DIPARTIMENTO DI MATEMATICA
DELL'UNIVERSITÀ DI PISA

Jesùs M. Ruiz

THE BASIC THEORY OF POWER SERIES

Edizioni ETS

DOTTORATO DI RICERCA IN MATEMATICA

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Second edition
revised and extended

Edizioni ETS

Preface

The aim of these notes is to cover the basic algebraic tools and results behind the scenes in the foundations of Real and Complex Analytic Geometry. The author has learned the subject through the works of many mathematicians, to all of whom he is indebted. However, as the reader will immediately realize, he was specially influenced by the writings of S.S. Abhyankar and J.-C. Tougeron. In any case, the presentation of all topics is always as elementary as it can possibly be, even at the cost of making some arguments longer. The background formally assumed consists of:

1) Polynomials: roots, factorization, discriminant; real roots, Sturm's Theorem, formally real fields; finite field extensions, Primitive Element Theorem.

2) Ideals and modules: prime and maximal ideals; Nakayama's Lemma; localization.

3) Integral dependence: finite ring extensions and going-up.

4) Noetherian rings: primary decomposition, associated primes, Krull's Theorem.

5) Krull dimension: chains of prime ideals, systems of parameters; regular systems of parameters, regular rings.

These topics are covered in most texts on Algebra and/or Commutative Algebra. Among them we choose here as general reference the following two:

- M. Atiyah, I.G. Macdonald: Introduction to Commutative Algebra, 1969, Addison-Wesley: Massachusetts; quoted [A-McD].
- S. Lang: Algebra, 1965, Addison-Wesley: Massachusetts; quoted [L].

Even quotations to these two books are kept to a minimum, avoiding them whenever a reasonably self-contained explanation could be provided. In this way many deep results can be obtained for rings of power series almost from scratch, giving them a highly geometrical meaning. Two examples of this are that localizations preserve regularity or that integral closures are finite. In fact, that is one of the goals of this book: to review the commutative algebra of power series in geometrical terms. The guidelines for this review are the Local Parametrization Theorem, the Nullstellensatz, and Zariski's Main Theorem. Furthermore, all the work is carried out both in the complex and the real case, showing the additional difficulties and the peculiarities of the latter.

All in all, the final hope is that this book will be of some help to those not acquainted with either the geometry behind local commutative algebra or the algebra behind local analytic geometry.

The notes are based on courses and seminars given by the author during many years, organized by the Departments of Algebra, Geometry, and Topology, at the Universidad Complutense de Madrid, with the support of the D.G.I.C.Y.T.. Several people have made these activities possible and enthralling, and have contributed to the actual writing of the book. First of all, Tomás Recio, who led the author into the beauty of Analytic Geometry. Also, Carlos Andradas, MariEmi Alonso, and José Manuel Gamboa, who provided enjoyable and fruitful discussions on the teaching and learning of power series, and Capi Corrales, who read word by word the first draft of the manuscript and suggested all kind of accurate corrections.

The final thanks are for MariPaz, to whom this book, and everything else, is dedicated.

Majadahonda, October 1992

Note to the revised edition: Since publication, this book has been most often used as text or reference in various courses and seminars by the author and colleagues. This experience has provided many minor and some major modifications and additions to the 1993 version. Besides misprints, typos and others, there are two main improvements worth to mention. First, the inclusion of a converse to Rückert's Parametrization Theorem, which was a central topic here; second, a full proof of Shiota's theorem that every power series can be made polynomial in at least two variables.

Majadahonda, October 2006

Contents

I Power Series

Summary. In this chapter we study in detail convergent and formal power series. We start by recalling the notion of convergence and its main properties: absolute convergence and reordering of convergent series. We then introduce formal and convergent series and discuss the operations with them: sum, multiplication, substitution and derivation. In particular we get the Identity Principle for functions associated to convergent power series. Finally we state and prove three essential results of the theory: Rückert's Division Theorem, Weierstrass's Preparation Theorem and Hensel's Lemma.

1 Series of Real and Complex Numbers

We will denote by $\sum_\nu a_\nu$ or $\sum a_\nu$ a series of elements of the field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Here the indices $\nu = (\nu_1, \dots, \nu_n)$ are elements of $\mathbb{N}^n$, and we will use the standard notations $|\nu| = \nu_1 + \cdots + \nu_n$ and $\nu! = \nu_1! \cdots \nu_n!$.

Definition 1.1 *The series* $\sum a_\nu$ converges to the element $c \in \mathbb{K}$ *if for every real number* $\varepsilon > 0$ *there is a finite set* $I_\varepsilon \subset \mathbb{N}^n$ *such that* $|\sum_{\nu\in I} a_\nu - c| < \varepsilon$ *for every finite set of indices* $I \supset I_\varepsilon$*. In that case we say that* c *is the* sum of the series *and we write* $c = \sum a_\nu$*.*

Remarks 1.2 *a)* For series with indices in $\mathbb{N}$ the convergence in the sense of Definition 1.1 implies the classical one, but not conversely: $\sum(-1)^k/k$ does not converge according to 1.1, nevertheless the limit $\lim_{p\to\infty}\sum_{1\le k\le p}(-1)^k/k$ does exist.

b) If $\sum a_\nu$ converges to $c \in \mathbb{K}$, then $c = \lim_{p\to\infty}\sum_{|\nu|\le p} a_\nu$.

c) Let $\sum a_\nu,\ \sum b_\nu$ converge to $c, d \in \mathbb{K}$ respectively, and consider $\lambda, \mu \in \mathbb{K}$. Then $\sum(\lambda a_\nu + \mu b_\nu)$ converges to $\lambda c + \mu d$.

d) Let $\sum a_\nu,\ \sum b_\nu$ be two convergent series of real numbers such that $\sum_{\nu\in I} a_\nu \le \sum_{\nu\in I} b_\nu$ for every finite set of indices I. Then $\sum a_\nu \le \sum b_\nu$.

