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On an oblique boundary value problem related to the Backus problem in Geodesy

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Abstract

We show the existence and uniqueness of a viscosity solution for an oblique nonlinear problem suggested by the study of the Backus problem on the determination of the external gravitational potential of the Earth from surface measurements of the modulus of the gravity force field.

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1. Introduction

We show the existence and uniqueness of a solution of the oblique boundary problem

$$\begin{cases} \Delta v = 0 & \text{in } B(0,1), \\ v + \dfrac{\partial v}{\partial \mathbf{n}} = \sqrt{(g^2 - |\nabla_s v|^2)_+} & \text{on } \partial B(0,1), \end{cases} \tag{1.1}$$

where $B(0,1)$ denotes the unit ball of $\mathbb{R}^3$.

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The study of this problem was suggested in a previous paper by the authors [8] in order to deal, after some transformations, with the Backus problem coming from Geodesy (see, for example, [2,6,11,12,18,35]) and Geomagnetism (see, [2–4,21]). Assuming the surface of the Earth (S) to be known, in Geodesy it is posed the problem of whether the *external* gravitational field of the Earth can be (or not) determined merely from measurements of its intensity on the Earth surface. If we denote the gravitational or Newtonian potential of the Earth by u, and g denotes the modulus of the force of gravity on S (in Geodesy g is simply called *gravity*), then by well-known properties of u (see, for example, [13, Chapter 1]) the problem can be formulated as

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ |\nabla u| = g & \text{on } S, \\ u(x) \to 0 & \text{as } x \to \infty, \end{cases} \tag{1.2}$$

where S is a closed surface in $\mathbb{R}^3$, Ω denotes its exterior domain and g is a given positive continuous function on S. Notice that we do not take into consideration the Earth rotation (for a more complete model see, for instance, [6]). This geodetic problem is quite realistic since the gravity can be easily measured both in land and sea, and by spatial positioning techniques the hypothesis concerning the knowledge of S is not far from being realistic too, nowadays. In Geomagnetism, we may formulate a completely analogous problem for the external magnetic field of the Earth.

In $\mathbb{R}^3$ space, the usual two-dimensional approach (the inversion defined by $z = x + iy \mapsto \bar{z}^{-1}$ where $\bar{z} = x - iy$ and $(x, y) \in \mathbb{R}^2$) is not feasible (we refer to [9] and [24] for this problem in $\mathbb{R}^2$). If we use the Kelvin transformation (see [1, Chapter 4]) the boundary condition is not preserved and it changes slightly. For example, if now v denotes the Kelvin transform of u and

$$S = \{\mathbf{x} = (x_1, x_2, x_3) \in \mathbb{R}^3 : x_1^2 + x_2^2 + x_3^2 = 1\}, \tag{1.3}$$

then it can be proved (see [27]) that (1.2) is equivalent to

$$\begin{cases} \Delta v = 0 & \text{inside } S, \\ \left(v + \dfrac{\partial v}{\partial \mathbf{n}}\right)^2 + |\nabla_s v|^2 = g^2 & \text{on } S, \end{cases} \tag{1.4}$$

where $\partial v / \partial \mathbf{n}$ is the outer normal derivative of v and $\nabla_s v$ denotes the tangential or surface gradient of v.

As far as the authors know, there is not yet a *global* existence theorem for (1.2). Some local existence theorems are known (see [6,16,32]): roughly speaking, *if g is close enough* (in a convenient Hölder space of functions) *to some g_0 such that $g_0 = |\nabla u_0|$ on S, where u_0 is a given, regular at infinity, harmonic function in Ω, then there is a function u close to u_0 solution of* (1.2). (Hereafter, by a solution of (1.2) we mean a function $u \in C^2(\Omega) \cap C^1(\bar{\Omega})$, vanishing at infinity, and satisfying pointwise both the Laplace equation and boundary condition.)

The following uniqueness result for problem (1.2) is well known (see [2,18]): *there is at most one solution of* (1.2) *whose normal derivative is strictly negative* (*or strictly positive*) *at each point of S*. In Section 2 of this paper, we generalize this result to functions with *nonpositive* (or *nonnegative*) normal derivative. Although our approach is the same as that followed by Backus (via the maximum principle), our generalization comes from a slightly more careful examination of the boundary condition.

Section 3 is devoted to the study of the oblique problem. The key idea connecting this problem and the Backus problem is based on the following simple remark: if (1.4) has a solution which satisfies $v + \partial v/\partial \mathbf{n} \geqslant 0$, then necessarily

$$v + \frac{\partial v}{\partial \mathbf{n}} = \sqrt{g^2 - |\nabla_s v|^2}.$$

The results of this paper improve and complete those of the previous paper by the authors [8].

Throughout this work we shall denote the real space of harmonic functions in an open subset Ω of $\mathbb{R}^N$ by $\mathscr{H}(\Omega)$. For unbounded Ω, $\mathscr{H}_\infty(\Omega)$ will denote the subset of $\mathscr{H}(\Omega)$ consisting of functions vanishing at infinity. If S is a closed surface in $\mathbb{R}^3$, we shall use the notation $C_+(S) = \{g \in C(S) : g(x) \geqslant 0 : \forall x \in S\}$.

2. Some results about the uniqueness of solutions

We start by recalling some elementary results

Lemma 2.1 (*Díaz et al. [8]*). *Let S be a closed surface in $\mathbb{R}^3$ and let Ω be the unbounded connected component of $\mathbb{R}^3 \backslash S$. Let $u \in \mathscr{H}_\infty(\Omega) \cap C^1(\bar{\Omega})$ not vanishing identically. Then*

$$\min(m, 0) < u(x) < \max(M, 0), \quad \forall x \in \Omega,$$

where $m = \min_S u$ and $M = \max_S u$.

