of C adherent to α into the connected components of $C \cap U_\alpha$, and we obtain $cc(C \cap U_\alpha) \geq cc_\alpha(C)$. □

Notes

The real spectrum was introduced by Coste-Roy at the end of the seventies ([Co-Ry1,2]) by model theoretic reasons. In particular, they showed the fundamental equivalence between constructibility and first order definability. There are now several very good presentations of the theory: [Lm3], [Be2], [Kn-Sch] and of course [B-C-R]. Here we refer to the latter and make some small modifications that fit our setting.

The first version of the finiteness theorem was given by Łojasiewicz [Ł1] without any reference to model theory; among the many other later proofs we quote here [Rc]. The real Nullstellensatz was proved independently and almost simultaneously by Dubois ([Du]) and Risler ([Rs]) at the end of the sixties, although a weak form of it had passed unnoticed in an earlier paper by Krivine ([Kv]). The first Positivstellensatz is due to Stengle ([St1]) and many others followed, as for instance several by Swan ([Sw]). The final unified treatment for all of them was given by Colliot-Thélène ([C-T]) in his contribution to the foundational 1981 real geometry conference at Rennes. The Hörmander-Lojasiewicz inequality in its abstract form was first published in [An-Br-Rz], but had already been proved by Coste (oral communication) using model theory. The notions of specialization and generization are classical in algebraic geometry and for real spectra were developed by Coste-Roy ([Co-Ry2]), who showed also their connection with real valuations; this was immediately pursued by Brumfiel in [Bf2]. However, the study of real valuations and their relationship with orderings is much older, appearing already in the papers of Baer ([Ba]) and Krull ([Kr]). Here we have not touched the classical theme of valuations and singularities, which has also a real version (see [An-Rz3]). The real going-up for integral extensions and the real going-down for polynomial extensions were proved in [Co-Ry2]. The systematic use of valuation rings associated to specializations was introduced by Schwartz ([Schw2]) and Delfs ([Df]) to study the sheaf of continuous abstract semialgebraic functions. The invention of this sheaf is largely due to Brumfiel ([Bf3,4]), with Schwartz's crucial finding that a definable continuous abstract function need not be semialgebraic. The progressive description given here tries to be closer to intuition. The cylindrical decomposition of Section 6 was introduced in [Al-An] to study constructibility in power series rings.

Finally, the notion of real strict localization was introduced by Roy ([Ry]) to be the real counterpart of Grothendieck's strict localization and the key ingredient in the construction of the sheaf of abstract Nash functions. This idea appeared first in [Ar-Mz]. All of this was later clarified in [Al-Ry]. Our Theorem 7.11 is an expanded version of the corresponding result of the latter paper, while Proposition 7.13 comes from [Rz7]. It is well worth mentioning here that the properties of real strict localizations are intimately tied to a very important open problem of the theory: the verification of a property called idempotency. This highly abstract property has been brought into new attention by Quarez [Qz]), who has shown that it is equivalent to Efroymson's extension and separation problems on Nash functions ([Ef]).

Chapter III. Spaces of Signs

Summary. Spaces of signs are defined by imposing four suplementary axioms on real spaces; then, spaces of orderings are a special class of these spaces of signs. This is done in Section 1, where we also define subspaces and draw the first consequences of our definitions. Section 2 contains the fundamental properties of forms, mainly in the case of spaces of orderings. The important notion of fan is introduced in Section 3, together with its elementary properties. In Section 4 we consider local spaces of orderings and localizations, which behave very much like in ring theory. Localizations are used in Section 5 to show that the real space associated to a commutative ring with unit is actually a space of signs, and also in Section 6, to prove that the subspaces of a space of signs are again spaces of signs.

1. The Axioms

Let (X, G) be a real space; we have a pairing

$$X \times G \to \mathbb{F}_3 \; ; \; (x, g) \mapsto g(x).$$

Since G separates points (Axiom $\mathbf{S}_3$) this pairing gives an inclusion $X \subset \widehat{G} = \mathrm{Hom}(G, \mathbb{F}_3)$. Now, $\widehat{G}$ inherits from $\mathbb{F}_3^G$ the product topology and $\widehat{G}$ is closed in $\mathbb{F}_3^G$. So $\widehat{G}$ is compact in this topology, which we call constructible. In fact, the constructible topology on X is just the induced one. Note also that a necessary condition for $\sigma \in \widehat{G}$ to be in X is that $\sigma(-1) = -1$. Now, for $x, y \in X$ we may form the product $\sigma = xy$ in $\widehat{G}$, which is no longer in X since $\sigma(-1) = 1$. However, products of odd order may again be in X, and it will be very important to study them. If G is a group, then $g^2 = 1$ for all $g \in G$ and $\widehat{G} = \mathrm{Hom}(G, \; \{+1, -1\}) = \mathrm{Hom}(G, \mathbb{C}^*)$ is the usual dual group when we provide G with the discrete topology (see [Mo]). This will be essential in Chapter IV.