Proposition 1.3 *A series* $\sum a_\nu$ *of non-negative real numbers converges if and only if there is* $M > 0$ *such that* $\sum_{\nu\in I} a_\nu < M$ *for every finite set of indices* $I \subset \mathbb{N}^n$*. In this case the sum of the series is the supremum of all those finite sums.*

Proof. If there is such an M, then the supremum $c = \sup\{\sum_{\nu\in I} a_\nu \mid I \text{ finite}\}$ exists. Let us see that the series converges to c. Indeed, for every $\varepsilon > 0$ there is a finite set of indices I_ε such that $c - \sum_{\nu\in I} a_\nu < \varepsilon$, and for every finite set $I \supset I_\varepsilon$ we have

$$c \ge \sum_{\nu\in I} a_\nu \ge \sum_{\nu\in I_\varepsilon} a_\nu,$$

Lemma 3.1 *Let $f \in \mathcal{F}_n$, $f \neq 0$. After a linear change of coordinates, f becomes regular of order $\omega(f)$ with respect to $\mathtt{x}_n$.*

Proof. Set $f = \sum a_\nu \mathtt{x}^\nu$, $p = \omega(f)$. Then $f_p = \sum_{|\nu|=p} a_\nu \mathtt{x}^\nu \neq 0$, and there are $c_1, \dots, c_{n-1} \in \mathbb{K}$ with $c = f_p(c_1, \dots, c_{n-1}, 1) \neq 0$ (otherwise the homogeneous polynomial f_p would be divisible by $\mathtt{x}_n - 1$). We now make the change of coordinates $\mathtt{x}_i = \mathtt{y}_i + c_i \mathtt{y}_n, 1 \leq i < n$, $\mathtt{x}_n = \mathtt{y}_n$ to get $g(\mathtt{y}) = f(\mathtt{y}_1 + c_1\mathtt{y}_n, \dots, \mathtt{y}_{n-1} + c_{n-1}\mathtt{y}_n, \mathtt{y}_n)$. Clearly $g(0, \dots, 0, \mathtt{y}_n) = f(c_1\mathtt{y}_1, \dots, c_{n-1}\mathtt{y}_{n-1}, \mathtt{y}_n)$ consists of the monomial $c\mathtt{y}_n^p$ plus terms of higher degrees. ■

Proposition 3.2 *(Rückert's Division Theorem) Let $\Phi \in \mathcal{O}_n$ a convergent power series, regular of order p with respect to $\mathtt{x}_n$. For every $f \in \mathcal{O}_n$ there exist $Q \in \mathcal{O}_n$ and $R \in \mathcal{O}_{n-1}[\mathtt{x}_n]$ with degree of $R < p$ such that $f = Q\Phi + R$. This conditions determine Q and R uniquely. Furthermore, if Φ is distinguished in x_n and $f \in \mathcal{O}_{n-1}[\mathtt{x}_n]$, then $Q \in \mathcal{O}_{n-1}[\mathtt{x}_n]$, too.*

The same result holds true when substituting $\mathcal{O}_n$ by $\mathcal{F}_n$ and $\mathcal{O}_{n-1}$ by $\mathcal{F}_{n-1}$.

Proof. Since Φ is regular of order p with respect to $\mathtt{x}_n$ we can write

$$\Phi = \varphi + c\mathtt{x}_n^p, \quad \varphi = \sum_{i=0}^{p} a_i(\mathtt{x}')\mathtt{x}_n^{p-i} + \mathtt{x}_n^{p+1} b(\mathtt{x}),$$

where $a_0, \dots, a_p \in \mathcal{O}_{n-1} = \mathbb{K}\{\mathtt{x}'\}$, $\mathtt{x}' = (\mathtt{x}_1, \dots, \mathtt{x}_{n-1})$ and $c \in \mathbb{K}$. Up to multiplication by $1/c$ we may assume $c = 1$.

Let $\rho = (\rho_1, \dots, \rho_n)$, $\rho_i > 0$. For $f = \sum a_\nu \mathtt{x}^\nu \in \mathcal{O}_n$ we will denote by $\|f\|$ the sum of the series $\sum |a_\nu| \rho^\nu$ when this sum exists and $\|f\| = +\infty$ otherwise. Let X be the set of all series $f \in \mathcal{O}_n$ with $\|f\| < +\infty$. Clearly, if ρ is small enough, X contains any prescribed finite collection of convergent power series; in particular $f, \varphi, b, a_i \in X$.

We define a map $T : X \to X$ as follows: if $Q \in X$ set

$$f - \varphi Q = R + \mathtt{x}_n^p T(Q),$$

where $R \in \mathcal{O}_{n-1}[\mathtt{x}_n]$, and the degree of R is $< p$. This map T is *contractive*, that is,

$$\operatorname{dist}(T(Q), T(Q')) < c \operatorname{dist}(Q, Q')$$

for any $Q, Q' \in X$, where $0 < c < 1$ (dist stands for the distance associated to the norm $\|\cdot\|$). Indeed, let

$$f - \varphi Q = R + \mathtt{x}_n^p T(Q), \quad f - \varphi Q' = R' + \mathtt{x}_n^p T(Q').$$

Then

$$\varphi(Q' - Q) = R - R' + \mathtt{x}_n^p (T(Q) - T(Q')).$$

II Analytic Rings and Formal Rings

Summary. We devote this chapter to the description of the categories of analytic and formal rings over $\mathbb{R}$ and $\mathbb{C}$. First, we develop Mather's formalism of finite and quasifinite homomorphisms. Then, we obtain Noether's Projection Lemma and further algebraic properties of these rings. Thus, we come to one fundamental construction: Abhyankar's and Rückert's Parametrization. Afterwards, we introduce the regularity ideals and prove Nagata's Jacobian Criteria. Finally, we discuss complexification, an essential tool to understand the differences between the real and the complex categories.

1 Mather's Preparation Theorem

Again we set $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. Given a ring A and a prime ideal $\mathfrak{p}$ of A, we will denote by $\kappa(\mathfrak{p})$ the residue field of $\mathfrak{p}$, that is, the quotient field of the ring $A/\mathfrak{p}$; if A is local, we will denote by $\mathfrak{m}_A$ its maximal ideal.