We also recall the *Hopf boundary point lemma*.

Lemma 2.2 (*Gilbarg and Trudinger [10, Lemma 3.4]*). *Let Ω be a domain in $\mathbb{R}^N$ and $u \in \mathscr{H}(\Omega)$. Let $x_0 \in \partial\Omega$ be such that*

(a) *u is continuous at x_0;*
(b) *$u(x_0) > u(x)$ for all $x \in \Omega$;*
(c) *$\partial\Omega$ satisfies an interior sphere condition at x_0.*

Then, the outer normal derivative of u at x_0, if it exists, satisfies the strict inequality

$$\frac{\partial u}{\partial \mathbf{n}}(x_0) > 0.$$

We are now in a position of proving our first theorem. We shall assume that S is regular enough as to apply Lemma 2.2, and by $\partial/\partial\mathbf{n}$ we shall mean the derivative along the normal of S pointing to the *exterior* of S. Let $F : \mathscr{H}_\infty(\Omega) \cap C^1(\bar{\Omega}) \to C_+(S)$ be the operator defined as

$$F(u) = \gamma(|\nabla u|),$$

where γ is the trace (or restriction to S) operator. Observe that $F(u) = F(-u)$.

Theorem 2.3. *Let $u \in \mathscr{H}_\infty(\Omega) \cap C^1(\bar{\Omega})$ with $\partial u/\partial \mathbf{n} \leqslant 0$ on S. Let $v \in \mathscr{H}_\infty(\Omega) \cap C^1(\bar{\Omega})$ be such that $F(v) \leqslant F(u)$. Then, either $v \equiv u$ or $v < u$ in $\bar{\Omega}$.*

Remark 2.1. Alternatively, if $\partial u/\partial \mathbf{n} \geqslant 0$ on S and $F(v) \leqslant F(u)$, then, either $v \equiv u$ or $v > u$ in $\bar{\Omega}$. Simply apply Theorem 2.3 to the functions $-u$ and $-v$.

Remark 2.2. If $u \in \mathscr{H}_\infty(\Omega) \cap C^1(\bar{\Omega})$ is such that $\partial u/\partial \mathbf{n} \leqslant 0$ $(\partial u/\partial \mathbf{n} \not\equiv 0)$ on S then $u > 0$ in $\bar{\Omega}$. In fact, by Lemma 2.1 we only need to show that $m = \min_S u > 0$. If $m \leqslant 0$ and $x_0 \in S$ is such that $u(x_0) = m$, then by Lemma 2.1 we have $u(x) > u(x_0)$. But Lemma 2.2 shows that in this case $\partial u/\partial \mathbf{n}(x_0) > 0$.

Remark 2.3. If we restrict the domain of definition of F to

$$D_-(F) = \{u \in \mathscr{H}_\infty(\Omega) \cap C^1(\bar{\Omega}) : \partial u/\partial \mathbf{n} \leqslant 0 \text{ on } S\}$$

then this theorem shows that F is *injective*. (The same remark holds if F is restricted to

$$D_+(F) = \{u \in \mathscr{H}_\infty(\Omega) \cap C^1(\bar{\Omega}) : \partial u/\partial \mathbf{n} \geqslant 0 \text{ on } S\}.)$$

In other words, if $u \in D_-(F)$ is a solution of problem (1.2), then it is the unique solution of (1.2) in $D_-(F)$. In addition, observe that if $u \in D_-(F)$ is the (unique) solution of (1.2) in $D_-(F)$, then $-u$ is the (unique) solution of (1.2) in $D_+(F)$.

Remark 2.4. It should be noted that the problem

$$\Delta u = 0 \text{ in } D, \quad |\nabla u| = g \text{ on } S, \tag{2.1}$$

where D is the *interior* of a closed surface S, which has no special relevance in Geodesy to the best of our knowledge, is completely different from (1.2). This interior problem has been studied by some authors (see, for example, [22,23]). Observe that for the interior problem (2.1) the assumption $\partial u/\partial \mathbf{n} \leqslant 0$ on S for the solution has no sense, since if $u \in \mathscr{H}(\Omega) \cap C^1(\bar{\Omega})$ then

$$\int_S \frac{\partial u}{\partial \mathbf{n}} \, \mathrm{d}s = 0$$

and so $\partial u/\partial \mathbf{n}$ necessarily changes sign on S unless u is constant.

Remark 2.5. Let $u \in D_-(F)$ be a solution of (1.2) (assumed to exist). Then Theorem 2.3 and Remark 2.1 also give us the following result: if $v \in \mathscr{H}_\infty(\Omega) \cap C^1(\bar{\Omega})$ is any other solution, then

$$-u < v < u \quad \text{in } \bar{\Omega}$$

or in other words, $-u$ and u would be the minimal and maximal solutions of (1.2), respectively.

