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Abstract

A set which can be defined by systems of polynomial inequalities is called *semialgebraic*. When such a description is possible locally around every point, by means of analytic inequalities varying with the point, the set is called *semianalytic*. If one single system of strict inequalitites is enough, either globally or locally at every point, the set is called *basic*. The topic of this work is the relationship between these two notions. Namely, we describe and characterize, both algebraically and geometrically, the obstructions for a basic semianalytic set to be basic semialgebraic. Then, we describe a special family of obstructions that suffices to recognize whether or not a basic semianalytic set is basic semialgebraic. Finally, we use the preceding results to discuss the effect on basicness of birational transformations.

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Key words and phrases: Semialgebraic and semianalytic sets, basic and generically basic sets; algebraic and analytic fans, trivialization and centers of an algebraic fan; henselization, amalgamation, going-down for fans; fan approximation lemma, birational blowing-down.

Introduction

Let X be a real algebraic variety. A subset $S \subset X$ is called semialgebraic if it can be written as

$$S = \bigcup_{i=1}^{r} \{x \in X \mid f_{i1}(x) > 0, \dots, f_{is_i}(x) > 0, g_i(x) = 0\}$$

for some polynomial functions f_{ij}, g_i on X. The "pieces"

$$S_i = \{x \in X \mid f_{i1}(x) > 0, \dots, f_{is_i}(x) > 0, g_i(x) = 0\}$$

are called *basic* semialgebraic sets, and are seen as the most elementary type of them. In fact, many properties of semialgebraic sets are first shown for basic sets, and then extended to the general case. Now, notice that the basic piece S_i can also be seen as a basic open semialgebraic subset of the algebraic set $X \cap \{g_i = 0\}$. This makes basic open semialgebraic sets the natural object of study. Quite a lot of literature has been devoted to them, specially since Bröcker published his celebrated paper [Br2] containing a characterization of basic semialgebraic sets as well as some estimates on the number of polynomials needed to describe them. He brought into evidence a beautiful interplay between semialgebraic sets, real spectra, spaces of orderings and reduced quadratic forms. These results have also been extended to the analytic category, both in the global setting, [AnBrRz1], for the so-called global semianalytic sets, as well as in the local one, that is, for semianalytic germs, [Rz2], [AnBrRz2]. In both cases, the definitions of basic open sets are obvious and similar to the given above. Also, the spirit of the proofs is identical to the one of Bröcker in the algebraic case, showing the depth of his ideas and the relationship just mentioned.

In these notes we apply once more the same general philosophy, but in a different context: instead of working either in the algebraic or analytic side, we want to understand the behaviour of basicness of semialgebraic sets when we go back and forth into the analytic category, considering them as semianalytic germs at all different points. The results obtained here have

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been announced in [AnRz5] under a purely geometric form which, hidding all technical algebraic aspects, tries to convey intuitive support for the main ideas involved.

To be more precise, let $S \subset X$ be a basic open semialgebraic set,

$$S = \{x \in X \mid f_1(x) > 0, \dots, f_s(x) > 0\}$$

where $f_1, \dots, f_s$ are polynomial functions on X. Obviously, for all $a \in X$, the germ S_a is basic open in the semianalytic category. This work originated from the question of when the converse is true. That is, suppose that $S \subset X$ is a semialgebraic set such that for all $a \in X$ the germ S_a is basic open seminanalytic,

Is then S basic semialgebraic?

The answer, in that generality, is in the negative. In fact, being basic semialgebraic is a "global property", and the obstructions for an open semialgebraic set S to be basic can very well be of a "non-local" nature, in the sense that they are not concentrated at any point, see Example 1.1 *b)*. There is no hope of recognizing these obstructions from our hypothesis that the S_a's are basic open semianalytic. Hence, the problem we are interested in can be stated in a much more accurate way as:

Assume that S_a is basic open semianalytic for all $a \in X$. Do there exist "local" obstructions for S to be basic open semialgebraic?

