

# An everyday strategy analysed via Control Theory

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**Abstract:** We give a simple mathematical proof of the popular strategy "don't put off for tomorrow what you can do today" by using the HUM method due to Jacques-Louis Lions for the controllability of linear systems.

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*Key words:* HUM method, popular strategy, constructive task.

## 1 Introduction

Many everyday strategies in many cultures are almost as traditional as language and one can find a natural establishment almost in every language spoken nowadays. The simple purpose of this short note is to analyze one of them by using the methods of Mathematical Control Theory. The concrete question we shall consider is as follows: when trying to make a certain task depending on time, shall we execute it immediately or execute it later avoiding any cost? Let us recall here what says the clever languish by Cervantes:

“no dejes para mañana lo que puedas hacer hoy”.

An English equivalent version could be as follows

“don't put off for tomorrow what you can do today”.

In order to formulate this strategy in a mathematical manner we idealize the task by means of the goal

$$x(T) = y_d, \tag{1.1}$$

where  $y_d \in \mathbb{R}$  represents the value of the state,  $x(t)$  (assumed well defined on an interval  $[t_0, T)$ ) which we want to attain. We represent our possible actions by means of the scalar control  $u(t)$ . What is peculiar to the above popular strategy is the comparison of the "energies" we must develop (i. e. the "energy required by our action") according the moment in which we execute such an action. Thus, we shall consider the cases of a family of control operators of the form  $B(t)u(t)$  with

$$B(t) = \chi_{[a,b]}(t) \tag{1.2}$$

(the characteristic function of the interval  $[a, b]$  in which we implement our control), where the interval  $[a, b]$ , contained in a larger interval  $[t_0, T]$  (with  $0 \leq t_0 < T$ ), is executed in different moments. More precisely, we shall analyze the optimality of the required controls  $u(\cdot)$  and  $u^*(\cdot)$  associated to two possible intervals,  $[a, b]$  and  $[a^*, b^*]$ , of the same executing length times (i.e.  $b-a = b^*-a^*$ ) but different starting executing times ( $a < a^*$ ). Since, as it is well-known, there is no uniqueness of the controls leading the state from a given initial datum to a final desired state, (1.1), we shall formulate our result in a well determined subclass of controls such as the one given through the application of the "HUM method" due to J.L. Lions [2] (see details in Section 2) and that we shall denote, in short, as the class of HUM-controls.

We shall assume also that our task has a "constructive nature". The simplest way to formulate it is by assuming that the state  $x(t)$  solves the Cauchy Problem

$$(CP) \begin{cases} x'(t) = Ax(t) + B(t)u(t) & t \in (t_0, T), \\ x(t_0) = 0, \end{cases}$$

for some positive constant  $A > 0$ . In particular, we know (see [1]) that for any choice of  $B(t) = \chi_{[a,b]}(t)$ , such HUM-control  $u$  does exist (i.e. problem (CP) and (1.1) is *completely controllable*), that  $u \in L^\infty(t_0, T)$  and that  $u$  minimizes the "Hilbert energy cost", in the sense that if  $v$  is any other control leading also the state from the same given initial datum to the same final desired state then

$$\|u\|_{L^2(t_0, T)} \leq \|v\|_{L^2(t_0, T)}.$$

Our main result, which gives a simple mathematical justification to this strategy, is the following:

**Theorem.** *Given  $t_0 < T$ ,  $y_d \in \mathbb{R}$ ,  $A > 0$  and  $B$ , defined by (1.2), if  $u(\cdot)$  and  $u^*(\cdot)$  are the HUM-controls associated to two controlling intervals,*

$[a, b]$  and  $[a^*, b^*]$  with the same executing length time (i.e.  $b - a = b^* - a^*$ ) but different starting executing time ( $a \neq a^*$ ), then  $a < a^*$  implies that

$$\|u\|_{L^\infty(t_0, T)} < \|u^*\|_{L^\infty(t_0, T)} \quad (1.3)$$

and

$$\|u\|_{L^2(t_0, T)} < \|u^*\|_{L^2(t_0, T)}. \quad (1.4)$$

## 2 Proof of the Theorem

As mentioned before the result can be proved for several classes of well-determined subclasses of controls but here we shall follow the adaptation of the so called "HUM method" of J.L. Lions in the spirit of the monograph Coron [1]. We consider the "adjoint retrograde problem" defined by

$$\begin{cases} \phi'(t) = -A\phi(t) & t \in (t_0, T), \\ \phi(T) = \phi_T. \end{cases}$$

(Obviously, in this so simple formulation, the transposition of the matrix  $A$  is trivially  $A^T = A \in \mathbb{R}$ ). We now solve the adjoint problem associated to a generic final datum  $\phi_T \in \mathbb{R}$  getting that  $\phi(t) = \phi_T e^{A(T-t)}$ . The main idea of the HUM method is to use the duality existing between the adjoint and the original problems. In our case it is described by means of the application  $\Lambda : \mathbb{R} \rightarrow \mathbb{R}$  given by  $\Lambda(\phi_T) = y_d$ . Moreover we know ([1]) that the HUM-control is defined by

$$u(t) = B(t)^\dagger \phi(t).$$

For the sake of completeness, we shall check directly and prove that, in our case,  $u$  has the concrete expression

$$u(t) = \chi_{[a, b]}(t) \phi_T e^{A(T-t)}.$$

Without loss of generality we can assume  $t_0 = 0$ . To check the complete controllability we must verify that if we denote by  $\mathcal{A} = \Lambda(\mathbb{R})$  to the "reachability set" then we have  $\mathcal{A} = \mathbb{R}$ . But since  $y_h(t) = C e^{At}$  is the general solution of the homogenous linear equation, by using the "variation of parameters method" we can find a particular solution  $y_p(t)$

$$y_p(t) = c(t) e^{At}$$

with

$$c(t) = \int_0^t \chi_{[a,b]}(s) \phi_T e^{A(T-2s)} ds,$$

i.e.

