Toda variedad de dimensión pura tiene un atlas finito.
(salvo curvas con infinitos ptos. de borde)

(1) X variedad, $\dim \equiv p$. Topología:

II Axioma

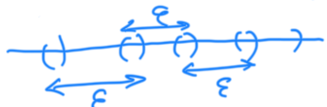
T_2

localmente $\approx \mathbb{R}^p \Rightarrow \exists$ bases de ent. $\underset{\substack{\wr\wr \\ \text{compactos de } \mathbb{R}^p}}{\text{compactos}}$ $\Big\} \Rightarrow \exists$ bases de entornos cerrados $=$ regular $\Big\} \Rightarrow$ metrizable $\Downarrow$ paracompacto

(2) Dimensión por recubrimiento, finitud de $\mathbb{R}^p$.

Def.- $\mathcal{U}=\{U_\alpha\}$ tiene orden $\le r$: $U_{\alpha_1}\cap\ldots\cap U_{\alpha_s}\neq\emptyset \Rightarrow s\le r$

(a) $\forall \varepsilon>0 \ \exists$



rec's abiertos $\begin{cases} \mathbb{R}=\bigcup_\alpha I^\varepsilon_\alpha, \ \operatorname{diam}(I^\varepsilon_\alpha)<\varepsilon, \ \operatorname{ord}\{I^\varepsilon_\alpha\}\le 2 \\ \mathbb{R}^p=\bigcup(I_{\alpha_1}\times\cdots\times I_{\alpha_p}) \ \operatorname{diam}(I^\varepsilon_{\alpha_1}\times\cdots\times I^\varepsilon_{\alpha_p})<n\varepsilon, \ \operatorname{ord}\{I_{\alpha_1}\times\cdots\times I_{\alpha_p}\}\le 2^p. \end{cases}$

(b) $K\subset\mathbb{R}^p$ compacto, $K=\bigcup_i \overset{ab}{U_i} \Rightarrow$ n° $\overset{>0}{\rho}$ de Lebesgue: $\forall A\subset K \ \operatorname{diam}(A)<\rho \Rightarrow \exists U_i\supset A$

$\Rightarrow \exists\varepsilon>0$: $\{A_\alpha=(I^\varepsilon_{\alpha_1}\times\cdots\times I^\varepsilon_{\alpha_p})\cap K\}$ refina $\{U_i\}$: $\forall\alpha \ A_\alpha\subset U_{i(\alpha)}$

$\Rightarrow V_i=\bigcup_{i=i(\alpha)} A_\alpha\subset U_i$ & $\operatorname{ord}\{V_i\}\le 2^p$.

(3) X variedad: $X=\bigcup_\alpha U_\alpha$ dominios de coord's, $D_\alpha=\overline{U_\alpha}\approx \overset{\text{compacto}}{K}\subset\mathbb{R}^p$

paracompacidad: $\{U_\alpha\}$ loc. finita, II Ax. $\{U_k\}$ subr. numerable

$\Downarrow$

$\{D_\alpha\}$ loc. finita $\Rightarrow$ $\{D_k\}$ loc. finita.

(4) $\exists \underset{\substack{\shortparallel \\ U_{0k}}}{\{U_k\}}>\{U_{1k}\}>\cdots>\underset{\ell\ge1}{\{U_{\ell k}\}}>$ rec. ab^{os} $U_{\ell-1k}\supset U_{\ell k}$ $\begin{cases} U_{\ell-1k}\setminus U_{\ell k}\subset D_\ell \\ \operatorname{ord}\{U_{\ell k}\cap D_\ell\}\le 2^p \end{cases}$ ⊛

Ind: $\{U_{\ell-1k}\}\Rightarrow\{U_{\ell k}\}$ $\quad$ 2(b), $D_\ell\approx \overset{\text{compacto}}{K}\subset\mathbb{R}^p$

(a) $D_\ell=\bigcup_k U_{\ell-1k}\cap D_\ell = \bigcup_k V_k$, $V_k \overset{ab}{\subset} D_\ell$, $V_k\subset U_{\ell-1k}$ & $\operatorname{ord}\{V_k\}\le 2^p$

