

Mapping Degree Theory

Enrique Outerelo
Jesús M. Ruiz

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Preface

This book springs from lectures on degree theory given by the authors during many years at the Departamento de Geometría y Topología at the Universidad Complutense de Madrid, and its definitive form corresponds to a three-month course given at the Dipartimento di Matematica at the Università di Pisa during the spring of 2006. Today mapping degree is a somewhat classical topic that appeals to geometers and topologists for its beauty and ample range of relevant applications. Our purpose here is to present both the history and the mathematics.

The notion of degree was discovered by the great mathematicians of the decades around 1900: Cauchy, Poincaré, Hadamard, Brouwer, Hopf, etc. It was then brought to maturity in the 1930s by Hopf and also by Leray and Schauder. The theory was fully burnished between 1950 and 1970. This process is described in Chapter I. As a complement, at the end of the book there is included an index of names of the mathematicians who played their part in the development of mapping degree theory, many of whom stand tallest in the history of mathematics. After the first historical chapter, Chapters II, III, IV, and V are devoted to a more formal proposition-proof discourse to define and study the concept of degree and its applications. Chapter II gives a quick presentation of manifolds, with special emphasis on aspects relevant to degree theory, namely regular values of differentiable mappings, tubular neighborhoods, approximation, and orientation. Although this chapter is primarily intended to provide a review for the reader, it includes some not so standard details, for instance concerning tubular neighborhoods. The main topic, degree theory, is presented in Chapters III and IV. In a simplified manner we can distinguish two approaches to the theory: the Brouwer-Kronecker degree and the Euclidean degree. The first is developed in Chapter III by differential means, with a quick diversion into the de Rham computation in cohomological terms. We cannot help this diversion: cohomology is too appealing to skip. Among other applications, we obtain in this chapter a differential version of the Jordan Separation Theorem. Then, we construct the Euclidean degree in

Chapter IV. This is mainly analytic and astonishingly simple, especially in view of its extraordinary power. We hope this partisan claim will be acknowledged readily, once we obtain quite freely two very deep theorems: the Invariance of Domain Theorem and the Jordan Separation Theorem, the latter in its utmost topological generality. Finally, Chapter V is devoted to some of those special results in mathematics that justify a theory by their depth and perfection: the Hopf and the Poincaré-Hopf Theorems, with their accompaniment of consequences and comments. We state and prove these theorems, which gives us the perfect occasion to take a glance at tangent vector fields.

We have included an assorted collection of some 180 problems and exercises distributed among the sections of Chapters II to V, none for Chapter I due to its nature. Those problems and exercises, of various difficulty, fall into three different classes: (i) suitable examples that help to seize the ideas behind the theory, (ii) complements to that theory, such as variations for different settings, additional applications, or unexpected connections with different topics, and (iii) guides for the reader to produce complete proofs of the classical results presented in Chapter I, once the proper machinery is developed.

We have tried to make internal cross-references clearer by adding the Roman chapter number to the reference, either the current chapter number or that of a different chapter. For example, III.6.4 refers to Proposition 6.4 in Chapter III; similarly, the reference IV.2 means Section 2 in Chapter IV. We have also added the page number of the reference in most cases.

One essential goal of ours must be noted here: we attempt the simplest possible presentation at the lowest technical cost. This means we restrict ourselves to elementary methods, whatever meaning is accepted for elementary. More explicitly, we only assume the reader is acquainted with basic ideas of differential topology, such as can be found in any text on the calculus on manifolds.

We only hope that this book succeeds in presenting degree theory as it deserves to be presented: we view the theory as a genuine masterpiece, joining brilliant invention with deep understanding, all in the most accomplished attire of clarity. We have tried to share that view of ours with the reader.