Of course, not very much can be said about the structure of (X, G) in this generality. So we are going to impose further conditions on (X, G).

Proposition and Definition 1.1 *The following conditions for (X, G) are equivalent:*

$0, \ldots, g_s(x) > 0\}$. But given such forms, Y does not meet the constructible set $C = \{x \in X \,|\, \rho(x) \neq \rho'(x)\}$, that is,

$$\emptyset = C \cap Y = C \cap \bigcap_{g \in G_1} \{g > 0\}.$$

Then, by compactness of the constructible topology, there are $g_1, \ldots, g_s \in G_1$ such that $\{g_1 > 0, \ldots, g_s > 0\} \cap C = \emptyset$. These g_i are the functions we sought.

3^{rd} step: Conclusion. By the preceding reductions, we can suppose that (X, G) is a space of orderings and $Y = \{g_1 > 0, \ldots, g_s > 0\}$. In order to prove MM for $(Y, G|Y)$ consider $\rho = \langle a_1, a_2, \ldots, a_n\rangle$, $\tau = \langle b_1, b_2, \ldots, b_m\rangle$, $h, c_2, \ldots, c_{n+m} \in G$ such that $(\rho + \tau)(x) = \langle h, c_2, \ldots, c_{n+m}\rangle(x)$ for all $x \in Y$. Then, consider the Pfister form $\varphi = \langle\langle g_1, \ldots, g_s\rangle\rangle$ and the products $\rho \otimes \varphi$, $\rho' \otimes \varphi$. Clearly,

$$\langle h, c_2, \ldots, c_{n+m}\rangle \otimes \varphi = \langle h, d_2, \ldots, d_l\rangle$$

for suitable $d_2, \ldots, d_l \in G$, and by Proposition 2.6 *a)*,

$$(\rho \otimes \varphi + \tau \otimes \varphi)(x) = \langle h, d_2, \ldots, d_l\rangle(x)$$

for all $x \in X$. Thus, by MM for (X, G), we find $f, g \in G$ such that $f \in D(\rho \otimes \varphi)$,$g \in D(\tau \otimes \varphi)$ and $h \in D(\langle f, g\rangle)$. Finally, by Proposition 2.6 *e)*, we have $f \in D_Y(\rho)$, $g \in D_Y(\tau)$, which completes the proof. □

Notes

The idea of studying the space of all orderings of an algebraic structure from an axiomatic viewpoint is due to Marshall, who was motivated by the case of spaces of orderings of fields ([Mr 1-5]). Later, it turned out that his axioms also applied to skew fields ([Ts], [Cr]), to local rings ([Kn1]), and even to ternary fields ([Kl]). The abstract setting for real spectra of rings that is presented here is new. Our axioms imitate more or less well known facts of rings. The meaning of the initials, surely guessed by the reader, are the following: **PE**=positive equation, **HŁ**=Hörmander-Łojasiewicz, and **MM**=Murray Marshall. In fact, Marshall discovered that all the pleasant properties of spaces of orderings of fields depend just on these axioms (which simplify considerably in the field case). The axiom **MM** is related Pfister's local-global principle for fields ([Pf]), plus an almost trivial local-global principle for rings ([Br4]; see Section 5, in particular Proposition 5.6, and also [Mr-Wa]). Recently, Marshall has found a different approach, enough to deal with spectra of rings ([Mr9]), by using one single axiom which implies **PE**, **HŁ** and **MM**; it has been shown later that this single axiom is in fact equivalent to the other three.

Most of the material of Section 1 is standard, but Proposition 1.16 seems to be new even for real spectra. The content of section 2 can be found in [Mr1], except for Proposition 2.10. The important Proposition 2.7 is due to

W. Scharlau ([Schl1,2]). For spaces of orderings of fields the notion of SAP space was introduced in [El-Lm]. These spaces were characterized by Prestel in the very important paper [Pr1]. The notion of fan was introduced in [Be-Kö]; however its importance was not yet clear at that time. The transcription of these notions to spaces of orderings is straightforward. Chain length was first defined in [Mr4], and is crucial for the proof of local-global principles in spaces of orderings (see Sections IV.5-6). Also the miraculous lemma is Marshall's contribution; it is the key for the classification of finite spaces of orderings in the axiomatic context. The material of Section 4 is new in the general case, but for real spectra of local rings it can be found already in [Kn1]. Concerning spaces of orderings, the result of Section 6 is in [Mr1].