Remark 2.6. Compare Theorem 2.3 and these remarks with [29, Theorem 1] and [28, Theorems 1 and 3], where similar results have been obtained for the problems

$$\Delta u = -2 \text{ in } \Omega \subset \mathbb{R}^N, \quad |\nabla u| = g \geqslant 0 \text{ on } \partial\Omega$$

and

$$\Delta u = f(u) \text{ in } \Omega \subset \mathbb{R}^N, \quad |\nabla u| = g \geqslant 0 \text{ on } \partial\Omega,$$

where Ω is a bounded domain in $\mathbb{R}^N$ and f satisfies

$$f'(s) \geqslant 0 \quad (f'(s) \not\equiv 0), \quad f(0) = 0.$$

Proof of Theorem 2.3. If $F(v) \leqslant F(u)$ then

$$\langle \nabla w, \nabla w + 2\nabla u \rangle \leqslant 0 \quad \text{on } S, \tag{2.2}$$

where $w = v - u$. If w does not vanish identically, then by Lemma 2.1 we have

$$\min(m, 0) < w(x) < \max(M, 0) \quad \forall x \in \Omega,$$

where $m = \min_S w$ and $M = \max_S w$. If we prove that $M < 0$ then we would obtain the desired result. Let $x_0 \in S$ be such that $M = w(x_0)$. If $M \geqslant 0$ then $w(x) < w(x_0)$ for all $x \in \Omega$, and hence $\partial w / \partial \mathbf{n}(x_0) < 0$. But this is in contradiction with inequality (2.2) at x_0. In fact, since the tangential gradient of w at x_0 is zero, (2.2) at x_0 becomes

$$\frac{\partial w}{\partial \mathbf{n}} \left(\frac{\partial w}{\partial \mathbf{n}} + 2 \frac{\partial u}{\partial \mathbf{n}} \right) \leqslant 0,$$

so if $\partial w / \partial \mathbf{n}(x_0) < 0$ then $2\partial u / \partial \mathbf{n} \geqslant -\partial w / \partial \mathbf{n} > 0$ at x_0, but this is not true. This completes the proof. □

It is also remarkable the following result (compare with [28, Theorem 1(iii)]).

Theorem 2.4. *Let u be the* (*unique*) *solution of* (1.2) *which satisfies* $\partial u / \partial \mathbf{n} \leqslant 0$ (*assumed to exist*). *Then*, *if v is any other solution of* (1.2) ($v \not\equiv \pm u$), *there are points* $x_0, \tilde{x}_0 \in S$ *such that*

$$\frac{\partial v}{\partial \mathbf{n}}(x_0) = -\frac{\partial u}{\partial \mathbf{n}}(x_0) \geqslant 0 \quad \textit{and} \quad \frac{\partial v}{\partial \mathbf{n}}(\tilde{x}_0) = \frac{\partial u}{\partial \mathbf{n}}(\tilde{x}_0) \leqslant 0.$$

Proof. Let $w = v - u$ and $\tilde{w} = v + u$. Then, by Theorem 2.3, $w < 0$ and $\tilde{w} > 0$ in $\bar{\Omega}$. Let $x_0 \in S$ be such that $w(x_0) = \min_S w$, and let $\tilde{x}_0 \in S$ be such that $w(\tilde{x}_0) = \max_S \tilde{w}$. Then, by Lemma 2.1 we have $w(x_0) < w(x)$ and $\tilde{w}(x) < \tilde{w}(\tilde{x}_0)$ in $\bar{\Omega}$. By the Hopf boundary point lemma we then conclude that

$$\frac{\partial w}{\partial \mathbf{n}}(x_0) > 0 \quad \text{and} \quad \frac{\partial \tilde{w}}{\partial \mathbf{n}}(\tilde{x}_0) < 0.$$

On the other hand, since $\nabla_s w(x_0) = \nabla_s \tilde{w}(\tilde{x}_0) = 0$, we have at x_0

$$\frac{\partial w}{\partial \mathbf{n}} \left(\frac{\partial w}{\partial \mathbf{n}} + 2 \frac{\partial u}{\partial \mathbf{n}} \right) = 0$$

and at $\tilde{x}_0$

$$\frac{\partial \tilde{w}}{\partial \mathbf{n}} \left(\frac{\partial \tilde{w}}{\partial \mathbf{n}} - 2 \frac{\partial u}{\partial \mathbf{n}} \right) = 0.$$

Hence, $\partial w/\partial \mathbf{n} + 2\partial u/\partial \mathbf{n} = 0$ at x_0 and $\partial \tilde{w}/\partial \mathbf{n} - 2\partial u/\partial \mathbf{n} = 0$ at $\tilde{x}_0$, and this completes the proof of this theorem. □

Example 2.1. Let S be the unit sphere in $\mathbb{R}^3$ (see (1.3)). Let c be an arbitrary positive constant. In this case, the functions $\pm c/r$, where $r = |x|$, are the radial solutions of (1.2). Let $u = c/r$. Since $\mathrm{d}u/\mathrm{d}r = -c < 0$ on S, then by Remark 2.3 the function u is the unique solution of (1.2) with $g = c$ which satisfies $\partial u/\partial \mathbf{n} \leqslant 0$. In addition, if v ($\not\equiv \pm u$) is any other solution of (1.2) with $g = c$, then by Remark 2.5 we have the estimate

$$|v| < c/r \quad \text{in } \bar{\Omega}.$$

By Theorem 2.4, there are points $x_0, \tilde{x}_0 \in S$ such that

$$\frac{\partial v}{\partial \mathbf{n}}(x_0) = c, \quad \frac{\partial v}{\partial \mathbf{n}}(\tilde{x}_0) = -c.$$

Since $|\partial v/\partial \mathbf{n}| \leqslant c$ on S, we conclude that $c = \max_S(\partial v/\partial \mathbf{n})$ and $-c = \min_S(\partial v/\partial \mathbf{n})$.