History

In the body of mathematics, the notion of degree stands as a beautiful achievement of topology and one of the main contributions of the twentieth century, which has been called the century of topology. In Chapter I we try to outline how the ideas that led to this fundamental notion of degree were sparked and came to light. It is only natural that such a task is biased by our personal opinions and preferences. Thus, it is likely that a specialist in, say, partial differential equations would present the tale in a somewhat different way. All in all, a choice must be made and ours is this:

- §1. *Prehistory*: Gauss, Cauchy, Liouville, Sturm, Kronecker, Poincaré, Picard, Bohl (1799–1910).
- §2. *Inception and formation*: Hadamard, Brouwer (1910–1912).
- §3. *Accomplishment*: Hopf, Leray, Schauder (1925–1934).
- §4. *Renaissance and reformation*: Nagumo, de Rham, Heinz (1950–1970).
- §5. *Axiomatization*: Führer, Amann, Weiss (1970–1972).
- §6. *Further developments*: Equivariant theory, infinite dimensions.

The presentation of these topics is mainly discursive and descriptive, rigorous proofs being deferred to Chapters II through V where there will be complete arguments for all the most classical results presented here.

1. Prehistory

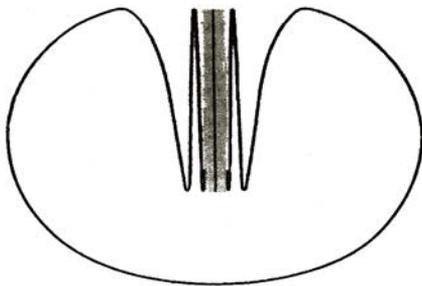
Roughly speaking, degree theory can be defined as the study of those techniques that give information *on the existence of solutions of an equation of the form $y = f(x)$* , where x and y dwell in suitable spaces and f is a map from one to the other. The theory also gives clues for *the number of solutions and their nature*. An important particular case is that of an equation $x = f(x)$, where f is a map from a domain D of a linear space into D itself: this is the so-called *Fixed Point Problem*.

Schönflies Theorem. *Let Γ be a Jordan curve in \mathbb{R}^2 and G_1 its interior. Then, for all $x \in G_1$ and all $a \in \Gamma$, there is a simple arc from x to a whose points other than a are all in G_1 .*

We say that a is *accessible* from G_1 . This finally led to the following characterization of Jordan curves:

If a compact set in \mathbb{R}^2 has two complementary domains, from which every point of the set is accessible, then the set is a Jordan curve.

Here we see the typical topological beast which is not a Jordan curve:



* * *

Let us now take a closer look at Hadamard's paper. In the first paragraph, Hadamard analyzes the proof above of the Jordan Separation Theorem, focusing on the part involving the order (= index) of a point with respect to the curve. He thus presents clearly what he wants to generalize. To do that, he defines, in the second paragraph, what he means by *a surface in the Euclidean space*. His notions are based on ideas introduced by Poincaré, at the beginning of the century, to treat polyhedra. We will not go into detail here, but in a sketchy way, we can think of an $(n - 1)$ -*surface in \mathbb{R}^n* as a subspace that decomposes into pieces homeomorphic to the standard $(n - 1)$ -simplex, which glue in a suitable way along their faces. Hadamard concludes this paragraph by stating *Green's Theorem*, which reduces a volume (triple) integral to a surface (double) one.

In the third paragraph, Hadamard defines the order of a point with respect to a closed oriented surface in \mathbb{R}^n . Let us suppose we are given a hypersurface S , with coordinates $(x_1, \dots, x_n) \in \mathbb{R}^n$ with respect to parameters (u_1, \dots, u_{n-1}) , such that the x_i 's have continuous partial derivatives with respect to the u_j 's (later, this condition is weakened to the mere existence of partial derivatives); suppose also that the origin is not a point of S . Then, Hadamard shows this key fact:

Manifolds

Here we describe the objects of our theory: manifolds and mappings. We will be dealing with differentiable manifolds: our methods are those of differential topology. The general definitions and some basic facts are gathered in §§1–2. In particular, we stress the distinction between differentiable and smooth structures. In §3 we state the essential *Sard-Brown Theorem* for differentiable mappings but prove it only for smooth mappings. This brings in the subtleties concerning finite or infinite differentiability, but we have chosen a most dramatic approach: to use the fact that every differentiable structure is in fact *smooth*. In §4 we discuss in depth the existence and two different constructions of *tubular neighborhoods and differentiable retractions*, both of which will be required later. Then, in §5 we prove a key result for degree theory: differentiable mappings that are homotopic are *differentiably* homotopic. This is essential, because even for smooth manifolds, we use our methods to study continuous mappings, neither smooth nor even differentiable. In §6 we consider a special type of homotopies: the so-called *diffeotopies*, often needed to move points in manifolds at will. Finally, in §7 we recall the basics of *orientation*, which play a crucial role in degree theory.