Chapter IV. Spaces of Orderings

Summary. This chapter contains a fully detailed presentation of the theory of spaces of orderings. In Section 1 we reformulate the notion of space of orderings to stress the connection with the duality of topological groups and the theory of reduced quadratic forms. Section 2 is devoted to sums and extensions of spaces of orderings and their basic properties. In Section 3 we introduce spaces of finite type and their trees, which support the use of induction in many proofs. The fundamental fact that the chain length of a space of orderings is bigger than or equal to that of any subspace is proved in Section 4. Also in this section we define solid fans, impervious fans, and places, which are essential to prove in Section 5 that finite chain length is equivalent to finite type. This is the key technical result of the theory. We prove in Section 6 the local-global principle that reduces problems on forms from the whole space to its finite subspaces. In Section 7 we draw the consequences: the representation theorem, the generation formula and the stability formula. Using these, we bound each invariant s, $\bar{s}$, t, $\bar{t}$, w, l in terms of any of the others. In particular, all of them are finite or infinite at a time. The final result of the section and the chapter is a local-global separation principle.

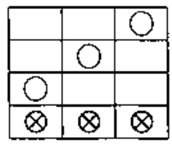
1. The Axioms Revisited

In this chapter we shall use equivalent definitions of prespace and space of orderings. The reader will soon realize that several hidden references to this alternative presentation have been scattered in the preceding chapters.

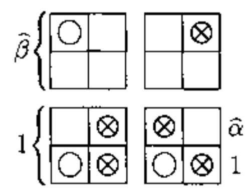
(1.1) Marshall's Axioms. Let G be a (multiplicative) abelian group *of exponent* 2, that is, such that $g^2 = 1$ for all $g \in G$, and let $\widehat{G} = \text{Hom}(G, \mathbb{C})$ the topological dual group of G for the discrete topology on G (see [Mo]). Then, $\sigma(g) \in \{+1, -1\}$ for all $\sigma \in \widehat{G}$ and $g \in G$, so that $\widehat{G} = \text{Hom}(G, \{+1, -1\})$. Furthermore, let us be given a *distinguished element* $-1 \in G$, $-1 \neq 1$, and a subset X of $\widehat{G}$. The pair (X, G) is a prespace of orderings if the following conditions hold:

$\mathbf{O}_1$: X is closed in $\widehat{G}$.

Examples 7.14 *a)* The separation of basic sets is no longer possible if $s(X) > 2$. Consider, for instance, the space $(X', G') = 3 \times E$ and $(X, G) = (X', G')[\mathbb{Z}_2 \times \mathbb{Z}_2]$. Then $s(X) = 3$ and the sets C, D whose elements are indicated by $\bigcirc$ and $\otimes$ respectively, are basic, but they cannot be separated, as one checks by inspection.



b) It is very likely that the bound $cl(Y) = 2^{s-1}$ in Theorem 7.12 is sharp. This is clear for $s \leq 2$ and for $s = 3$ consider the space $(X, G) = ((E + E)[\mathbb{Z}_2] + (E + E)[\mathbb{Z}_2])[\mathbb{Z}_2]$ and the subsets C and D as indicated below



Here C and D cannot be separated but their intersections with any subspace of chain length 3 can be separated. □

Notes

The ideas and results of this chapter are mainly due to Marshall, who introduced abstract spaces of orderings and studied them in five important papers ([Mr1-5]). However, in the special situation of spaces of orderings of fields, most of the constructions and important results had been known long before. They were discovered on the way of studying quadratic forms and Witt rings ([Wt2]) over formally real fields, a theory that became very appealing after Pfister's fundamental result ([Pf], [Le-Lo]). An excellent presentation of this theory can be read in Lam's book [Lm1].

So, let us first make a few comments on what was known in the field situation. Of course, the notion of subspace is classical, and corresponds to the notion of precone ([Se]). As was already mentioned, fans were introduced by Becker and Köpping. The basic constructions, their properties, and spaces of finite type, appear implicitely in [Br3] and more explicitely in [Cr] (see also [Mz]). The theory of extensions of spaces of orderings is rooted to the Baer-Krull theorem ([Ba], [Kr]) and Springer's notion of residue forms ([Sp]). The paper [Br3] contains also the main result of Section 5 in the case of finite spaces of orderings, whose proof is much shorter if one works over fields. The same methods apply to the general case of finite chain length, which corresponds to a field that admits only finitely many real places (see also [Bw-Mr]). Theorem 6.4 comes from [Be-Br]. There, it is deduced from another local-global principle of [Br1] that had been showed by methods due to Prestel ([Pr1]). There is a different proof using the chain length as in the abstract approach, thus reducing the problem to the case of a field with finitely many real places, where the methods of [Br3] can be applied (see [Bw-Mr]). The representation theorem is also due to Becker and Bröcker ([Be-Br]) after it was conjectured by Brown ([Bw]). The first definition of the stability index s of a field appeared in [Br1] as the $\mathbb{F}_2$-dimension of the cokernel of the total signature homomorphism. That this stability index has also the meaning we use here was pointed out by a letter of Knebusch. The stability formula (Corollary 7.4) appeared in [Br2]. Finally, Theorem 7.11 is due to Schwartz [Schw].