In the next theorem, we shall denote the interior domain to S by D. We recall that if D is *starshaped* (with respect to $0 \in D$) then $\langle x, \mathbf{n}\rangle \geqslant 0$ on S.

Theorem 2.5. *Let S be a closed surface in $\mathbb{R}^3$ such that D is starshaped. Let u be a solution of* (1.2). *Then*

$$u \leqslant (D^2/d)\|g\|_\infty,$$

where $D = \max_S |x|$, $d = \mathrm{dist}(0, S)$, *and* $\|g\|_\infty = \max_S g$.

Proof. On S we have

$$|\nabla u(x)| \leqslant \frac{D^2}{r^2}\|g\|_\infty \quad (r = |x|).$$

Let $v = c/r$ where $c = D^2\|g\|_\infty$. Since $|\nabla v| = c/r^2$, then

$$|\nabla u| \leqslant |\nabla v| \quad \text{on } S.$$

Since in addition $v \in \mathscr{H}(\mathbb{R}^3\backslash 0)$ and $\langle \nabla v, \mathbf{n}\rangle = -cr^{-3}\langle x, \mathbf{n}\rangle \leqslant 0$ on S, by Theorem 2.3 we have

$$u \leqslant \frac{c}{r} \leqslant \frac{c}{d}$$

and this completes the proof. □

We also have the following gradient bound for solutions of (1.2).

Theorem 2.6. *Let u be a solution of* (1.2). *Then we have*

$$\sup_{\Omega} |\nabla u| \leqslant \|g\|_\infty.$$

Proof. It is easy to check that

$$\Delta(|\nabla u|^2) = 2\,\mathrm{Tr}(\mathrm{M}_u^2) \geqslant 0,$$

where M_u is the matrix whose entries are the second derivatives of u. (For a more general result concerning the subharmonic character of some powers of $|\nabla u|$, where $u \in \mathscr{H}(\Omega)$, see [33].) As in the proof of Lemma 2.1, let $B(0, R)$ denote an open ball centered at the origin $0 \in D$ of radius R and containing S. By the maximum principle for subharmonic functions (see [10, Theorem 2.3]), in the connected open set $\Omega_R = \Omega \cap B(0, R)$ we then have

$$|\nabla u(x)|^2 \leqslant \sup_{\partial\Omega_R} |\nabla u|^2 = \max(\|g\|_\infty^2, M(R)),$$

where $M(R) = \max_{\partial B(0,R)} |\nabla u|^2$. Since $M(R) \to 0$ as $R \to \infty$ (see [34, Section 23.2]), letting $R \to \infty$ we obtain the desired result. $\square$

Without any restriction on the sign of the normal derivative of the solution, it is clear that if u is a solution of (1.2) then $-u$ is a solution as well. We then could wonder if these functions u and $-u$ are the only solutions of the problem. In general, the answer is in the negative as it was proved by Backus (see [3]). In fact, let $\widetilde{\mathscr{H}}_\infty(\Omega)$ be the subset of $\mathscr{H}_\infty(\Omega) \cap C^1(\bar{\Omega})$ consisting of functions z not vanishing identically and such that the oblique boundary value problem

$$\begin{cases} \Delta w = 0 & \text{outside } S, \\ \langle \nabla w, \nabla z\rangle = 0 & \text{on } S, \\ w(x) \to 0 & \text{as } x \to \infty \end{cases} \tag{2.3}$$

has a nontrivial $C^2(\Omega) \cap C^1(\bar{\Omega})$ solution. Since $|\nabla u| = |\nabla v|$ if and only if

$$\langle \nabla(u - v), \nabla(u + v)\rangle = 0,$$

we then have the following.

Proposition 2.7. $\widetilde{\mathscr{H}}_\infty(\Omega) \neq \emptyset$ *if and only if there exist two functions* $u, v \in \mathscr{H}_\infty(\Omega) \cap C^1(\bar{\Omega})$ $(u \not\equiv \pm v)$ *such that* $|\nabla u| = |\nabla v|$ *on* S.

Proof. Let $z \in \widetilde{\mathscr{H}}_\infty(\Omega)$ and let w be a nontrivial solution of (2.3). Define $u = (w + z)/2$ and $v = (w - z)/2$. Then $u, v \in \mathscr{H}_\infty(\Omega) \cap C^1(\bar{\Omega})$ and

$$|\nabla u|^2 = \tfrac{1}{2}(|\nabla w|^2 + |\nabla z|^2) = |\nabla v|^2$$

on S.

On the other hand, if u and v $(u \not\equiv \pm v)$ are such that $|\nabla u| = |\nabla v|$ on S, then $u + v \in \widetilde{\mathscr{H}}_\infty(\Omega)$. This completes the proof. $\square$

In the case of a sphere, Backus proved (see [3]) that $\widetilde{\mathscr{H}}_\infty(\mathbb{R}^3\backslash\bar{B}(0,R)) \neq \emptyset$. In fact he found nontrivial solutions of (2.3) by choosing $z = x_3/r^3 \in \mathscr{H}_\infty(\mathbb{R}^3\backslash\{0\})$. See [17] for a related topic.

Remark 2.7. If S is smooth enough, it should be observed that if $z \in \widetilde{\mathscr{H}}_\infty(\Omega)$ then ∇z is tangential to S in some set $T \subset S$. In fact, if $T = \emptyset$ then it follows that the only solution of (2.3) is $w = 0$ (see, for example, [25]). In the above example of Backus the tangential set T is the equator of the sphere.