1. Differentiable mappings

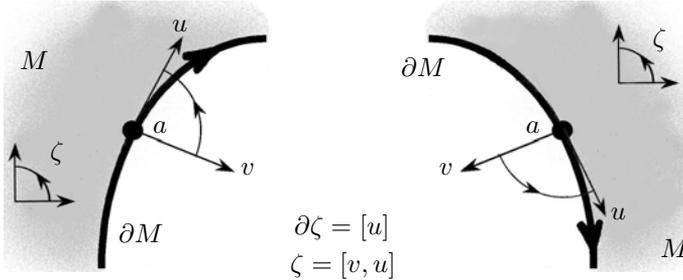
In this section we recall the basic notions concerning differentiability. In particular we discuss partitions of unity and bump functions.

(1.1) Differentiable mappings. Let $U \subset \mathbb{R}^p$ be an open set and let $r = 1, \dots, \infty$. A differentiable \mathcal{C}^r function $f : U \rightarrow \mathbb{R}$ is a function whose partial derivatives $\frac{\partial^k f}{\partial x_{i_1} \dots \partial x_{i_k}}$ exist and are continuous for all $k \leq r$. We use the terminology *smooth* when $r = \infty$. A mapping $f : X \rightarrow Y$ of arbitrary sets $X \subset \mathbb{R}^p$ and $Y \subset \mathbb{R}^q$ is \mathcal{C}^r if for every point $x \in X$ there are \mathcal{C}^r functions $\bar{f}_j : U \rightarrow \mathbb{R}$, $1 \leq j \leq q$, defined on an open neighborhood U of x in \mathbb{R}^p such that $\bar{f} = (\bar{f}_1, \dots, \bar{f}_q)$ coincides with f on $U \cap X$; we say that \bar{f} is a local \mathcal{C}^r extension of f .

In (1), $\mathbf{x}_1 \circ \gamma : [0, \varepsilon] \rightarrow [0, \infty)$ must be increasing, and $t_1 = (\mathbf{x}_1 \circ \gamma)'(0)$ is positive; hence u is inward. In (2) we get an outward vector, because $\mathbf{x}_1 \circ \gamma : (1 - \varepsilon, 1] \rightarrow [0, \infty)$ is decreasing.

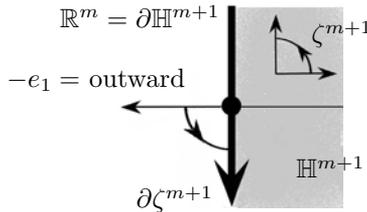
Finally, we define an orientation on $T_x(\partial M)$ by

$$\partial\zeta_x = [u_1, \dots, u_{m-1}] \quad \text{if} \quad \zeta_x = [v, u_1, \dots, u_{m-1}], \quad v \in T_x M \text{ outward.}$$



Again, some straightforward computations confirm that this indeed defines an orientation $\partial\zeta$ on ∂M .

Examples 7.8. (1) The orientation of $\mathbb{R}^m \equiv \mathbb{R}^m \times \{0\}$ as the boundary of $\mathbb{H}^{m+1} : x_1 \geq 0$ is the “wrong” one! Indeed, \mathbb{H}^{m+1} carries the canonical orientation ζ^{m+1} , but since e_1 is inward tangent, we get $\partial\zeta^{m+1} = -\zeta^m$.



(2) Let us look at the unit sphere \mathbb{S}^m as the boundary of the ball $\sum_i x_i^2 \leq 1$. Then the vector $\nu(x) = x$ is outward tangent to the ball, and we see that the orientation of the sphere as boundary of the ball is the canonical orientation defined in II.7.5(2), p. 88.