We have been consciously ambiguous about the meaning of "local" and "non-local" obstructions. The precise definition requires the use of *fans* and *valuations*. In fact, first of all, the question of whether S is basic open semialgebraic can be translated, by means of the *tilde* operator (Sections 3,4), into the question of whether the corresponding constructible set $\tilde{S}$ of the *real spectrum* $\operatorname{Spec}_r(\mathcal{P}(X))$, is basic open, where $\mathcal{P}(X)$ denotes the ring of polynomial functions on X. Now, Bröcker's result, [Br3], characterizes basic open constructible subsets C of the real spectrum $\operatorname{Spec}_r(A)$ of any commutative ring A, in terms of distinguished subsets of $\operatorname{Spec}_r(A)$ with a certain combinatorial structure, called *fans*. In particular, the fans of $\operatorname{Spec}_r(\mathcal{P}(X))$ can be seen as the obstructions mentioned above for a constructible set to be basic. A well known deep result asserts that the elements of a fan can specialize to at most two points of X. Thus, a *local fan* should mean that all its elements specialize to the same point $a \in X$, while a *non-local fan* is one whose elements specialize to two different points (may be at infinite). Since the expressions "local" and "non-local" may induce to confussions with other notions, we will not use them anymore, talking instead of *fans with 1 specialization point* and *fans with 2 specialization points*, or for short *1pt-fans* and *2pt-fans*. This gives a precise meaning to the above sentences.

Similarly, the question of whether the germ S_a is basic semianalytic, is translated into the question of whether the *constructible* set $\tilde{S}_a$ of the *real spectrum* $\operatorname{Spec}_r(\mathcal{O}_{X,a})$ is basic, where $\mathcal{O}_{X,a}$ denotes the ring of germs of analytic functions on X_a. Here we can apply, again, Bröcker's fan criterium, this time dealing with fans of $\operatorname{Spec}_r(\mathcal{O}_{X,a})$. Thus we have the following diagram of related topics:

Semialgebraic subsets of X	$\longleftrightarrow$	Constructible subsets of $\operatorname{Spec}_r(\mathcal{P}(X))$	$\longleftrightarrow$	Fans of $\operatorname{Spec}_r(\mathcal{P}(X))$
Semianalytic germs at a	$\longleftrightarrow$	Constructible subsets of $\operatorname{Spec}_r(\mathcal{O}_{X,a})$	$\longleftrightarrow$	Fans of $\operatorname{Spec}_r(\mathcal{O}_{X,a})$

Hence, comparing basicness between the analytic and the algebraic categories becomes comparing the fans of $\operatorname{Spec}_r(\mathcal{O}_{X,a})$ and $\operatorname{Spec}_r(\mathcal{P}(X))$. Thus, the question above can be more properly stated as:

Which are the fans of $\operatorname{Spec}_r(\mathcal{P}(X))$ *that are restriction of fans of* $\operatorname{Spec}_r(\mathcal{O}_{X,a})$*?*

These fans are called *analytic*. As pointed out above, an easy necessary condition for a fan $F \subset \operatorname{Spec}_r(\mathcal{P}(X))$ to be analytic is that F must be a 1pt-fan, so that we can state the final form of our problem as:

Which are the 1pt-fans of $\operatorname{Spec}_r(\mathcal{P}(X))$ *that are analytic?*

In this work we give a complete answer to this question, both in algebraic and in geometric terms. However, to reach that point, quite a bit of background and technicalities are needed. This is the reason why these notes have become so thick that we have changed our original idea and have developed them as a monograph. Indeed, first of all, to understand the mentioned "fan criterion", we need to introduce some background on real spectra, the tilde operator, and the relationship between fans and basic semialgebraic sets. All this is done as shortly as possible in Sections 1-5.

Also, being faced to work with fans, we need the most useful tool to handle them: valuation theory. In fact, it is the connection between valuations and fans what allows us to construct and manipulate the latter. The main points about this interplay are collected in Sections 6 and 7. Specially important is another precious result by Bröcker: the so-called *trivialization theorem for fans*, [Br1], which guarantees not only the existence of valuation rings compatible with a given fan, F, but also that some of them, W, trivializes F, that is, the fan F_W induced in the residue field k_W of W has at most two elements. In Section 7 we pay special attention to the notion of centers of a fan: they are the centers at X of the valuation rings compatible with it. Among them, the *trivialization centers*, that is, the centers of the valuation rings trivializing F, will be of utmost importance for the geometric characterization of analytic fans in Section 14.

Going further, remember that our problem is to study whether a given 1pt-fan F of $\mathcal{P}(X)$ extends to the analytic ring $\mathcal{O}_{X,a}$. Thus, as it is becoming customary, we introduce an intermediate step in between, namely the henselization $\mathcal{P}(X)^h_a$ of the localization of $\mathcal{P}(X)$ at a, giving rise to the notion of the *henselization of a local fan*. This is explained in Section 8. So, we study first the extension of F to $\mathcal{P}(X)^h_a$, and second from this ring to $\mathcal{O}_{X,a}$. Quite surprisingly the difficulties arise at the first level, while the extension from the henselian to the analytic ring goes smoothly, as we prove in Section 9. This is a consequence of the powerful M. Artin's approximation theorem.