Now let us return to the situation of abstract spaces of orderings. Compared with Marshall's original papers, we have made several changes and suplements. Of course, the first three sections are more or less standard. The notion of chain length in Section 4 is one of Marshall's major inventions in order to make the theory work. For the proof of Proposition 4.1 we follow closely the exposition in [Lm2], which uses ideas of Leep. The notions of solid and impervious fans are new; we hope that they clarify a little bit the content of [Mr4]. Section 5 is the only one where we borrow Marshall's presentation [Mr2,4]. The representation theorem can also be shown without going through the more involved local-global principles [Mr3], but these are needed anyway, as for the generation formula (Theorem 7.3). This latter was easily derived from [Mr4] in [Br6], and then used for the first geometric applications. The connection between stability and the other invariants like w, l and t, was discovered in [Br7], where one finds the estimates of Corollaries 7.8 and 7.9 *a)*. That of Corollary 7.9 *b)* is due to Marshall, while in [Br7] is a bound for disjoint unions, which is even worse. Theorem 7.12 is taken from [Br8], but the remarkable bound for the chain-length is new.

Chapter V. The Main Results

Summary. In an arbitrary space of signs X, generation of basic sets, stability indices, representation of functions by signatures and separation of closed sets reduce to the corresponding problems in the spaces of orderings V^* associated to the subvarieties V of X. This general principle is proved in Sections 1, 2 and 3. Also, in Sections 1 and 2, we obtain criteria for a set to be basic open or to be principal open, and extend to X the inequalities among the invariants s, $\bar{s}$, t, $\bar{t}$, w and l, which were already known for spaces of orderings. Section 4 is devoted to the notions of real divisor and regularity in X. Using them we can bound from below s and $\bar{s}$. Moreover, we compare basicness of an open set and its closure, resp. of a closed set and its interior. Finally, Artin-Lang spaces are introduced in Section 5, jointly with the tilde operator: this is the notion that makes the abstract theory fruitful of geometric applications.

1. Stability Formulae

Let (X, G) be a space of signs. We are going to prove a generation formula, which splits into two parts. Firstly, we consider local spaces of signs, and then we come to the global situation. To start with we show:

Proposition 1.1 *Let (X, G) be local and let F be a non trivial fan in the space of orderings $(X_{\max}, G^*)$ associated to (X, G). Then there exists a subvariety $W \subset X$ such that $\operatorname{Adh}_Z(x) = W$ for all $x \in F$. If, moreover, F is finite, then $(G|F) \setminus \{0\} = G^*|F$, and F is a fan in (X, G).*

The situation here is different from that in Remark III.3.13, since $(X_{\max}, G^*)$ need not be a subspace of (X, G) and so it is not clear whether F is a fan in X.

Proof. Let $x_1, x_2 \in F$ and assume that $\operatorname{Adh}_Z(x_1) \neq \operatorname{Adh}_Z(x_2)$. We can choose $x_3, x_4 \in F$ such that $F' = \{x_1, x_2, x_3, x_4\}$ is a four element fan. Let V be the minimal non–empty Z-closed set of X. There are two cases:

Case 1: $\operatorname{Adh}_Z(x_i) = V$ for some $i = 1, \dots, 4$. By Proposition III.1.16 we find $g \in G$ with $g(x_1) = g(x_2) = g(x_3) = 1$ and $g(x_4) = -1$. Also $g \in G^*$. Contradiction.