(3) A case of relevance for later discussion is that of a cylinder. Let M be an oriented boundaryless manifold and consider the cylinder $M' = [0, 1] \times M$ oriented as described in II.7.1, p. 85: if u_1, \dots, u_m give the orientation ζ_x at $x \in M$, at $(t, x) \in M'$ take the orientation

$$\xi_{(t,x)} = [(1, 0), (0, u_1), \dots, (0, u_m)].$$

The Brouwer-Kronecker degree

Here we present the beautiful *Brouwer-Kronecker degree theory* for arbitrary proper mappings from the differentiable viewpoint. To start with, §§1 and 3 contain the construction of the Brouwer-Kronecker degree. Also, we use it to prove the *Fundamental Theorem of Algebra*. In §2 we take a small diversion to describe the degree through integration by means of the *de Rham cohomology*. Next, we discuss in §4 the extension of the theory to arbitrary differentiable manifolds, which is based on the computation of the degree of a C^1 mapping. In §5 we define the so-called *Hopf invariant*, for mappings between spheres of unequal dimension, which is based upon the notion of *link coefficient*. We devote §6 to a beautiful application of the theory: the *Jordan Separation Theorem*. We show that every closed differentiable hypersurface disconnects the Euclidean space, and from this we deduce that such a hypersurface has a global equation and is orientable. We conclude the chapter by proving in §7 the famous *Brouwer Fixed Point Theorem*, jointly with some interesting consequences, and the equally important theorem by Brouwer concerning the existence on spheres of vector fields without zeros.

1. The degree of a smooth mapping

Here we present the notion of degree for smooth data. The first step towards the definition of the Brouwer-Kronecker degree is as follows:

Proposition 1.1. *Let $f : M \rightarrow N$ be a proper smooth mapping of two oriented, boundaryless, smooth manifolds of dimension m . Then each regular value $a \in R_f \subset N$ of f has an open neighborhood $V \subset R_f$ such that for every $b \in V$, the inverse image $f^{-1}(b)$ is a finite set and the integer*

$$d(f, b) = \sum_{x \in f^{-1}(b)} \text{sign}_x(f)$$

only depends on a .

Proof. First note that for any regular value b the compact set $f^{-1}(b)$ is discrete. This follows because f is a local diffeomorphism at each $x \in f^{-1}(b)$

Indeed, consider the curve germ $\gamma(t) = x + tu$. Then

$$\pi_p \circ \gamma(t) = p + \lambda(t)(x + tu - p)$$

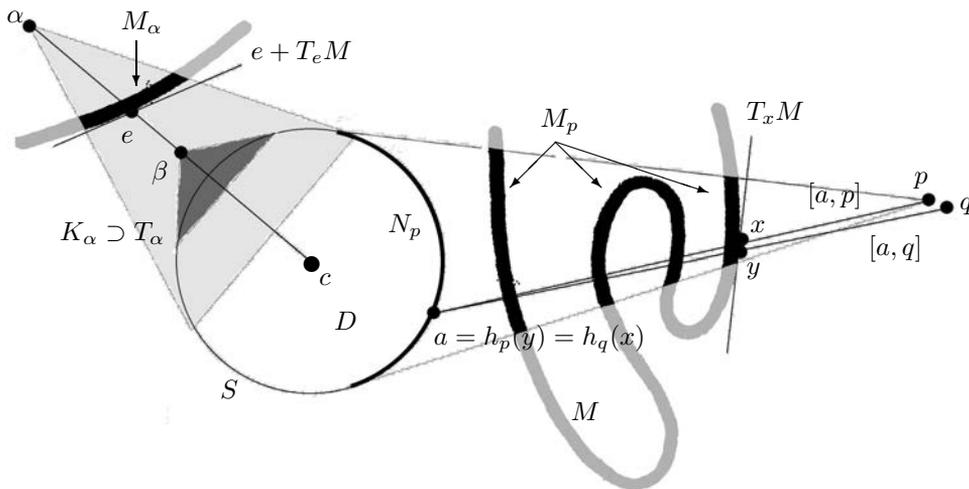
for some smooth $\lambda(t)$, and

$$d_x \pi_p(u) = (\pi_p \circ \gamma)'(0) = \lambda'(0)(x - p) + \lambda(0)u = v.$$

Since $\pi_p(x) = p + \lambda(0)(x - p) \in S$ and $p \notin S$, we have $\lambda(0) \neq 0$, and v is vanishes if and only if u is parallel to $x - p$.