Remark 2.8. The following question, posed by Backus [3], seems to still remain open: let $u \in \mathscr{H}_\infty(\Omega) \cap C^1(\bar{\Omega})$; how many functions v are there in $\mathscr{H}_\infty(\Omega) \cap C^1(\bar{\Omega})$ such that $|\nabla v| = |\nabla u|$ on S? (see [36]).

3. On the oblique boundary value problem

In this section we shall restrict ourselves to the simplest case of a sphere (1.3) and consider the equivalent problem (1.4). Specifically, we are interested in the boundary value problem

$$\begin{cases} \Delta v = 0 & \text{in } \Omega = B(0,1), \\ v + \dfrac{\partial v}{\partial \mathbf{n}} = \sqrt{(g^2 - |\nabla_s v|^2)_+} & \text{on } \partial\Omega, \end{cases} \tag{3.1}$$

where

$$(g^2 - |\nabla_s v|^2)_+ = \max\{(g^2 - |\nabla_s v|^2), 0\}.$$

(Hereafter we shall exclude the case $g \equiv 0$, since if $g \equiv 0$, by Theorem 3.15, the only solution of Problem (3.1) is $v \equiv 0$.)

Before to deal with the existence and uniqueness of solutions let us examine the relationship between problems (3.1) and (1.4). We have

Lemma 3.1. *Let v be the solution* (*assumed to exist*) *of* (3.1). *If*

$$|\nabla_s v| \leqslant g \quad \text{on } \partial\Omega \tag{3.2}$$

then v is the unique solution of (1.4) *such that $v + \partial v/\partial \mathbf{n} \geqslant 0$ on $\partial\Omega$.*

In addition, if v does not satisfy (3.2) *then the boundary value problem* (1.4) *has no solutions satisfying $v + \partial v/\partial \mathbf{n} \geqslant 0$ on $\partial\Omega$.*

Remark 3.1. In the first part of this lemma, the uniqueness of v is clear from Remark 2.3. In fact, the function v in (1.4) is the Kelvin transform of u, that is to say

$$v(x) = |x|^{-1} u(|x|^{-2} x),$$

so we have on $\partial\Omega$

$$v + \partial v/\partial \mathbf{n} = -\partial u/\partial \mathbf{n} \tag{3.3}$$

and then $v + \partial v/\partial \mathbf{n} \geqslant 0$ if and only if $\partial u/\partial \mathbf{n} \leqslant 0$.

Remark 3.2. For an arbitrary positive constant c, if $g(x) \equiv c$ then $v \equiv c$ is the unique solution of (3.1). Since $\nabla_s v \equiv 0$, by Lemma 3.1 we then conclude that $v \equiv c$ is the unique solution of (1.4) satisfying $v + \partial v/\partial \mathbf{n} \geqslant 0$ on $\partial\Omega$. Compare this result with Example 2.1.

Therefore, what we want to prove is that indeed problem (3.1) has a solution and we have (3.2) for the solution of (3.1). Then we could conclude an existence theorem for (1.4). We have still not proved these things but we state the following.

Conjecture 3.1. *Problem* (3.1) *has a unique solution* $v \in C^2(\Omega) \cap C^1(\bar{\Omega})$. *In addition*, v *satisfies* (3.2).

This conjecture is based on the remainder results of this section. With respect to condition (3.2) we have the following result.

Proposition 3.2. *Let* $g \in C_+(\partial\Omega)$ *and let* v *be a classical solution* (*assumed to exist*) *of* (3.1). *Then*

$$\{x \in \partial\Omega : |\nabla_s v| < g\} \neq \emptyset.$$

Proof. Assume, on the contrary, that $|\nabla_s v| \geqslant g$ on $\partial\Omega$. Then we have

$$v + \partial v/\partial \mathbf{n} = 0 \quad \text{on } \partial\Omega.$$

With the same argument used in the proof of Theorem 3.15 we now conclude that $v \equiv 0$ in $\bar{\Omega}$, and this would imply that $g \equiv 0$. The proof is complete. □

Remark 3.3. If $g > 0$ the conclusion of this proposition directly follows from the fact that if $v \in C^1(\bar{\Omega})$, as we are assuming, then the tangential gradient of v vanish at the points of the boundary where the harmonic function v reaches its maximum and minimum values.

We now introduce the following sets:

$$A_- = \{x \in \partial\Omega : |\nabla_s v| < g\}$$

and

$$A_+ = \{x \in \partial\Omega : |\nabla_s v| > g\}.$$

In order to obtain some more information about these sets, we shall use the following identity which can be inferred from an integral identity due to Rellich [31]; see also [30, (2.14)] and compare with [14, (1.1.4)].

Proposition 3.3. *Let* $v \in \mathscr{H}(\Omega) \cap C^1(\bar{\Omega})$, *where* $\Omega = B(0, 1)$ *in* $\mathbb{R}^N$ $(N \geqslant 2)$. *Then*,

$$(N-2)\int_\Omega |\nabla v|^2 \, \mathrm{d}x = \int_{\partial\Omega} \left(|\nabla_s v|^2 - \left| \frac{\partial v}{\partial \mathbf{n}} \right|^2 \right) \mathrm{d}s.$$

Since by Green's first identity we have

$$\int_\Omega v\Delta v\,\mathrm{d}x + \int_\Omega |\nabla v|^2\,\mathrm{d}x = \int_{\partial\Omega} v\,\frac{\partial v}{\partial \mathbf{n}}\,\mathrm{d}s$$

if $v \in \mathscr{H}(\Omega)$, then

$$\int_{\partial\Omega} v\,\frac{\partial v}{\partial \mathbf{n}}\,\mathrm{d}s = \int_\Omega |\nabla v|^2\,\mathrm{d}x \geqslant 0.$$