Definition 1.1 *An* analytic *(resp. a* formal*)* ring over $\mathbb{K}$ *is a ring isomorphic to* $\mathbb{K}\{\mathtt{x}\}/I$ *(resp.* $\mathbb{K}[[\mathtt{x}]]/I$*) with* $\mathtt{x} = (\mathtt{x}_1, \dots, \mathtt{x}_n)$*; we will usually not specify "over* $\mathbb{K}$*". If* A, B *are two analytic (resp. formal) rings, an* analytic *(resp. a* formal*)* homomorphism $A \to B$ *is a homomorphism of* $\mathbb{K}$*-algebras. The field* $\mathbb{K}$ *is called the* coefficient field*.*

Proposition 1.2 *Every analytic (resp. a formal) ring is a noetherian local ring. The canonical homomorphism* $\mathbb{K} \to A/\mathfrak{m}_A$ *is an isomorphism, and* $A = \mathbb{K} + \mathfrak{m}_A$*.*

Proof. We will only give the proof in the analytic case, that of the formal case being analogous with the obvious changes. This will be done systematically all through this chapter.

First of all, since there is a certain surjective homomorphism $\mathcal{O}_n \to A$, the general case follows immediately from the case $A = \mathcal{O}_n$. Hence we suppose $A = \mathcal{O}_n$.

We argue by induction on n. If $n = 0$ the result is trivial, so we let $n > 0$ and I an ideal $\neq 0$ of $\mathcal{O}_n$. Choose $\Phi \in I$, $\Phi \neq 0$. By Lemma I.3.1, we may assume that Φ is regular of order, say, p with respect to $\mathtt{x}_n$. By Rückert's Division Theorem (Proposition I.3.2), the ring $\mathcal{O}_n/\Phi\mathcal{O}_n$ is generated by $1, \mathtt{x}_n, \dots, \mathtt{x}_n^{p-1}$ as an $\mathcal{O}_{n-1}$-module. Since $\mathcal{O}_{n-1}$ is noetherian by induction hypothesis, we deduce that $\mathcal{O}_n/\Phi\mathcal{O}_n$ is a noetherian $\mathcal{O}_{n-1}$-module ([A-McD 6.5, 6.2]). Thus $I/\Phi\mathcal{O}_n$ is finitely generated as an $\mathcal{O}_{n-1}$-module, say by the classes of $f_1, \dots, f_s \in I$. In this situation $f_1, \dots, f_s, \Phi$ generate I.

Finally, the assertions concerning the coefficient field $\mathbb{K}$ are immediate. ■

In particular $\delta = c^2$, and from Cramer's Rule we obtain

$$\delta b_i = cD_i(y_1, \dots, y_p, t_1, \dots, t_p),$$

where D_i is a polynomial with coefficients in $\mathbb{Z}$. We now recall that the t_j's are integral over A as θ is, and note that analogously the y_j's are integral over B as y is. But since $A \to B$ is finite, B is integral over A ([A-McD 5.1]), and the conclusion is that all those elements are integral over A. Whence every $\delta b_i \in K$ is integral over A and, since A is integrally closed in K, $\delta b_i \in A$. Thus we get

$$\delta y = \delta b_0 + \delta b_1 \theta + \cdots + \delta b_{p-1}\theta^{p-1} \in A + A\theta + \cdots + A\theta^{p-1}.$$

■

Corollary 3.3 *The ring $\overline{B}$ is a finitely generated B-module.*

Proof. In fact, $\delta\overline{B}$ is a sub A-module of $M = A + A\theta + \cdots + A\theta^{p-1}$, which is finitely generated over A. Since A is noetherian, M is noetherian, too, and $\delta\overline{B}$ is finitely generated over A. Finally, if $\delta g_1, \dots, \delta g_m \in \delta\overline{B}$ generate $\delta\overline{B}$ over A, then $g_1, \dots, g_m$ generate $\overline{B}$ over A, and, consequently, over B. ■

Now we come back to the situation of Noether's Projection Lemma:

Proposition 3.4 *(Rückert's Parametrization) Let $\mathfrak{p}$ a prime ideal of $\mathcal{O}_n$ of height r, and put $d = n - r$. After a linear change of coordinates the following conditions hold:*

a) The canonical homomorphism $\mathcal{O}_d = A \to B = \mathcal{O}_n/\mathfrak{p}$ is injective and finite.

b) The class $\theta_{d+1} = \mathtt{x}_{d+1} \mod \mathfrak{p}$ is a primitive element of the quotient field L of B over the quotient field K of A.

c) The irreducible polynomial over K of $\theta_j = \mathtt{x}_j \mod \mathfrak{p} \in \mathfrak{m}_B$ is a distinguished polynomial $P_j \in \mathcal{O}_d[\mathtt{x}_j]$ of degree p_j $(d < j \leq n)$.

d) For every $j = d+2, \dots, n$, there is a polynomial $Q_j \in \mathcal{O}_d[\mathtt{x}_{d+1}]$ of degree $< p_{d+1}$ such that

$$\delta \mathtt{x}_j - Q_j \in \mathfrak{p},$$

where $\delta \in \mathcal{O}_d \setminus (0)$ is the discriminant of P_{d+1}.

e) For every integer $q \geq \max\{p_{d+2}, \dots, p_n, \sum_{j=d+2}^{n}(p_j - 1)\}$ we have

$$\delta^q \mathfrak{p} \subset I = \{P_{d+1}, \delta\mathtt{x}_{d+2} - Q_{d+2}, \dots, \delta\mathtt{x}_n - Q_n\}\mathcal{O}_n \subset \mathfrak{p}.$$

As usual, there is an analogous statement in the formal case.

III Normalization

Summary. This chapter is devoted to the study of integral closures in the categories of analytic and formal rings. The main goal is the description of normalizations. We then give two applications: uniformization of 1-dimensional rings, introducting quadratic transforms, and Newton-Puiseux's Theorem.