We can reformulate the above remark by saying that $d_x \pi_p$ is *injective on any linear hyperplane L transversal to $x - p$* .

Now, pick a point $e \in M$ and a line ℓ through e not tangent to M (that is, $\ell \not\subset e + T_e M$). This choice guarantees that e is isolated in $\ell \cap M$ (II.2.12, p. 60): there is a segment $[\alpha, c] \subset \ell$ containing $e (\neq \alpha, c)$, such that $[\alpha, c] \cap M = \{e\}$. In particular, c is not in the closed set M , and there is an open ball D centered at c with $M \cap \overline{D} = \emptyset$.



Note that for D small enough, π_α restricts to a diffeomorphism $h_\alpha : M_\alpha = M \cap T_\alpha \rightarrow N_\alpha$. Indeed, for a first choice of D , the preceding remarks on the derivatives of conic projections tell that $d_e h_\alpha$ is a linear isomorphism. Consequently, h_α is a local diffeomorphism at the point e , and the claim follows by shrinking D .

Once this setting is ready, consider an arbitrary point $p \notin M \cup \overline{D}$ and the corresponding π_p, K_p, T_p, N_p . If there is some point $a \in N_p$ such that $[a, p] \cap M = \emptyset$, we write

$$w_2(p) = 0.$$

Degree theory in Euclidean spaces

In this chapter we develop *Euclidean Degree theory*. This could be done in full generality for proper mappings on closures of arbitrary open sets. However, here we have chosen the simplest way, enough for the applications, and restricted ourselves to bounded open sets. The reader will find in §§1–3 that everything can be done with a little calculus (and some very ingenious arguments); on the other hand, it is quite straightforward to extend the construction to the general unbounded case, a profitable exercise we recommend to the reader. Along the way we include Gauss's proof of the *Fundamental Theorem of Algebra* and a refinement of the *Brouwer Fixed Point Theorem*. Next, in §4, we come back to the origins and define the *winding number*, which is the agent behind the scenes for the equivalence of the Brouwer-Kronecker and Euclidean degrees. In §5, we deduce the famous *Borsuk-Hirsch Theorem* and an important consequence: the *Invariance of Domain Theorem*. After this, §6 contains one fundamental computation: the *Multiplication Formula* for the Euclidean degree of a composite mapping. Finally, that formula is used in §7 to deduce a most general purely topological version of the *Jordan Separation Theorem* and from it the *Invariance of Domain Theorem* once again.

1. The degree of a smooth mapping

From now on, D stands for a bounded open set in \mathbb{R}^{m+1} , and $X = \overline{D} \setminus D$ stands for its (topological) boundary. Our purpose here is to define the degree of a continuous mapping $\overline{D} \rightarrow \mathbb{R}^{m+1}$. This we will do by the usual method: first we consider smooth mappings, and then we approximate by them any given continuous mapping. Here, the reader will recognize some familiar arguments, which are repeated to make the construction fully self-contained.

Note that, \overline{D} being compact, a continuous mapping $f : \overline{D} \rightarrow \mathbb{R}^{m+1}$ is proper (hence closed), and in particular $f(\overline{D})$ and $f(X)$ are closed in \mathbb{R}^{m+1} . Furthermore, the *norm*

$$\|f\| = \max\{\|f(x)\| : x \in \overline{D}\}$$

perpendicular to the others, hence independent. Thus we conclude that $\Delta(s)$ has no zero; hence it has constant sign:

$$(2) \quad \Delta(s) \text{ has the same sign at } s = 0 \text{ and } s = 1.$$

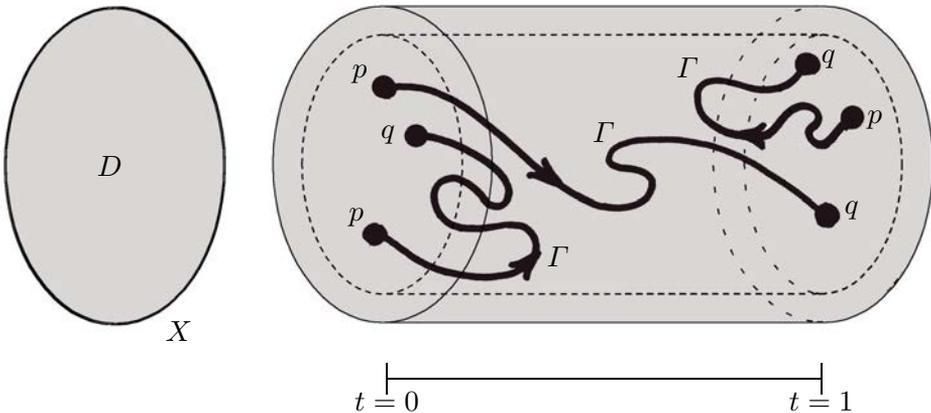
Next, we look again at the equations in (1), which tell how to combine the columns in our determinant to get

$$(3) \quad \frac{dt}{ds} \cdot \Delta = \begin{vmatrix} \left\| \frac{d\gamma}{ds} \right\|^2 & \frac{dx_1}{ds} & \dots & \frac{dx_{m+1}}{ds} \\ 0 & \frac{\partial \varphi_1}{\partial x_1} & \dots & \frac{\partial \varphi_1}{\partial x_{m+1}} \\ \vdots & \vdots & & \vdots \\ 0 & \frac{\partial \varphi_{m+1}}{\partial x_1} & \dots & \frac{\partial \varphi_{m+1}}{\partial x_{m+1}} \end{vmatrix} = \left\| \frac{d\gamma}{ds} \right\|^2 \det \left(\frac{\partial \varphi_i}{\partial x_j}(\gamma(s)) \right).$$

After this preparation, we compute degrees:

$$(4) \quad \begin{cases} d(H_0, D, a) = \sum_k \sum_{(0,y) \in \Gamma_k} \text{sign}_y(H_0), \\ d(H_1, D, a) = \sum_k \sum_{(1,z) \in \Gamma_k} \text{sign}_z(H_1). \end{cases}$$

For that computation, we are only interested in the components $\Gamma = \Gamma_k$ that meet the levels $t = 0, 1$. Those components are the bordered components among the Γ_k 's, that is, the components diffeomorphic to a compact interval; hence they have two distinct boundary points, $\gamma(0) = p$ and $\gamma(1) = q$, both at those t -levels. The following picture shows the various possibilities:



Clearly, the required equality of degrees $d(H_0, D, a) = d(H_1, D, a)$ will come from the following facts:

The Hopf Theorems

In this chapter we formulate and prove the theorems that fully complete degree theory: the degree is the only homotopy invariant for spheres. This must be formulated for both the Brouwer-Kronecker and the Euclidean degrees, although in essence the two cases are the same. These are the *Hopf Theorems*. As these theorems refer to mappings into spheres, we start the chapter by constructing in §1 various basic examples of them. Then in §2 we prove the first Hopf Theorem, that *two mappings into a sphere of the same (Brouwer-Kronecker) degree are homotopic*; hence, in particular, the cohomotopy group $\pi^m(M)$ of any m -manifold and the homotopy group $\pi_m(\mathbb{S}^m)$ of the m -sphere are \mathbb{Z} . In §3 we deduce the same result for the Euclidean degree, which has the virtue of determining the cohomotopy groups $\pi^m(X)$ for some more exotic spaces X of dimension m , like the topological circle. These Hopf Theorems require the use of diffeotopies. Next, in §4 we come back to the *Hopf invariant* and define the *Hopf fibrations* to study some homotopy groups $\pi_k(\mathbb{S}^m)$ for $k > m$. Then we turn to the theory of tangent vector fields, and in §5 we discuss the essential notion of *index of a tangent vector field at an isolated zero*, using both the Brouwer-Kronecker and the Euclidean degrees. This notion is best illustrated by the *gradients of Morse functions*, which we describe in §6. Finally, we conclude the chapter and the book by presenting in §7 another deep theorem also named after Hopf, the *Poincaré-Hopf Index Theorem*, that computes the index of a tangent vector field. We also see how the *Gauss-Bonnet Formula* follows from this.