Notes

The content of the whole chapter is new. We present here the final generalization of the geometric results which are collected, for instance, in [Br10]. Sometimes these generalizations are straightforward, but often new ideas are also involved. In the case of local rings, Proposition 1.1 is due to Knebusch [Kn1], while Corollaries 1.2 and 1.3 appear in [Br6]. Theorem 1.4 has a long history. First of all, Bröcker showed in [Br6] that for a real algebraic variety of dimension d over a real closed field $s = d$ for $d \leq 3$ and $s \leq d(d-2)(d-4)\cdots$ in general. Later this was improved a little bit (see [B-C-R]), until the break through was done in 1988 when Scheiderer observed the usefulness of the elementary Corollary III.2.8. Soon afterwards, Scheiderer ([Sch1]) and Bröcker (unpublished) found proofs that $s = d$ for arbitrary d. The latter proof had the advantage that it could be easily generalized to our Theorem 1.4 for arbitrary noetherian rings ([Br9]). Other approaches were given by Mahé ([Mh]), who presented a direct attack in the geometric case, and Marshall ([Mr6]), who showed Theorem 1.4 for arbitrary rings (commutative with unit). In the geometric setting Corollaries 1.5-1.10 appear all in [Br10]. Theorems 2.1 and 2.5 are new, even in the geometric context. In [Mr7] Marshall uses the minimal length of a separating family for a semialgebraic set, an invariant which is closely related to our ℓ. The rest of Section 2 appeared for the geometric setting in [Br10]. The separation question has a very long history in real geometry, starting with Mostowski's counterexample and separation theorem ([Mw]) (see also [Co], and [Br-St] for a quantitative approach). The content of Section 3, in the geometric situation, is in [Br8]. More direct methods have been used by Acquistapace, Broglia, Fortuna, Galbiati, Tognoli and Vélez ([Ac-Bg-Vz], [Bg-To], [Ft-Gl]). Proposition 4.3 generalizes a result of Scheiderer ([Sch1]), and Propositions 4.5, 4.6 an earlier one by Bröcker ([Br10]). Clearly, Definition 5.1 is the abstract version of the Artin-Lang theorem [Lg1]. The ultrafilter theorem (Theorem 5.3) appeared for the first time explicitely in Brumfiel's book ([Bf1]), although it might have been known before. Finally, Remark 5.4 *c)* has its roots in E. Artin's work on Hilbert's 17th problem ([A]). A slightly different axiomatic approach to real spectra which covers partially the same main results is due to Marshall ([Mr8]).

Chapter VI. Spaces of Signs of Rings

Summary. In view of the global stability formulae (Corollary V.1.6), and the canonical decomposition (III.1.9), the computation of stability indices in the space of signs of a ring reduces to estimations of the size of fans of residue fields of that ring. We obtain in Section 1 such estimations, via real valuations, after proving the so-called trivialization theorem for fans. Then, in Section 2, we deduce upper bounds for the stability index of a field extension in terms of the ground field. These bounds are sharp when the ground field is real closed or the rational numbers field, as follows from the lower bounds discussed in Section 3. In section 4 we generalize the previous upper bounds to algebras. The results are specially good for algebras over a field, which are the matter of Section 5; again, we obtain the best estimations over a real closed field and over the rationals. Section 6 is devoted to totally archimedean rings, which are the abstract counterparts of compact spaces. These rings have two special features: firstly, their complexity bounds are low, and, secondly, generation of basic sets and separation are characterized by multilocal conditions. We end in Section 7 with the translation to concrete semialgebraic geometry of most of the abstract results obtained so far. We as well discuss several examples and counterexamples to questions raised in earlier chapters.

1. Fans and Valuations

Throughout this section, K denotes a formally real field and we set $\Sigma = \sum K^2$. Let X be the space of signs $(\mathrm{Spec}_r(K), G)$ associated to K. As described in Example IV.1.4, $G = K^*/\Sigma^*$, and the subspaces Y of X correspond bijectively to the precones $T \subset K$ by the formulas