1 Integral Closures

We will denote by $\mathbb{K}' \supset \mathbb{K}$ either of the extensions: $\mathbb{C} = \mathbb{C}$, $\mathbb{C} \supset \mathbb{R}$, $\mathbb{R} = \mathbb{R}$. First of all we prove:

Lemma 1.1 *Let A be an analytic (resp. a formal) ring over $\mathbb{K}$ and $B \supset A$ an integral domain, which is a finite A-module. Then B is an analytic (resp. a formal) ring over $\mathbb{K}' \supset \mathbb{K}$.*

Proof. By Noether's Projection Lemma (Proposition II.2.6) we may assume $A = \mathbb{K}\{\mathtt{x}\}$, $\mathtt{x} = (\mathtt{x}_1, \dots, \mathtt{x}_d)$; since B is a finite A-module, it is an integral extension of A. Let now $\mathfrak{n}$ be any maximal ideal of B. By the properties of integral extensions ([A-McD 5.8]) the ideal $\mathfrak{n} \cap A$ is the maximal ideal $\mathfrak{m}_A$ of A.

Consider an element $z \in B$. There is a monic polynomial $H \in A[\mathtt{z}]$ such that $H(z) = 0$. By the Preparation and Division Theorems we have $H = H' \cdot H''$, where H' is a distinguished polynomial and H'' a polynomial which is a unit in the ring of power series. This means

$$H'(\mathtt{x}, \mathtt{z}) = a_p + \cdots + a_1 \mathtt{z}^{p-1} + \mathtt{z}^p \in A[\mathtt{z}], \quad a_1, \dots, a_p \in \mathfrak{m}_A$$

and

$$H''(\mathtt{x}, \mathtt{z}) = b_q + \cdots + b_1 \mathtt{z}^{q-1} + \mathtt{z}^q \in A[\mathtt{z}], \quad b_q \notin \mathfrak{m}_A.$$

Since B is a domain, there are two possibilities:

- $H'(z) = 0$. Then $z^p = -(a_p + \cdots + a_1 z^{p-1}) \in \mathfrak{m}_A B \subset \mathfrak{n}$.
- $H''(z) = 0$. Then $z(b_2 + \cdots + z^{q-1}) = -b_q$ is a unit in A, which implies that z is a unit in B.

This shows that B is a local ring, and $\mathfrak{n}$ is its maximal ideal. On the other hand the residue field $\mathbb{K}' = B/\mathfrak{n}$ is a finite extension of $A/\mathfrak{m} = \mathbb{K}$, and $\mathbb{K}' \supset \mathbb{K}$ is either $\mathbb{C} = \mathbb{C}$ or $\mathbb{C} \supset \mathbb{R}$ or $\mathbb{R} = \mathbb{R}$. We now
Claim. B contains $\mathbb{K}'\{\mathtt{x}\}$ and is a finite $\mathbb{K}'\{\mathtt{x}\}$-module.

2-real, which is false. Consider now a series $h(\mathtt{t}^{1/p}) \in \mathbb{C}\{\mathtt{t}^*\}$ with $h \in \mathbb{C}\{\mathtt{x}\}$. We write $h = f + \sqrt{-1}g$ with $f, g \in \mathbb{R}\{\mathtt{x}\}$ and get $h(t^{1/p}) = f(\mathtt{t}^{1/p}) + \sqrt{-1}g(\mathtt{t}^{1/p})$. ■

Proposition 4.4 *(Newton-Puiseux's Theorem) We have:*

a) The field $\mathbb{C}(\{\mathtt{t}^*\})$ (resp. $\mathbb{C}((\mathtt{t}^*))$) is algebraically closed.

b) The field $\mathbb{R}(\{\mathtt{t}^*\})$ (resp. $\mathbb{R}((\mathtt{t}^*))$) is real closed.

Proof. By Lemma 4.4 $\sqrt{-1} \notin \mathbb{R}(\{\mathtt{t}^*\})$, and $\mathbb{C}(\{\mathtt{t}^*\}) = \mathbb{R}(\{\mathtt{t}^*\})[\sqrt{-1}]$. These two conditions imply that the two assertions in the statement are equivalent (general theory of formally real fields [L XI.2]) Hence we will only prove *a)*.

We have to see that every polynomial $P \in \mathbb{C}(\{\mathtt{t}^*\})[\mathtt{y}]$ of degree $p \geq 1$ has some root. Multiplying by a common denominator of the coefficients we may assume that $P \in \mathbb{C}\{\mathtt{t}^*\}[\mathtt{y}]$. Then, by considering the polynomial

$$a^{p-1}P(\mathtt{y}/a),$$

where a is the coefficient of the monomial of maximal degree p, we can suppose that P is monic. We have hence

$$P = \mathtt{y}^p + a_1(\mathtt{t}^{1/q})\mathtt{y}^{p-1} + \cdots + a_p(\mathtt{t}^{1/q}),$$

with $a_1, \ldots, a_p \in \mathbb{C}\{\mathtt{x}\}$. We set now

$$P^* = \mathtt{y}^p + a_1(\mathtt{x})\mathtt{y}^{p-1} + \cdots + a_p(\mathtt{x}) \in \mathbb{C}\{\mathtt{x}, \mathtt{y}\}.$$