1. Mappings into spheres

In this section we discuss several methods for constructing smooth mappings into spheres, or to modify one already given.

To start with, we exhibit simple examples of mappings of arbitrary degree.

Example 1.1. *Let $\mathbb{S}^m \subset \mathbb{R}^{m+1}$ be the standard unit sphere. Then, for every integer d there are mappings $f_d : \mathbb{S}^m \rightarrow \mathbb{S}^m$ of degree d .*

Since all constant mappings have degree 0, we assume $d \neq 0$.

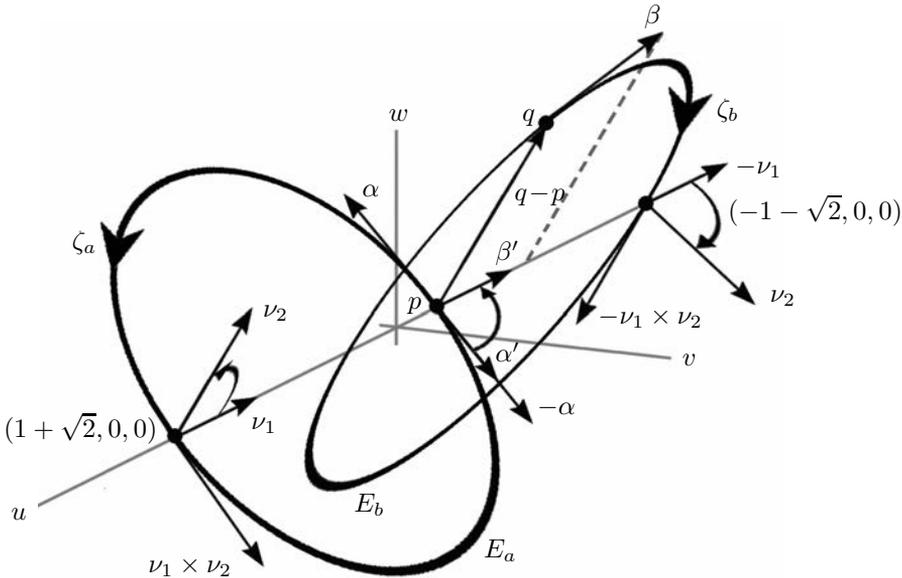
An easy computation shows that the poles of \mathbb{S}^2 , $a = (0, 0, -1)$ and $b = (0, 0, 1)$ (both $\neq (-1, 0, 0) = f(0, 0, 0, -1)$), are two regular values with inverse images:

$$\begin{cases} C_a = f^{-1}(a) = \mathbb{S}^3 \cap \{y_1 + x_2 = y_2 + x_1 = 0\}, \\ C_b = f^{-1}(b) = \mathbb{S}^3 \cap \{y_1 - x_2 = y_2 - x_1 = 0\}. \end{cases}$$

By projection from the south pole we get in \mathbb{R}^3 two ellipses E_a and E_b :

$$E_a : \begin{cases} v + w = 0, \\ (u - 1)^2 + 2v^2 = 2, \end{cases} \quad E_b : \begin{cases} v - w = 0, \\ (u + 1)^2 + 2v^2 = 2. \end{cases}$$

They are depicted below, with the orientations ζ_a and ζ_b that they carry as inverse images.