$$T = \{t \in K \mid t = 0 \text{ or } t\Sigma^* \subset Y^\perp\},$$

$$Y = \{\sigma \in X \mid \sigma(t) = +1 \text{ for all } t \in T\};$$

we have then $G|Y = K^*/T^*$ and write $Y = X/T$. Concerning fans we have

Notes

The algebraic study of formally real fields goes back to E. Artin and Schreier ([A],[A-S]). Shortly after that, Baer ([Ba]) and Krull ([Kr]) discovered the connection between orderings and valuations of fields (see Theorem 1.3). As mentioned before, the notion of fan was introduced by Becker and Köpping in [Be-Kö], where one also finds Proposition 1.1. The fan trivialization theorem is due to Bröcker ([Br2]). Here we present a different proof which also works in the p-adic situation. The remaining results of Section 1 appeared in [Br1], as well as most of Section 2. However, the proof of Theorem 2.7 is taken from [An-Be], and Proposition 2.10 is due to Marshall-Walter ([Mr-Wa]). Also Section 3 is partially done in [Br1], but the more delicate Propositions 3.5 and 3.7 are new. The results of Setion 4 are new too. They simplify a lot earlier steps ([An-Br-Rz]) towards the calculation of stability indices of semianalytic sets. Except for finitely generated algebras over the rationals, the applications given in Section 5 were known before. In particular, the equality in Proposition 5.3 is due to Scheiderer ([Sch1]). Again, the study of archimedean rings in Section 6 is new material. However, multilocal properties appeared already in [Br4]. The fact that an open semialgebraic set can always be described by strict inequalities over the field generated by the coefficients of any starting description was proved by Dickmann ([Dk]) answering a question of Bröcker ([Br5]); his original proof belongs to model theory. Examples 7.1-7.3 and 7.10 are well-known, except perhaps for the ones over the rationals. In Example 7.1 we recover the pictures of 'geometric fans' described in detail in Section I.4.4. Much work is done towards proving that obstructions coming from fans show up already in these geometric fans ([An-Rz1,2,4]). This can be extended to finite subspaces, in order to treat separation questions ([Ac-An-Bg]). Thus one can take a step to obtaining constructible results ([Ac-Bg-Vz], [Ac-An-Bg]). The effort to visualize valuative obstructions can be traced back to Schülting ([Schü]; see also [Br-Schü] and [Al-Gm-Rz]). Proposition 7.4 is taken from [Br10], and "cutting off hairs" was first proved over the reals in [Rz2]. The striking Propositions 7.9-7.11 seem to be new.

Chapter VII. Real Algebra of Excellent Rings

Summary. This chapter is devoted to excellent rings, and contains the results that allow to extend what we have already seen for semialgebraic sets to semianalytic sets. In Sections 1 and 2 we collect the commutative algebra needed later. Very few proofs are given, since almost everything can be found in our general references [Mt] and the more elementary [At-Mc], [Bs-Is-Vg]; an important exception is our proof that local-ind-etale limits of excellent rings are again excellent. In addition, we state without proof the fundamental Rotthaus's theorem on M. Artin's approximation property. In Section 3 we characterize the extension of prime cones under completion, a crucial result for all that follows. The curve selection lemma which is proved in Section 4 has many important applications: existence theorems for valuations and fans (Section 5), and constructibility of closures (Section 6) are some. It is also needed in Section 7 for the proof of another key theorem: the real going-down for regular homomorphisms. After this, we characterize local constructibility of connected components in Section 8.

1. Regular Homomorphisms

We start by setting the terminology and reviewing quickly some notions and facts from commutative algebra which are essential for our purposes. Many results are given without proof, but we include references where they can be found. We also present some proofs that are surely well known to specialists but for which we have not a precise or accesible reference. Remember that all rings are assumed to be commutative with 1.

Definition 1.1 *Let A be a ring. An A-module M is called* flat *if for any exact sequence of A-modules*

$$N' \to N \to N''$$

the corresponding tensorized sequence

$$M \otimes N' \to M \otimes N \to M \otimes N''$$

is exact.

follow the more basic reference [Mt]. Our proof of Theorem 2.5 is new as far as we know, although the theorem itself goes back to Seydi ([Sy]). Rotthaus's important theorem on the approximation problem for excellent henselian rings containing $\mathbb{Q}$ appeared in [Rt], solving a conjecture raised by M. Artin in [Ar2]. M. Artin himself proved the approximation property for henselizations of finite algebras and for analytic algebras [Ar1,2]. The final culmination of this approximation theory was achieved by Popescu in [Po1-3], and then clarified by Spivakovsky in [Sk]. We refer the reader to [Te] for a complete survey of the theory and its applications. Among these, there are several concerning complexity and other problems in real geometry ([Co-Rz-Sh], [Qz]).

Extension of orderings under completion was shown in [Rz3], as the essential step towards the solution of Hilbert's 17th problem for compact global analytic sets; the same idea and motivation can be found in [Jw2]. The proof given here is simpler since it uses approximation, not fully available at that time. Curve selection lemmas are classical in algebraic and analytic geometry and appear in a variety of different forms. The formulation of Theorem 4.2 is a quite technical improvement of [Rz8] and implies (in the real case) most algebraic and geometric versions which were scattered in the literature (see [Ls], [Me], [Rn], [Ro]). Proposition 5.1 on real dimension is taken from [Rz6,8], and generalizes the previous results for algebras finitely generated over fields. The existence of real valuations (Proposition 5.2) has been the matter of many papers, among which we quote [Lg1] as a classic and the more recent [Rb], [An] and [An-Rz3]. Constructibility of closures was proved first for power series rings in [Al-An] and then for excellent rings in [An-Br-Rz]; the proof given in this book is slightly shorter since it uses Proposition 5.2. Example 6.2 is due to Gamboa ([Gm]). The real going-down theorem was proved in [Rz9], and it has been a many-sided tool since ([Sch3 §3]). In fact, the result itself was the answer to a question of Alonso-Andradas who needed it to prove Proposition 8.3 on constructibilty of connected components for formal power series rings ([Al-An]). Apart from this use of the real going-down, their proof is very different from ours, and Theorem 8.4 and Corollary 8.5 are new. Finally, Proposition 8.6 is taken from [Rz7]; for a far reaching generalization see [Sch3].