Let $c \in \mathbb{C}$ be a root of multiplicity, say, $q \geq 1$ of the polynomial $P^*(0, \mathtt{y}) \in \mathbb{C}[\mathtt{y}]$. After the change $\mathtt{y} = \mathtt{y} + c$ we are reduced to the case $c = 0$, and from Hensel's Lemma (Proposition I.3.4) we get a factorization $P^* = QQ'$, where Q, Q' are monic polynomials of $\mathbb{C}\{\mathtt{x}\}[\mathtt{y}]$, Q has degree q and $Q(0, \mathtt{y}) = \mathtt{y}^q$. In particular, Q is a distinguished polynomial. By Remark II.2.2 we can factorize Q into irreducible distinguished polynomials $Q_1, \ldots, Q_s$ which are irreducible as series in $\mathbb{C}\{\mathtt{x}, \mathtt{y}\}$. Pick Q_1 for instance, and put $A = \mathbb{C}\{\mathtt{x}, \mathtt{y}\}/Q_1$. This analytic ring is a domain of dimension 1, because Q_1 is irreducible, and its normalization is isomorphic to $\mathbb{C}\{\mathtt{z}\}$ (Proposition 3.2). Thus we get an injective analytic homomorphism $A \to \mathbb{C}\{\mathtt{z}\}$, and from this another

$$\varphi : \mathbb{C}\{\mathtt{x}, \mathtt{y}\} \to \mathbb{C}\{\mathtt{z}\}$$

whose kernel is $Q_1\mathbb{C}\{\mathtt{x}, \mathtt{y}\}$. Then $\varphi(\mathtt{x}) \neq 0$, since otherwise $\mathtt{x}$ would belong to $Q_1\mathbb{C}\{\mathtt{x}, \mathtt{y}\}$ and Q_1 would divide $\mathtt{x}$. Thus $\varphi(\mathtt{x}) \in \mathbb{C}\{\mathtt{z}\}$ is a series of order $m > 0$, and by II.4.7 we can compose φ with an automorphism of $\mathbb{C}\{\mathtt{z}\}$ to get

$$\varphi(\mathtt{x}) = \mathtt{z}^m.$$

IV Nullstellensätze

Summary. This chapter is devoted to the real and complex Nullstellensätze. In the complex case the Nullstellensatz is a direct consequence of Rückert's Parametrization. In the real case two other results are essential, the Homomorphism Theorem and the solution to Hilbert's 17th Problem. We consider only the analytic case, remarking that everything can be done analogously in the formal one.

1 Zero Sets and Zero Ideals

According to Definitions and Notations III.4.1, let $\mathbb{K}\{\mathtt{t}^*\}$ be the ring of convergent Puiseux series over $\mathbb{K}=\mathbb{R}$ or $\mathbb{C}$, $\mathbb{K}(\{\mathtt{t}^*\})$ its quotient field and $\mathfrak{m}^*$ its maximal ideal. We will also use the notations

$$W=\mathbb{K}\{\mathtt{t}^*\},\ U=\mathfrak{m}^*,\ F=\mathbb{K}(\{\mathtt{t}^*\}).$$

(1.1) Fix an integer $n\geq 0$ and put $D=U\times\cdots\times U\subset F^n$. Let $f\in\mathcal{O}_n$. Then for every $x(\mathtt{t})=(x_1(\mathtt{t}),\dots,x_n(\mathtt{t}))\in D$ the substitution $f(x(\mathtt{t}))$ is a well defined convergent Puiseux series. Indeed, up to an isomorphism τ_p (III.4.1) this is only a substitution of ordinary convergent power series. An *associated function* can thus be defined by

$$D=U\times\cdots\times U\to W\subset F: x(\mathtt{t})\mapsto f(x(\mathtt{t})).$$

An intuitive way to see this construction is the following. The field F is an extension of $\mathbb{K}$ which contains an *infinitesimal* $\mathtt{t}$, that is, an element which belongs to every neighborhood of 0 in $\mathbb{K}$. Then D is a neigborhood of the origin in F^n which is small enough to be contained in the convergence domain $D(f)$ of any series $f\in\mathcal{O}_n$.

Definition 1.2 *Let I be an ideal of $\mathcal{O}_n$. The* zero set *of I is the set*

$$\mathcal{Z}(I)=\{x(\mathtt{t})\in D\,|\,f(x(\mathtt{t}))=0\ \text{ for all }\ f\in I\}.$$

The familiar properties of this operator are:

Proposition 1.3 *Let I,J be two ideals of $\mathcal{O}_n$. Then:*

a) $\mathcal{Z}(I)\subset\mathcal{Z}(J)$ *if* $I\supset J$.

b) $\mathcal{Z}(I\cdot J)=\mathcal{Z}(I\cap J)=\mathcal{Z}(I)\cup\mathcal{Z}(J)$.

c) $\mathcal{Z}(I+J)=\mathcal{Z}(I)\cap\mathcal{Z}(J)$.

5 Hilbert's 17th Problem

Let $W = \mathbb{R}\{\mathtt{t}^*\}$ be the ring of convergent Puiseux series, $F = \mathbb{R}(\{\mathtt{t}^*\})$ its quotient field and $U = \mathfrak{m}^*$ its maximal ideal. Fix $n \geq 1$ and consider the set $D = U \times \cdots \times U \subset F^n$ (1.1)

Definition 5.1 *Let Y be a subset of D. A power series $f \in \mathbb{R}\{\mathtt{x}\}$, $\mathtt{x} = (\mathtt{x}_1, \ldots, \mathtt{x}_n)$, is called* positive semidefinite *on Y if $f(x(\mathtt{t})) \geq 0$ for every $x(\mathtt{t}) \in Y$.*

Now Hilbert's 17th Problem makes sense for power series with real coefficients, and its solution is, as expected, that a positive semidefinite power series is always a sum of squares. But this has to be formulated in a suitable way.

Proposition 5.2 *Let I be an ideal of $\mathbb{R}\{\mathtt{x}\}$, $\mathtt{x} = (\mathtt{x}_1, \ldots, \mathtt{x}_n)$, and $f \in \mathbb{R}\{\mathtt{x}\}$. Then the following assertions are equivalent:*

a) f is positive semidefinite on $\mathcal{Z}(I)$.

b) There are $p \geq 1$ and $h_1, \ldots, h_r, g_1, \ldots, g_s \in \mathbb{R}\{\mathtt{x}\}$ such that

$$f\left(f^{2p} + h_1^2 + \cdots + h_r^2\right) = g_1^2 + \cdots + g_s^2 \mod I.$$

c) There are $h, g_1, \ldots, g_s \in \mathbb{R}\{\mathtt{x}\}$ such that

$$fh^2 = g_1^2 + \cdots + g_s^2 \mod I$$

and $\mathcal{Z}(h) \subset \mathcal{Z}(f)$.