By Proposition 3.3 we then have ($N > 2$)

$$\int_{\partial\Omega} v\,\frac{\partial v}{\partial \mathbf{n}}\,\mathrm{d}s = \frac{1}{N-2}\int_{\partial\Omega}\left(|\nabla_s v|^2 - \left|\frac{\partial v}{\partial \mathbf{n}}\right|^2\right)\mathrm{d}s. \tag{3.4}$$

Since on the other hand, we can write

$$\begin{aligned}\int_{\partial\Omega}\left(v\,\frac{\partial v}{\partial \mathbf{n}} + \left|\frac{\partial v}{\partial \mathbf{n}}\right|^2\right)\mathrm{d}s &= \int_{\partial\Omega}\left(v + \frac{\partial v}{\partial \mathbf{n}}\right)^2\mathrm{d}s - \left[\int_{\partial\Omega} v^2\,\mathrm{d}s + \int_{\partial\Omega} v\,\frac{\partial v}{\partial \mathbf{n}}\,\mathrm{d}s\right]\\ &\leqslant \int_{\partial\Omega}\left(v + \frac{\partial v}{\partial \mathbf{n}}\right)^2\mathrm{d}s, \end{aligned} \tag{3.5}$$

combining (3.4) and (3.5), we have proved the following inequality.

Corollary 3.4. *Let $v \in \mathscr{H}(\Omega) \cap C^1(\bar{\Omega})$, where $\Omega = B(0,1)$ in $\mathbb{R}^N$ ($N > 2$). Then,*

$$\frac{1}{N-2}\int_{\partial\Omega}\left(|\nabla_s v|^2 + (N-3)\left|\frac{\partial v}{\partial \mathbf{n}}\right|^2\right)\mathrm{d}s \leqslant \int_{\partial\Omega}\left(v + \frac{\partial v}{\partial \mathbf{n}}\right)^2\mathrm{d}s. \tag{3.6}$$

If $N = 3$ and v is a solution of (3.1), then from (3.6) we get

$$\int_{\partial\Omega} |\nabla_s v|^2\,\mathrm{d}s \leqslant \int_{\partial\Omega} (g^2 - |\nabla_s v|^2)_+\,\mathrm{d}s.$$

Then

$$\int_{A_-} |\nabla_s v|^2\,\mathrm{d}s + \int_{A_+} |\nabla_s v|^2\,\mathrm{d}s \leqslant \int_{A_-} (g^2 - |\nabla_s v|^2)\,\mathrm{d}s \tag{3.7}$$

and hence

$$2\int_{A_-} |\nabla_s v|^2\,\mathrm{d}s + \int_{A_+} |\nabla_s v|^2\,\mathrm{d}s \leqslant \int_{A_-} g^2\,\mathrm{d}s.$$

Since

$$\int_{A_+} |\nabla_s v|^2\,\mathrm{d}s > \int_{A_+} g^2\,\mathrm{d}s,$$

then we can state the following.

Proposition 3.5. *Let $g \in C_+(\partial\Omega)$ and let v be a classical solution* (*assumed to exist*) *of* (3.1). *Then*

$$\int_{A_-} g^2 \, \mathrm{d}s > \int_{A_+} g^2 \, \mathrm{d}s \tag{3.8}$$

and, *in particular $m(A_-) > 0$. Moreover*,

$$m\left(\left\{x \in \partial\Omega : |\nabla_s v| \leqslant \frac{g}{\sqrt{2}}\right\}\right) > 0 \tag{3.9}$$

and

$$\int_{A_+} |\nabla_s v|^2 \, \mathrm{d}s < \|g\|^2_{L^2(\partial\Omega)}. \tag{3.10}$$

(*Here $m(C)$ denotes the surface-area measure of a set $C \subseteq \partial\Omega$.*)

Proof. That $m(A_-) > 0$ comes from (3.8). To prove (3.9) we use the decomposition $A_- = B_1 \cup B_2$ where

$$B_1 = \left\{x \in A_- : \frac{g}{\sqrt{2}} < |\nabla_s v|\right\}$$

and

$$B_2 = \left\{x \in \partial\Omega : \frac{g}{\sqrt{2}} \geqslant |\nabla_s v|\right\}.$$

From inequality (3.7) we deduce that

$$\begin{aligned} 0 \leqslant \int_{A_+} |\nabla_s v|^2 \, \mathrm{d}s &\leqslant \int_{A_-} (g^2 - 2|\nabla_s v|^2) \, \mathrm{d}s \\ &= \int_{B_1} (g^2 - 2|\nabla_s v|^2) \, \mathrm{d}s + \int_{B_2} (g^2 - 2|\nabla_s v|^2) \, \mathrm{d}s. \end{aligned} \tag{3.11}$$

Calling $f(x) = g^2(x) - 2|\nabla_s v(x)|^2$ for $x \in \partial\Omega$, it is obvious that $f(x) < 0$ on B_1, whereas $f(x) \geqslant 0$ on B_2. Then, if $m(B_2) = 0$ we arrive at a contradiction since $m(B_1 \cup B_2) = m(A_-) > 0$. Inequality (3.10) immediately follows from (3.7). The proof is complete. □

Remark 3.4. If $g > 0$, (3.9) also follows from Remark 3.3.