(b) $U_{\ell k}=(U_{\ell-1k}\setminus D_\ell)\cup V_k\subset U_{\ell-1,k} \Rightarrow$ ⊛

(i) ab° de X ; key: $x\in V_k=A_k\cap D_\ell$ ($A_k \overset{ab}{\subset} X$) $\Rightarrow A_k\cap U_{\ell-1k}\subset(U_{\ell-1k}\setminus D_\ell)\cup V_k$ & $U_{\ell k}$ entorno de x ✓

(ii) $X=\bigcup_k U_{\ell k}$: $x\in X \Rightarrow \begin{cases} x\in D_\ell \Rightarrow \exists V_k\ni x \Rightarrow x\in U_{\ell k} \\ x\notin D_\ell \Rightarrow \exists U_{\ell-1\,k}\ni x \Rightarrow x\in U_{\ell-1\,k}\setminus D_\ell\subset U_{\ell k}\ \checkmark \end{cases}$

(5) $\{V_k=\bigcap_\ell U_{\ell k}\}$ rec. ab° de X, ord$\{V_k\}\leq 2^p$.

(a) V_k ab° : $x\in V_k$, $\{D_\ell\}$ rec. loc. finito $\exists U^x\cap D_\ell=\emptyset$ $\ell>\ell_0$ & $x\in D_{\ell_0}$

$\Rightarrow \ell>\ell_0: U^x\cap(U_{\ell-1\,k}\setminus U_{\ell k})=\emptyset \Rightarrow U^x\cap U_{\ell-1\,k}=U^x\cap U_{\ell k}$

$\Rightarrow x\in U^x\cap V_k=U^x\cap U_{\ell_0 k}\underset{ab^\circ}{\subset}X \Rightarrow V_k$ es entorno de x. ✓

(b) $X=\bigcup V_k$: $x\in X$ $\ell_0=\max\{\ell: x\in D_\ell\}$ $\exists k: x\in U_{\ell_0 k}$

$\Rightarrow \underset{\ni x}{U_{\ell_0 k}}\setminus U_{\ell_0+1\,k}\subset \underset{\not\ni x}{D_{\ell_0+1}}$, $\Rightarrow \underset{\ni x}{U_{\ell_0+1\,k}}\setminus U_{\ell_0+2\,k}\subset \underset{\not\ni x}{D_{\ell_0+2}} \Rightarrow \underset{\ni x}{U_{\ell_0+1\,k}}\ \cdots \Rightarrow \underset{\ni x}{U_{\ell k}}$ $\forall \ell\geq\ell_0$

$\Rightarrow x\in\bigcap_{\ell\geq\ell_0}U_{\ell k}=\bigcap_\ell U_{\ell k}=V_k$ (los $U_{\ell k}$ son una sucesión decreciente)

(c) ord$\{V_k\}\leq 2^p$

$x\in V_{k_1}\cap\ldots\cap V_{k_r}$, $\exists D_\ell\ni x \Rightarrow x\in D_\ell\cap \underset{\subset U_{\ell k_1}}{V_{k_1}}\cap\ldots\cap \underset{\subset U_{\ell k_r}}{V_{k_r}}$ $\left.\begin{matrix} \\ \text{ord}\{U_{\ell k}\cap D_\ell\}\leq 2^p\end{matrix}\right\}\Rightarrow r\leq 2^p$ ✓

(6) Truco de Milnor (según Palais). $\theta_k=\text{dist}(\cdot, X\setminus V_k)$ (X metrizable) $\{\theta_k>0\}=V_k$

$\forall r\geq 1$ $A=\underset{\neq's}{\{k_1,\ldots,k_r\}}$: $W_{rA}=\left\{\begin{matrix}\theta_{k_1}>0,\ldots,\theta_{k_r}>0\\ \theta_{k_1}>\theta_k,\ldots,\theta_{k_r}>\theta_k : k\notin A\end{matrix}\right\}\subset V_{k_1}\cap\ldots\cap V_{k_r}$