Proof. *a)$\Rightarrow$b)* If $f(0) \neq 0$, then *a)* implies $f(0) > 0$, and f is a square in $\mathbb{R}\{\mathtt{x}\}$. Thus we can assume $f(0) = 0$. Consider a new indeterminate $\mathtt{z}$ and the canonical inclusion $\mathbb{R}\{\mathtt{x}\} \subset \mathbb{R}\{\mathtt{x}, \mathtt{z}\}$. Let $f_1, \ldots, f_m \in \mathbb{R}\{\mathtt{x}\}$ be generators of I. We claim that

$$\mathtt{z} \in \mathcal{J}\left(\mathcal{Z}\left(f + \mathtt{z}^2, f_1, \ldots, f_m\right)\right).$$

Indeed, if $(x(\mathtt{t}), z(\mathtt{t})) \in \mathcal{Z}\left(\mathtt{z}^2 + f, f_1, \ldots, f_m\right)$, then $f_i(x(\mathtt{t})) = 0$ for all i, so that $x(\mathtt{t}) \in \mathcal{Z}(I)$ and by *a)* $f(x(\mathtt{t})) \geq 0$. Moreover $z(\mathtt{t})^2 + f(x(\mathtt{t})) = 0$ and consequently $z(\mathtt{t}) = 0$. Thus our claim is proved. Now, by Risler's Nullstellensatz (Proposition 4.1), there are $q \geq 1$ and $\alpha, \beta_j, F_i \in \mathbb{R}\{\mathtt{x}, \mathtt{z}\}$ such that

$$\mathtt{z}^{2q} + \sum_i F_i(\mathtt{x}, \mathtt{z})^2 = \alpha(\mathtt{x}, \mathtt{z})(\mathtt{z}^2 + f(\mathtt{x})) + \sum_j \beta_j(\mathtt{x}, \mathtt{z}) f_j(\mathtt{x}).$$

Furthermore, multiplying by $\mathtt{z}^2$ if necessary, we may assume that q is odd: $q = 2p+1$ with $p \geq 1$.

We next note that every series $F \in \mathbb{R}\{\mathtt{x}, \mathtt{z}\}$ can be uniquely written in the form

$$F(\mathtt{x}, \mathtt{z}) = G(\mathtt{x}, \mathtt{z}^2) + \mathtt{z} H(\mathtt{x}, \mathtt{z}^2);$$

V Approximation Theory

Summary. The central result of this chapter is M. Artin's Approximation Theorem of formal solutions of analytic systems, which we obtain in Section 3. To prove it we need a generalization of the classical Implicit Functions Theorem due to Tougeron (Section 1). This generalization has its own independent interest, which we illustrate in Section 2 with several consequences concerning the equivalence of power series and polynomials. Next we deduce (Section 4) the excellent behaviour of analytic rings under completion. Finally, we introduce in Section 5 the Nash rings, which are the smallest subrings of analytic rings that share with them all the nice properties proved so far.

1 Tougeron's Implicit Functions Theorem

Set $\mathtt{x} = (\mathtt{x}_1, \dots, \mathtt{x}_n)$, $\mathtt{y} = (\mathtt{y}_1, \dots, \mathtt{y}_p)$, and

$$A = \mathbb{K}\{\mathtt{x}\} \text{ (resp. } \mathbb{K}[[\mathtt{x}]]), \quad B = \mathbb{K}\{\mathtt{x}, \mathtt{y}\} \text{ (resp. } \mathbb{K}[[\mathtt{x}, \mathtt{y}]]).$$

We fix an element $F = (F_1, \dots, F_q) \in B^q$ with $F(0,0) = 0$, and look for a solution $\mathtt{y}_1 = y_1(\mathtt{x}), \dots, \mathtt{y}_p = y_p(\mathtt{x})$ of the system

$$F(\mathtt{x}, \mathtt{y}_1, \dots, \mathtt{y}_p) = 0,$$

under assumptions milder than the ones of the classical Implicit Functions Theorem. Of course those conditions will involve the Jacobian matrix

$$\lambda = \left(\frac{\partial F_i}{\partial \mathtt{y}_j}(\mathtt{x}, 0) \right).$$

(1.1) The matrix λ defines a homomorphism of A-modules $\lambda : A^p \to A^q$. Namely, let $\{\varepsilon_1, \dots, \varepsilon_p\}$ and $\{e_1, \dots, e_q\}$ be the canonical bases of A^p and A^q. Then,

$$\lambda(\varepsilon_j) = \sum_{i=1}^{q} \frac{\partial F_i}{\partial \mathtt{y}_j}(\mathtt{x}, 0) e_i = \left(\frac{\partial F_1}{\partial \mathtt{y}_j}(\mathtt{x}, 0), \dots, \frac{\partial F_q}{\partial \mathtt{y}_j}(\mathtt{x}, 0) \right).$$

Furthermore, we consider the A-module

$$M = A^q / \mathrm{Im}(\lambda).$$

With these notations, an element $\delta \in A$ belongs to the annihilator $\mathrm{Ann}(M)$ of M if and only if there are power series $\alpha_{ij} \in A$ such that

$$\delta e_i = \lambda \left(\sum_{j=1}^{p} \alpha_{ij} \varepsilon_j \right) = \lambda(\alpha_{i1}, \dots, \alpha_{ip});$$

Proposition 2.7 *(Shiota) Every power series $f \in \mathbb{K}\{\mathtt{x}_1, \dots, \mathtt{x}_n\}$, $n \geq 2$, is equivalent to a polynomial in two indeterminates $g \in \mathbb{K}\{\mathtt{x}_1, \dots, \mathtt{x}_{n-2}\}[\mathtt{x}_{n-1}, \mathtt{x}_n]$.*

In particular, every power series in two indeterminates is equivalent to a polynomial.