Chapter VIII. Real Analytic Geometry

Summary. In this chapter we apply all the previous results to the study of semianalytic sets in real analytic manifolds. In Section 1 we settle the terminology concerning *global* analytic functions and sets. Sections 2 and 3 are devoted to the local theory, that is, to *germs at points*. We review there several classical results in the framework of real spaces, with some technical suplements that will be needed later. In Section 4 we obtain the algebraic properties of the various rings of *global* analytic functions that will be used in the sequel. Sections 5 to 7 are devoted to the Artin-Lang property, the complexity and the constructibility of topological operations. This is the concrete reward for all preceding abstract work. In Section 8 we put it all together for the nicest case, that of *germs at compact sets*.

1. Semianalytic Sets

Let Ω be a real analytic manifold, which we always suppose paracompact and Hausdorff. Let $\mathcal{O}(\Omega)$ be the ring of global analytic functions on Ω.

Definition 1.1

a) A global analytic set *is a subset $X \subset \Omega$ of the form*

$$X = \{x \in \Omega \mid f_1(x) = \cdots = f_r(x) = 0\}$$

for some analytic functions $f_1, \dots, f_r \in \mathcal{O}(\Omega)$.

b) A global analytic function on a closed set $X \subset \Omega$ *is a function $f : X \to \mathbb{R}$ which is the restriction of a global analytic function on Ω.*

These definitions differ from the classical ones, which had local nature. In fact, there are sets (even compact sets) $X \subset \mathbb{R}^n$ that, near every point, are the zero set of an analytic function (depending on the point), but that are not global analytic sets.

Notes

The study of real analytic sets was started in the fifties by Bruhat, Cartan and Whitney, [Bh-Ca1,2], [Wh-Bh], who settled the basic properties concerning complexification, dimension, irreducible components, and regularity. Cartan's paper [Ca], where the important theorems A and B are proved for real manifolds, is the threshold of the study of global real analytic sets. This paper contains also a detailed exposition of the coherence problems in the real case. Other important results concerning global analytic sets and functions are Grauert's embedding theorem ([Gr]) and Frisch's finiteness theorems for coherent sheaves ([Fr]) which imply the fact that the rings of real analytic function germs at compact semianalytic sets are noetherian.

Semianalytic sets where first systematically studied by Lojasiewicz in [Lo1, 2], whose motivation was the division problem for distributions. The local basic properties of semianalytic sets were later condensed in Hironaka's rectiliniarization theorem ([Hk2]), a byproduct of his desingularization theorems ([Hk1]). This in fact led Hironaka to the study of subanalytic sets, a category discovered also by Hardt ([Ht1,2]) and the matter of a great interest ever since. Our approach, based on cylindrical decompositions (Proposition 3.1), follows the pattern of [Bt-Mm], [Rz4] and [Fz-Rc-Rz]. The bijection between semianalytic set germs and constructible subsets of the real spectrum of the ring of analytic function germs appeared in the two latter papers. The main improvement here is Proposition 3.3, which describes very carefully the behaviour of connected components of set germs in order to get information useful in the global setting.

The real Nullstellensatz and the solution to Hilbert's 17th problem for germs were obtained by Risler ([Rs]), and then a series of similar or somehow improved versions were obtained by different authors ([Ls], [Me], [Rn], [Rz1]). The global counterparts of these results came much later, after the first attempts by Adkins and Leahy ([Ad], [Ad-Lh]) and the answers in dimension 2 by Bochnak-Risler ([Bo-Rs]) and Jaworski ([Jw1]), or under strong additional assumptions by Bochnak-Kucharz-Shiota ([Bo-Ku-Sh]). Then, Jaworski ([Jw2]) and Ruiz ([Rz3]) solved independently the two problems in the compact case; the presence of real spectra in these solutions was clarified in [Rz5,6]. The characterization of sums of $2n$-th powers comes from [Rz11], although there the global result was proved directly, without describing explicitely the connection local-global. For non-singular compact analytic surfaces the same characterization had been obtained by Kucharz ([Ku]). The study of the topology of global semianalytic sets was first done in the compact case in ([Rz7]). The only known results without any compactness assumption correspond to dimension 2 ([Cs-An]). Our statements are given consistently under weakened compactness assumptions, in the form already proposed in [Rz3] and [Rz10]. We do it by means of filters of semianalytic sets associated to prime cones. This idea, inspired by Gillman-Jerison's famous book [Gi-Je], was first used in dimension 1 in [An-Be], and afterwards in arbitrary dimension in [Cs1,2], [Cs-An] and [Jw3]. We present here a mixed simplified version of Castilla's and Jaworski's formulations.