Although we have not proved that $A_+ = \emptyset$, the Proposition 3.5 can be considered as a partial result in this direction.

Now we concentrate our attention on the existence of solutions of (3.1). It should be noted that in contrast to problem (1.4), problem (3.1) is *oblique* using the terminology followed in [19]. In fact, for a general formulation

$$\begin{cases} \Delta u = 0 & \text{in } \Omega, \\ G(x, u, \nabla u) = 0 & \text{on } \partial\Omega \end{cases} \tag{3.12}$$

the problem (3.12) is *oblique* if, at $\Gamma = \partial\Omega \times \mathbb{R} \times \mathbb{R}^3$, the following inequality is satisfied:

$$\chi = \langle G_{\mathbf{p}}, \mathbf{n} \rangle > 0, \tag{3.13}$$

where $G_{\mathbf{p}}$ denotes the (possibly weak) partial derivative with respect to $\mathbf{p}$ when G is expressed in dummy variables $(x, z, \mathbf{p}) \in \Gamma$. Note that in the case of the original Backus problem (1.2) $G(x, z, \mathbf{p}) = |\mathbf{p}|$ and then G is oblique if and only if $\langle \mathbf{p}, \mathbf{n} \rangle > 0$ (i.e. the condition depends on $\partial u/\partial \mathbf{n}$ which is a priori unknown; the same can be said for problem (1.4)).

Lemma 3.6. *Problem* (3.1) *is oblique*.

Proof. In these variables the boundary operator in (3.1) is given by

$$G(x, z, \mathbf{p}) = z + \langle \mathbf{p}, \mathbf{n} \rangle - \sqrt{(g^2(x) - |\mathbf{p}_t|^2)_+}, \tag{3.14}$$

where $\mathbf{p}_t = \mathbf{p} - \langle \mathbf{p}, \mathbf{n} \rangle \mathbf{n}$ is the tangential projection of $\mathbf{p}$. Differentiating G with respect to $\mathbf{p}$ we get that for any prescribed (x, z)

$$G_{\mathbf{p}} = \begin{cases} \mathbf{n} & \text{if } |\mathbf{p}_t| > g(x), \\ \mathbf{n} + \dfrac{\mathbf{p}_t}{\sqrt{g^2(x) - |\mathbf{p}_t|^2}} & \text{if } |\mathbf{p}_t| < g(x) \end{cases}$$

and this proves that $\chi = 1$. □

Remark 3.5. It is interesting to note the following property of the operator (3.14). Let $\lambda > 0$. Observing that

$$\langle \mathbf{p} + \lambda \mathbf{n}, \mathbf{n} \rangle = \langle \mathbf{p}, \mathbf{n} \rangle + \lambda$$

and since the tangential projections of $\mathbf{p}$ and of $\mathbf{p} + \lambda \mathbf{n}$ coincide, then we have

$$G(x, z, \mathbf{p} + \lambda \mathbf{n}) - G(x, z, \mathbf{p}) = \lambda \tag{3.15}$$

for all $(x, z, \mathbf{p}) \in \Gamma$. From (3.15) we can conclude that the function $G(x, z, \mathbf{p})$ is strictly increasing with respect to $\mathbf{p}$ in the normal direction to $\partial\Omega$ at x. Barles [5] has recently proved that nonlinear boundary value problems with this property have, under some other additional conditions, a unique *viscosity solution* (see [5, Section I]) in $C(\bar{\Omega})$.

Although G given by (3.14) is not regular enough as to may apply a known existence theorem for oblique nonlinear boundary value problems (see [19]), it seems possible to approach G by more regular functions G_ε and to obtain an existence theorem for (3.1) by passing to the limit. In fact, let $\varepsilon > 0$ and consider the following modified problem

$$\begin{cases} \Delta v_\varepsilon = 0 & \text{in } \Omega, \\ v_\varepsilon + \dfrac{\partial v_\varepsilon}{\partial \mathbf{n}} = \sqrt{\varepsilon + (g^2 - |\nabla_s v_\varepsilon|^2)_+} & \text{on } \partial\Omega. \end{cases} \tag{3.16}$$

Firstly, we have the following uniqueness result for the problems (3.16).

Lemma 3.7. *Let* $\varepsilon \geqslant 0$. *Then problem* (3.16) *has at most one classical solution.*

Proof. Exactly the same argument used to prove Theorem 3.15 can be used here to prove this lemma. □

In addition, we have the following monotonicity (with respect to ε) result.

Lemma 3.8. *If* $0 \leqslant \varepsilon_1 < \varepsilon_2$ *then*

$$v_{\varepsilon_1} < v_{\varepsilon_2},$$

where v_{ε_1} *and* v_{ε_2} *are the solutions* (*assumed to exist*) *of* (3.16) *for* ε_1 *and* ε_2, *respectively.*

Proof. At some point $x_0 \in \partial\Omega$ the harmonic function $z = v_{\varepsilon_1} - v_{\varepsilon_2}$ takes its maximum value. Since at this point we have

$$\varepsilon_1 - \varepsilon_2 = (z + \partial z/\partial \mathbf{n})(w + \partial w/\partial \mathbf{n})$$

(where $w = v_{\varepsilon_1} + v_{\varepsilon_2}$) and $w + \partial w/\partial \mathbf{n} > 0$, then at x_0 we have

$$z + \partial z/\partial \mathbf{n} < 0.$$

Hence we necessarily conclude that $z(x_0) < 0$, and this completes the proof of this lemma. □

We also have the following a priori estimates.