Proof. We will use another different jacobian ideal. To define it, fix a factorization $f = f_1^{\alpha_1} \cdots f_p^{\alpha_p}$ into irreducible power series, and let I_f denote here the ideal generated by the $(p+1)$-minors of the matrix

$$\begin{pmatrix} \frac{\partial f_1}{\partial \mathtt{x}_1} & \cdots & \frac{\partial f_1}{\partial \mathtt{x}_p} & f_1 & \cdots & 0 \\ \vdots & & \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_p}{\partial \mathtt{x}_1} & \cdots & \frac{\partial f_p}{\partial \mathtt{x}_p} & 0 & \cdots & f_p \\ 0 & \cdots & 0 & \alpha_1 & \cdots & \alpha_p \end{pmatrix}$$

We will need later the fact, easily checked by direct computation, that linear changes of coordinates are compatible with this definition of I_f. Also, consider $f^* = f_1 \cdots f_p$. Now, we claim

$$f^*, \frac{\partial f^*}{\partial \mathtt{x}_1}, \dots, \frac{\partial f^*}{\partial \mathtt{x}_n} \in \sqrt{I_f}.$$

Indeed, by complexification (II.5.3) we can suppose $\mathbb{K} = \mathbb{C}$. Then, by Rückert's Nullstellensatz (Proposition IV.2.1) we must see that

$$f^*(x(\mathtt{t})) = \frac{\partial f^*}{\partial \mathtt{x}_1}(x(\mathtt{t})) = \cdots = \frac{\partial f^*}{\partial \mathtt{x}_n}(x(\mathtt{t})) = 0$$

for $x(\mathtt{t}) \in \mathcal{Z}(I_f)$. To that end, denote by $h_i \in I_f$ the minor corresponding to the i-th column and the last p columns. An easy computation gives:

$$\pm h_i = \sum_{j=1}^{p} \alpha_j f_1 \cdots \frac{\partial f_j}{\partial \mathtt{x}_i} \cdots f_p.$$

Hence, $\partial f / \partial \mathtt{x}_i = f_1^{\alpha_1 - 1} \cdots f_p^{\alpha_p - 1} h_i \in I_f$, and $(\partial f / \partial \mathtt{x}_i)(x(\mathtt{t})) = 0$. This holds for $1 \leq i \leq n$, and, by Corollary IV.2.5, we deduce $f(x(\mathtt{t})) = 0$, so that $f_j(x(\mathtt{t})) = 0$ for some $j = 1, \dots, p$. From this we already get $f^*(x(\mathtt{t})) = 0$. Furthermore, when we substitute $\mathtt{x} = x(\mathtt{t})$ in h_i and in the derivatives of f^* all summands containing f_j vanish, and we obtain

$$0 = h_i(x(\mathtt{t})) = \alpha_j f_1(x(\mathtt{t})) \cdots \frac{\partial f_j}{\partial \mathtt{x}_i}(x(\mathtt{t})) \cdots f_p(x(\mathtt{t})) = \alpha_j \frac{\partial f^*}{\partial \mathtt{x}_i}(x(\mathtt{t})).$$

Whence, our claim is proved.

Our second claim is that $\mathrm{ht}(I_f) \geq 2$, which is proved using the trick of Remark IV.2.6 *c)*. Suppose $\mathrm{ht}(I_f) = 1$, so that there is a height 1 prime ideal $\mathfrak{p} \supset I_f$. Since

VI Local Algebraic Rings

Summary. We study in this chapter some local properties of real and complex algebraic varieties. This study consists of the comparison of the so-called local algebraic rings and their completions. In Section 1 we define the local algebraic rings and their Nash, analytic and formal completions, and check the typical flat behaviour. In Section 2 we prove Chevalley's Theorem concerning completions of local algebraic domains. Section 3 is devoted to Zariski's Main Theorem stating that the completion of a local algebraic normal domain is normal. In Section 4 we describe the completion of the normalization of a local algebraic domain. Finally, we obtain in Section 5 Efroymson's Theorem, which deals with the implications of adding reality assumptions to Chevalley's and Zariski's statements.

1 Local Algebraic Rings

We introduce in this section several local rings attached to a point of an algebraic subset of the affine space. These local rings are the suitable tools to study and compare algebraic and analytic properties at a given point. Clearly, we can assume without loss of generality that the point is the origin, and we will do so from now on.

Let $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, and consider indeterminates $\mathtt{x} = (\mathtt{x}_1, \dots, \mathtt{x}_n)$. We denote by $\mathcal{R}_n$, $\mathbb{K}_0[\mathtt{x}]$ or $\mathbb{K}_0[\mathtt{x}_1, \dots, \mathtt{x}_n]$ the localization of the polynomial ring $\mathbb{K}[\mathtt{x}]$ at the maximal ideal generated by $\mathtt{x}_1, \dots, \mathtt{x}_n$. Clearly

$$\mathcal{R}_n = \{f/g \mid f, g \in \mathbb{K}[\mathtt{x}],\ g(0) \neq 0\}.$$

This is a local regular ring of dimension n with maximal ideal

$$\{h \in \mathcal{R}_n \,|\, h(0) \neq 0\},$$

and the canonical inclusions

$$\mathcal{R}_n \subset \mathcal{N}_n \subset \mathcal{O}_n \subset \mathcal{F}_n$$

are local homomorphisms.

Definition 1.1 *A* local algebraic ring over $\mathbb{K}$ *is a ring isomorphic to* $\mathbb{K}_0[\mathtt{x}]/I$ *with* $\mathtt{x} = (\mathtt{x}_1, \dots, \mathtt{x}_n)$*; usually we will not specify "over* $\mathbb{K}$*". If* A, B *are two local algebraic rings, a* local algebraic homomorphism $A \to B$ *is a homomorphism of* $\mathbb{K}$*-algebras. The field* $\mathbb{K}$ *is called the* coefficient field.

Proposition 5.4 *Let $<$ be an ordering of A. The following assertions are equivalent:*

a) $<$ *is a local ordering.*

b) $<$ *extends to some* $A_i^* = A^*/\mathfrak{q}_i$.

If this is the case, $\mathfrak{q}_i$ *corresponds via the bijection of Proposition 4.4 to the unique maximal ideal* $\mathfrak{n}_i$ *of the normalization* B *of* A *which is convex with respect to* $<$.