The first results on minimal generation of global semianalytic sets appeared in [An-Br-Rz] for global basic open semianalytic subsets of compact global analytic sets. The germ case was treated there as a preliminary tool. Our exact computations were not possible then, since the break through for spaces of orderings had not been done yet. Furthermore, the treatment given in Section 6 is much simpler, as well as more general. It yields very precise results for boundary bounded sets. Quite remarkably, the exact value of the invariant $\bar{s}$ for basic closed semianalytic set germs remains unknown up to one unit, except for planar set germs ([D-C]). Recently, Delzell ([Dz2]) found a surprising application of basicness of semianalytic germs to a classical question on Hilbert's 17th problem (see [Dz1]). Finally we can only mention the problem of comparing the different complexities defined respectively by regular, Nash or analytic functions. For this involved topic, closely related to M. Artin's approximation theory, we refer the reader to [An-Rz5,6], [Rz-Sh] and [Co-Rz-Sh].

Bibliography

[Ab] S. Abhyankar: On the valuations centered in a local domain. *Amer. J. Math.* **78**, 321-348 (1956).

[Ac-An-Bg] F. Acquistapace, C. Andradas, F. Broglia: Geometric obstructions to separation of semialgebraic sets. *To appear.*

[Ac-Bg-Vz] F. Acquistapace, F. Broglia, P. Vélez: An algorithmic criterion for basicness in dimension 2. *Manuscripta math.* **85**, 45-66 (1994)

[Ad] W.A. Adkins: A real analytic Nullstellensatz for two dimensional manifolds. *Boll. U.M.I.* **14B**, 888-903 (1977)

[Ad-Lh] W.A. Adkins, J.V. Leahy: A global real analytic Nullstellensatz. *Duke Math. J.* **43**, no. 1, 81-86 (1976)

[Al-An] M.E. Alonso, C. Andradas: Real spectra of complete local rings. *Manuscripta math.* **58**, 155-177 (1987)

[Al-Gm-Rz] M.E. Alonso, J.M. Gamboa, J. Ruiz: Ordres sur les surfaces réelles. *C.R. Acad. Sci. Paris* **298**, 17-19 (1984)

[Al-Ry] M.E. Alonso, M.-F. Roy: Real strict localizations. *Math. Z.* **194**, 429-441 (1987)

[An] C. Andradas: Specialization chains of real valuation rings. *J. Algebra* **124**, 437-446 (1989)

[An-Be] C. Andradas, E. Becker: A note on the real spectrum of analytic functions on an analytic manifold of dimension 1. *In* Real analytic and algebraic geometry, Proceedings, Trento 1988. *Springer LNM* **1420**, 1-21 (1990)

[An-Br-Rz] C. Andradas, L. Bröcker, J.M. Ruiz: Minimal generation of basic open semianalytic sets. *Invent. math.* **92**, 409-430 (1988)

[An-Rz1] C. Andradas, J.M. Ruiz: More on basic semialgebraic sets. *In* Real algebraic geometry, Proceedings, Rennes 1991. *Springer LNM* **1524**, 128-139 (1992)

[An-Rz2] C. Andradas, J.M. Ruiz: Low dimensional sections of basic semialgebraic sets. *Illinois J. Math.* **38**, no. 2, 303-326 (1994)

[An-Rz3] C. Andradas, J.M. Ruiz: On local uniformization of orderings. *In* Recent advances in real algebraic geometry and quadratic forms, Proceedings of the RAGSQUAD Year, Berkeley 1990-1991. *AMS Contemporary Math.* **155**, 19-46 (1994)

[An-Rz4] C. Andradas, J.M. Ruiz: Ubiquity of Lojasiewicz's example of a nonbasic semialgebraic set. *Michigan Math. J.* **41**, 465-472 (1994)

[An-Rz5] C. Andradas, J.M. Ruiz: Algebraic versus analytic basicness. *In* Real analytic and algebraic geometry, Proceedings, Trento 1992, 1-19. Walter de Gruyter, Berlin New York 1995

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