(a) $W_{rA}\underset{ab}{\subset}X$: $x\in W_{rA}$

$\left.\begin{matrix}\{V_k\}<\{U_k\}\\ \text{loc. finito}\end{matrix}\right\}\Rightarrow \exists U^x\cap V_k=\emptyset$ $k\neq n_1,\ldots,n_s \Rightarrow \theta_k|_{U^x}\equiv 0$ $k\neq n_1,\ldots,n_s$

$\Rightarrow W_{rA}\cap U^x=\{\theta_{k_1}>0,\ldots,\theta_{k_r}>0;\ \theta_{k_j}>\theta_k,\ k=n_i\notin A\}\underset{ab}{\subset}X$

finito desigualdades

(b) $W_{rA}\cap W_{rA'}=\emptyset$: $\left.\begin{matrix}\exists k\in A\setminus A' \Rightarrow \theta_k>\theta_{k'} \text{ en } W_{rA}\\ \exists k'\in A'\setminus A \Rightarrow \theta_{k'}>\theta_k \text{ en } W_{rA'}\end{matrix}\right\}$ incompatibles.

(c) $\bigcup_{r,A}W_{r,A}=X$: $x\in X \Rightarrow$ ordenamos los $\theta_k(x)\neq 0$: $\underbrace{\theta_{k_1}(x)=\cdots=\theta_{k_r}(x)}_{A}>\theta_{k_{r+1}}(x)\geq\cdots\geq\theta_{k_s}(x)$

$\Rightarrow x\in W_{rA}$ ✓

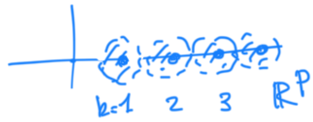
(d) $W_{rA}\subset V_{k_1}\cap\ldots\cap V_{k_r}$ & ord$\{V_k\}\leq 2^p \Rightarrow W_{r,A}=\emptyset$ $\forall r>2^p$

$\Rightarrow X=\bigcup_{r=1}^{2^p}W_r$, $W_r=\bigcup_A W_{rA}$

<u>CONCLUSIÓN</u>: W_r es dominio de coordenadas & $\{W_1,\ldots,W_{2^p}\}$ un atlas finito de X.

En efecto $W_{rA} \subset V_k \subset U_k$ dominio de coord.
$\exists k \in A$

$$\begin{tikzcd} U_k \arrow[d, "\varphi_k\ \text{difeo}"'] \arrow[dr, bend left, "h_k"] & \\ B_k \underset{ab}{\subset} \mathbb{R}^p \arrow[r, "\text{difeo}", "\psi_k"'] & \{\|x-k\|<1/2\} \overset{ab}{\subset} \mathbb{R}^p \end{tikzcd}$$

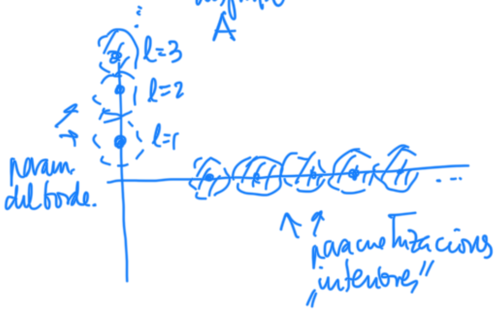


$\Rightarrow h|_{W_{rA}} = h_k|_{W_{rA}}$ define un homeo $h: \underset{\substack{\text{disjunta} \\ A}}{\bigcup} W_{rA} \to \mathbb{R}^p$. ∎

Caso $\partial \neq \emptyset$: $B_k \subset \mathbb{H}^p \xrightarrow[\psi_k]{\text{difeo}} \{\|x-\ell\|<1/2, x_1 \geq 0\}$

$\underline{p \geq 2}$

$\Downarrow$



Una curva con borde infinito no tiene atlas finito: cada pto del borde requiere una carta.

Una curva con borde finito sí lo tiene: se toma uno finito del interior y se le añade una carta para cada pto del borde.