Proof. The implication *b)* $\Rightarrow$ *a)* is an immediate consequence of Proposition 5.3. For the converse implication it is enough to prove the case of the Nash completion, by Proposition V.4.9 *c)*, and to do it, we start with the particular case $A = \mathcal{R}_d$, $A^* = \mathcal{N}_d$.

We will use Serre's Criterion as stated in the proof of Proposition V.4.9 *c)*: let $f_1, \dots, f_m \in \mathcal{R}_d$ be positive in $<$ and let us see that the equation

$$f_1 \mathtt{y}_1^2 + \cdots + f_m \mathtt{y}_m^2 = 0 \tag{1}$$

has in $\mathcal{N}_d$ only the trivial solution.

Indeed, for every real number $\varepsilon > 0$ the polynomial

$$h_\varepsilon = \varepsilon - (\mathtt{x}_1^2 + \cdots + \mathtt{x}_d^2)$$

is positive in $<$ (because $h_\varepsilon(0) > 0$ and $<$ is local). Then, by *E. Artin's Specialization Theorem* ([L XI.3 Lem.2]), we have

$$\{x \in \mathbb{R}^d \mid h_\varepsilon(x) > 0, f_1(x) > 0, \dots, f_m(x) > 0\} \neq \emptyset.$$

As this holds for every $\varepsilon > 0$ we deduce that the origin is adherent to the open set

$$Z = \{x \in D \mid f_1(x) > 0, \dots, f_m(x) > 0\},$$

where $D \subset \mathbb{R}^d$ is an open polycylinder centered at the origin on which $f_1, \dots, f_m$ are well defined analytic functions (such a D exists because the f_i's are rational functions whose denominators do not vanish at the origin).

Let now $g_1, \dots, g_m \in \mathcal{N}_n \subset \mathcal{O}_n$ be a solution of the equation (1), and consider the associated functions

$${}^a g_i : D(g_i) \to \mathbb{R}, \ 1 \le i \le m$$

(see I.2). Since $f_1 g_1^2 + \cdots + f_m g_m^2 = 0$ in $\mathcal{O}_d$, this holds the same for the associated functions on some non-empty open neighborhood of the origin

$$U \subset D \cap D(g_1) \cap \cdots \cap D(g_m).$$

It follows that each ${}^a g_i$ vanishes on the open set $U \cap Z \subset D(g_i)$. But Z is adherent to the origin, so that $U \cap Z \neq \emptyset$, and from the Identity Principle (Proposition I.2.9) we deduce $g_i = 0$. This ends the proof of the particular case.

We next apply to $\mathfrak{p}$ Noether's Projection Lemma for polynomials ([A-McD §5 Ex.16]), and can assume the following conditions:

Bibliographical Note

1. The contents of I.1 and I.2 are standard. We have been inspired by [Abhyankar] and [Grauert Fritzsche]; also by [Cartan 1962].
2. The proof of the Division Theorem given in III.3 is adapted from [Gersten], with the modifications required to work in the convergent case. A similar (but not so short) proof appears in [Tougeron].
3. Mather's Finiteness Theorem (II.1) and Noether's Projection Lemma (II.2) follow [Abhyankar] and [Tougeron], with the changes needed to avoid quotations from Commutative Algebra.
4. The construction presented in II.3 of a regular system of parameters of a prime ideal of power series is close to the one given in [Abhyankar], with some differences motivated by later technical needs. Of course, we have also used [Gunning Rossi]. The converse to Rückert's parametrization comes from [El Khadiri Tougeron]
5. The treatment of Jacobians given in II.4 follows that of [Tougeron]
6. The discussion of complexification presented in II.5 should be considered folklore, although we do not know any complete reference. The best one would be [Cartan 1957].
7. The study of integral closures (III.1) and normalizations (III.2) comes essentially from [Abhyankar] and [Tougeron]. The details concerning the real case and complexifications are again folklore, without previous explicit reference that we know.
8. The treatment of uniformization in dimension 1 given in III.3 is surely well-known to specialists. We have tried a most straightforward approach, with some hidden mention to valuation theory.
9. Our proof of Newton-Puiseux's Theorem is also well-known to specialists. The standard reference is [Walker]; the real case is explicitely formulated in [Lassalle].
10. The use of parametrizations as points and the subsequent proof of Rückert's Nullstellensatz (IV.1 and IV.2) appears in [Tougeron]. It is connected to the model theoretic approach of [Robinson].
11. The Homomorphism Theorem and the Real Nullstellensatz obtained in IV.3 and IV.4 are inspired in [Lassalle]. In fact, this is a reformulation of the original ideas of [Risler], with some simplifications. In addition, the results are a little stronger than in the prior literature.
12. The same can be said of the solution to Hilbert's 17th Problem described in IV.5. Here we have also robbed some ideas from [Stengle].
13. The qualitative and quantitative remarks in IV.5 concerning sums of squares come from [Bochnak Risler] and [Motzkin]. In the first case our proof is different, and in the second we also apply some trick from [Choi Lam].
14. Sections V.1 and V.2 are based on [Tougeron]. We have discarded the techni-

calities as much as possible. The discussion of the equivalence problem (V.2) is done along the same lines as [Shiota], with some simplifications taken from [Kucharz].
15. Algebraic power series and Nash rings are introduced and studied in V.5 following [Lazzeri Tognoli]. They provide an intermediate step between algebraic and analytic geometry, which is often useful to connect both fields.
16. The proofs of Chevalley's Theorem, Zariski's Main Theorem and the comparison of completions and normalizations of VI.2, VI.3 and VI.4 profit from this idea: by means of algebraic power series and Jacobian Criteria we can simplify Zariski's classical arguments ([Zariski Samuel]).
17. The same remark applies to our proof of Efroymson's Theorem in VI.5, which is simpler than the original one in [Efroymson]. Furthermore, this approach leads to a better result and describes more accurately the situation in the real case.

Finally, we would like to comment on the names chosen for many results in the book. They fall in either of the following cases: authors of a statement close to ours, authors of the ideas that led naturally to our formulation, or authors of a framework in which our presentation fits comfortably.

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