Let us describe ζ_a by means of II.7.6, p. 89. We localize f using (i) the projection from the south pole in \mathbb{S}^3 and (ii) the projection from the north pole in \mathbb{S}^2 . Since both projections reverse orientation (II.7.5(3), p. 88), we get the right orientation in the ellipse E_a , which for this localization has the equations

$$\begin{cases} g_1 = \frac{1}{\delta}(4u^2 + 4v^2 - 4w^2 - t^2) = 0, \\ g_2 = \frac{1}{\delta}(8uw - 4vt) = 0, \end{cases} \quad \text{where } \begin{cases} t = 1 - u^2 - v^2 - w^2, \\ \delta = (2 - t)^2 - 8vw - 4ut. \end{cases}$$

The denominator δ is important (at least its sign), because it affects the derivative of the localization $g = (g_1, g_2)$ of f . Let ν_1 and ν_2 be the gradients of g_1 and g_2 , respectively. Then the inverse image is oriented by

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For prerequisites, we recommend the quite *ad hoc* texts [4], [10], [16], and [18]. On the other hand, there is a wealth of literature on degree theory and related topics; we suggest as further reading the following books:

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Symbols

$y = f(x)$	1	$\bar{g} = \psi \circ g \circ \varphi^{-1} : K \rightarrow L $	24
$z^n + a_1 z^{n-1} + \dots + a_n = 0$	2	$d(f)$	25
$J_{x_0}^{x_1}(f)$	3	$F(t, x) = \frac{tf(x) - (1-t)x}{\ tf(x) - (1-t)x\ }$	27
$Z(z) = X(x, y) + iY(x, y)$	3	\mathbb{S}_+^n	28
$\frac{1}{2\pi i} \int_{\Gamma} \frac{Z'(z)}{Z(z)} dz$	3	$\ell(K_1, K_2)$	28
$N = x_0 J_{y_0}^{y_1}(\Delta)$	3	$\chi(M)$	30
$\frac{1}{2}\delta = \mu_1 - \mu_2$	4	$\Phi_h(x) = x - \mathcal{F}_h(x)$	34
$w = \begin{vmatrix} \frac{\partial F}{\partial x} & \frac{\partial F}{\partial y} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{vmatrix}$	5	$d(\Phi, W, 0) = d(\Phi_h, W_M, 0)$	34
$w(\Gamma, a) = \frac{1}{2\pi i} \int_{\Gamma} \frac{dz}{z-a}$	5	$x - \mathcal{F}(x) = 0$	34
$w(f(\Gamma), 0) = \frac{1}{2\pi i} \int_{f(\Gamma)} \frac{d\zeta}{\zeta}$	6	dist	36
$w(f(\Gamma), 0) = \sum_k w(\Gamma, a_k) \alpha_k$	6	$d(f, G, a)$	36
$w(f(\Gamma), 0) = \sum_k \alpha_k$	6	$\int_{\mathbb{R}^m} \Phi(x) dx = 1$	38
$E(k, \ell)$	8	$d(y(x), \Omega, z)$	38
$A(k, \ell)$	8	$\int_{\overline{\Omega}} \Phi(y(x) - z) J(y(x)) dx$	38
$\#E(k, \ell) - \#A(k, \ell)$	8	$\mathbb{M}(\mathcal{W})$	41
$\chi(F_0, F_1, \dots, F_n)$	8	$\mathcal{M}_0(\Omega)$	41
$\frac{1}{2}(\#E(k, \ell) - \#A(k, \ell))$	8	$\partial\Omega = \overline{\Omega} \setminus \Omega$	41
$V(z) = \frac{x-x^0}{\ x-x^0\ ^3}$	11	f, f_{∞}, f^*	43
$\int_{F_0=0} \langle V, \nu \rangle dS$	11	$d^*(f, U, 0)$	44
$w = N_1 - N_2$	17	$d^*(f, U)$	44
$\overline{D}^n = \{x \in \mathbb{R}^n : \ x\ \leq 1\}$	20	$\rho : \mathbb{S}^1 \rightarrow GL(V)$	46
$\partial\overline{D}^n = \mathbb{S}^{n-1}$	20	$\mathfrak{A}, \alpha = (\alpha_r)_{r \geq 0}$	46
$K, K , \dot{K} $	23	Deg(f, Ω)	46
		Deg(f, Ω) = $\Sigma(d^*(f, \Omega))$	47
		$\mathbb{S}^1 * a$	47

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