$$c(t) = \begin{cases} 0 & \text{if } t \leq a, \\ \int_a^t \phi_T e^{A(T-2s)} ds & \text{if } a \leq t \leq b, \\ \int_a^b \phi_T e^{A(T-2s)} ds & \text{if } t \geq b. \end{cases} \quad (2.5)$$

By imposing the initial condition  $y(0) = 0$  we get that

$$y(T) = \phi_T \left( -\frac{1}{2A} \right) [e^{A(T-2b)} - e^{A(T-2a)}] e^{AT} = \frac{\phi_T}{2A} e^{2AT} (e^{-2Aa} - e^{-2Ab}) = y_d,$$

which is true if and only if

$$y_d = \mu \phi_T,$$

with

$$\mu = e^{2AT} (e^{-2Ab} - e^{-2Aa}).$$

Obviously  $\mu \neq 0$ . In conclusion, we get that

$$\Lambda(\phi_T) = \mu \phi_T = y_d$$

As  $\Lambda$  is linear and  $\mu \neq 0$  then  $\Lambda(\mathbb{R}) = \mathbb{R}$ . Thus we can apply the HUM Theorem of J.L Lions (see, e.g. [1]) and get the complete controllability. We now proceed to compare the  $L^\infty(t_0, T)$ -norm of the concrete expressions of the HUM-controls  $u(\cdot)$  and  $u^*(\cdot)$ . Since they are dependent on the value  $\phi_T$  and there exists a bijection between this and the value  $y_d$  we obtain:

$$-\frac{1}{2A} \phi_T e^{2AT} (e^{-2Ab} - e^{-2Aa}) = y_d, \text{ i.e. } \phi_T = \frac{-2Ae^{-2AT}}{e^{-2Ab} - e^{-2Aa}} y_d.$$

Thus

$$u(t) = \chi_{[a,b]}(t) \frac{2Ae^{-2AT}}{e^{-2Aa} - e^{-2Ab}} y_d.$$

But

$$e^{-2Ab} - e^{-2Aa} = e^{-2Aa} (e^{-2A(b-a)} - 1),$$

and so

$$u(t) = \chi_{[a,b]}(t) \frac{2Ae^{A(2a-T-t)}}{1 - e^{-2A(b-a)}} y_d.$$

Analogously

$$u^*(t) = \chi_{[a',b']}(t) \frac{2Ae^{A(2a'-T-t)}}{1 - e^{-2Al}} y_d.$$

If we introduce now

$$\alpha = \frac{2Ay_d}{1 - e^{-2Al}},$$

then the HUM-controls are

$$u(t) = \chi_{[a,b]}(t) \alpha e^{A(2a-T-t)}$$

and

$$u^*(t) = \chi_{[a',b']}(t) \alpha e^{A(2a'-T-t)},$$

and a direct computation leads to the strict inequality (1.3). In order to prove the "Hilbert energy inequality" (1.4) we point out that

$$\|u\|_{L^2(0,T)}^2 = \int_0^T |B(s)u(s)|^2 ds = \phi_T^2 \int_a^b e^{2A(T-s)} ds,$$

and

$$\|u^*\|_{L^2(0,T)}^2 = \int_0^T |B(s)u^*(s)|^2 ds = \phi_T^2 \int_{a^*}^{b^*} e^{2A(T-s)} ds.$$

Then, for every  $0 \leq \alpha, \beta \leq T$

$$\int_\alpha^\beta e^{2A(T-s)} ds = -\left(\frac{e^{2A(T-s)}}{2A}\right)_{s=\alpha}^{s=\beta} = \frac{1}{2A} e^{2AT} (e^{-2A\alpha} - e^{-A\beta}) > 0.$$

But we can write

$$\|u\|_{L^2(0,T)}^2 = \frac{\phi_T^2}{2A} e^{2AT} (e^{-2Aa} - e^{-A(a+l)}) = \frac{\phi_T^2}{2A} e^{2AT-2Aa} (1 - e^{-Al}) = C e^{-2Aa},$$

and

$$\|u^*\|_{L^2(0,T)}^2 = \frac{\phi_T^2}{2A} e^{2AT} (e^{-2Aa^*} - e^{-A(a^*+l)}) = \frac{\phi_T^2}{2A} e^{2AT-2Aa^*} (1 - e^{-Al}) = C e^{-2Aa^*},$$

with  $C = \frac{1}{2A} e^{2AT} (1 - e^{-Al}) > 0$  and so, again,  $a < a^*$  implies the strict inequality (1.4). ■

**Remark.** Many generalizations and variants are possible (to be published elsewhere).

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## References

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