Thus, continuing with our process, we have to determine which 1pt-fans of $\mathcal{P}(X)$ extend to the henselization. This is done in Section 12, where we obtain the algebraic characterization of analytic fans: a fan F is analytic if and only if the *amalgamation property* holds for some valuation ring compatible with F, or, to put it in short, some valuation ring compatible with F *amalgamates*. This roughly means that there is a suitable extension of the residue field k_V of V to which the induced fan F_V extends. Unfortunately, it turns out that this property, whose definition and basic facts are introduced in Section 11, is quite hard to check and to deal with, except when the fan F_V has at most two elements. However, a detailed analysis of the extension of valuation rings to the henselization, shows that if F is analytic, then amalgamation always holds for the biggest valuation ring W_F trivializing F. The proof of this result appears in Section 12.

This opens the door to the geometric interpretation of amalgamation. Indeed, curiously enough, under these circumstances, amalgamation can be checked by looking at directed families of real algebraic sets Y, such that the centers of W_F at them are finite coverings of the center of W_F in X. This leads to the definition of finite coverings associated to a fan F (Section 13), and later, to the announced geometric characterization of analytic fans (Section 14), which roughly says that a fan is non-analytic if it becomes a 2pt-fan in some finite covering.

Let us explain the last sentence a bit more. Although basicness of semialgebraic sets S is of birational nature, analyticity of fans is not. For instance it may very well happen that a 1pt-fan F becomes a 2pt-fan after blowing-up X, and conversely, 2pt-fans can be made 1pt-fans by blowing-down. This can be used to translate, at convenience, obstructions to basicness from non-local settings to local ones and conversely. In Section 17, we develope an instance of this, showing that all the obstructions for S to be basic can be concentrated at one point a in a suitable model S', so that S is basic semialgebraic if and only if S'_a is basic semianalytic.

Finally, in Sections 15 & 16 we approach the problem from a different point of view: we introduce approximation of fans. Roughly speaking, if a given fan F produces an obstruction for a given semialgebraic set S to be basic semialgebraic, then any F' close enough to F will produce an obstruction too. In particular, if any fan could be approximated by a 2pt-fan, this would imply that analytic fans are redundant to decide whether S is basic semialgebraic or not. Thus, a natural question is to decide which 1pt-fans can be approximated by 2pt-fans. First, we find a dense distinguished family of fans: the so-called *parametric fans*. This is a consequence of the approximation lemma proven in Section 15: any fan can be arbitrarily approximated by a parametric fan. Therefore, parametric fans suffice to check basicness of semialgebraic sets. Without entering in the formal definition, we just say that they are fans obtained by pulling back trivial fans by some special discrete valuation rings.

Then, in Section 16, we prove that approximation by parametric fans behaves also well with respect to analyticity, in the sense that analytic (resp. non-analytic) fans, can be arbitrarily approximated by analytic (resp non-analytic) parametric fans. Also, we see that any fan F can be approximated by a 2pt-fan, except in case all the trivialization centers of F are reduced to one point. Moreover, if F is parametric fan verifying this condition, then any close enough approximation F' also verifies it, and, in particular, it is a 1pt-fan. This indicates, somehow, that these *strictly 1pt-fans* contain the essential analytic information.

In order to put this idea into the geometric terms of the beginning of this introduction, consider the Alexandroff compactification $X^* = X \cup \infty$ of X. Let $S \subset X$ be an open semialgebraic set which is basic semianalytic at all points $a \in X$. Then S is basic semialgebraic if and only if it is basic semianalytic at ∞ and verifies Bröcker's fan criterion for 2pt-fans.

This work was started in 1990, at the Berkeley campus of the University of California, during the Special Year on Real Algebraic Geometry and Quadratic Forms, organized by Professors T.Y. Lam and R. Robson. We enjoyed their hospitality and are glad to thank them the oportunity to participate in the seminars that were held there. We also thank RAGSQUADers who attended to the very first presentation of these ideas which only more than three years later have been brought down into their final form. We also acknowledge to the institutions that made this work possible: the Universidad Complutense of Madrid that supported our stay in Berkeley by a grant del Amo, the N.S.F. and the D.G.I.C.Y.T., inside whose research project PB 89-0379-C02-02 this paper has been developed. Last, we are grateful to our colleagues from Madrid, who, first, took over their shoulders our teaching duties, allowing thereby our leave, and, second, listened patiently our explanations and enriched them with their comments.

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