Lemma 3.9. *Let* v_ε *be the solution* (*assumed to exist*) *of* (3.16) *where* $\varepsilon \geqslant 0$. *Then*

(a) $v_\varepsilon > 0$;
(b) $v_\varepsilon \leqslant \sqrt{\varepsilon + \|g\|_\infty^2}$ *in* $\bar{\Omega}$;
(c) $\left|\frac{\partial v_\varepsilon}{\partial \mathbf{n}}\right| \leqslant \sqrt{\varepsilon + \|g\|_\infty^2}$ *in* $\bar{\Omega}$.

Proof. Although the proof of this lemma follows similar arguments used in [27, Lemmas 1, 2 and 3], here, we include this proof for sake of completeness. If $\varepsilon = 0$ then part (a) follows from Remark 2.2 and (3.3) if on $\partial\Omega$ $v_\varepsilon + \partial v_\varepsilon/\partial \mathbf{n} \not\equiv 0$. If $v_\varepsilon + \partial v_\varepsilon/\partial \mathbf{n} \equiv 0$ on $\partial\Omega$, then $v_\varepsilon \equiv 0$ in $\bar{\Omega}$ and g should vanish identically. If $\varepsilon > 0$ then $v_\varepsilon + \partial v_\varepsilon/\partial \mathbf{n} > 0$ on $\partial\Omega$ and (a) also follows from Remark 2.2 and (3.3). Let $x_0 \in \partial\Omega$ be a point where v_ε achieves its maximum value; then, at this point we have

$$v_\varepsilon + \partial v_\varepsilon/\partial \mathbf{n} = \sqrt{\varepsilon^2 + g^2}.$$

Since $\partial v_\varepsilon/\partial \mathbf{n}(x_0) \geqslant 0$ then we conclude (b). Finally, part (c) follows from (a), (b) and the following inequalities (on $\partial\Omega$):

$$\partial v_\varepsilon/\partial \mathbf{n} \leqslant v_\varepsilon + \partial v_\varepsilon/\partial \mathbf{n} \leqslant \sqrt{\varepsilon^2 + \|g\|_\infty^2} \tag{3.17}$$

and

$$\partial v_\varepsilon/\partial \mathbf{n} \geqslant -v_\varepsilon.$$

The upper bound for $v_\varepsilon + \partial v_\varepsilon/\partial \mathbf{n}$ in (3.17) is obtained by observing that

$$\begin{aligned}(v_\varepsilon + \partial v_\varepsilon/\partial \mathbf{n})^2 &= \varepsilon + (g^2 - |\nabla_s v_\varepsilon|^2)_+ \\ &= \begin{cases} \varepsilon \leqslant \varepsilon + \|g\|_\infty^2 & \text{if } |\nabla_s v_\varepsilon|^2 \geqslant g^2, \\ \varepsilon + g^2 - |\nabla_s v_\varepsilon|^2 \leqslant \varepsilon + \|g\|_\infty^2 & \text{if } |\nabla_s v_\varepsilon|^2 \leqslant g^2. \end{cases} \qquad \square\end{aligned}$$

Now in order to get an existence theorem for the modified problem (3.16) we shall use an existence theorem obtained by Lieberman and Trudinger (see, [19, Corollary 7.7]) for the boundary value problem

$$\begin{cases} a^{ij}(x,u,\nabla u)D_{ij}u + a(x,u,\nabla u) = 0 & \text{in } \Omega, \\ G(x,u,\nabla u) = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.18}$$

where $D^2u = (D_{ij}u)$ denotes the Hessian matrix of the function u, and where Ω is a smooth bounded domain in $\mathbb{R}^N$. In fact, they have proved that if $G \in C^{0,1}(\bar{\Omega} \times \mathbb{R} \times \mathbb{R}^N)$ is *oblique*, then under certain natural structure conditions the boundary value problem (3.18) has a $C^{1,\beta}(\bar{\Omega}) \cap C^2(\Omega)$ solution for some $\beta > 0$.

In our case, in the variables $(x, z, \mathbf{p})$ the boundary operator in (3.16) is given by

$$G_\varepsilon(x,z,\mathbf{p}) = z + \langle \mathbf{p}, \mathbf{n}\rangle - \sqrt{\varepsilon + (g^2(x) - |\mathbf{p}_t|^2)_+}. \tag{3.19}$$

Like in problem (3.1), the boundary operator (3.19) is oblique.

About the regularity of our modified boundary operator we have the following.

Lemma 3.10. *Let* $\varepsilon > 0$. *If* $g \in C^{0,1}(\partial\Omega)$ *then* $G_\varepsilon \in C^{0,1}(\Gamma)$.

Proof. Let $(x, z, \mathbf{p}), (x', z', \mathbf{p}') \in \Gamma$. Then

$$G_\varepsilon(x,z,\mathbf{p}) - G_\varepsilon(x',z',\mathbf{p}') = (z - z') + \langle \mathbf{p} - \mathbf{p}', \mathbf{n}\rangle + (\zeta' - \zeta),$$

where

$$\zeta = \sqrt{\varepsilon + (g^2(x) - |\mathbf{p}_t|^2)_+}$$

and

$$\zeta' = \sqrt{\varepsilon + (g^2(x') - |\mathbf{p}'_t|^2)_+}.$$

We may consider different possibilities. If $|\mathbf{p}_t|^2 \geqslant g^2(x)$ and $|\mathbf{p}'_t|^2 \geqslant g^2(x')$, then $\zeta' - \zeta = 0$. If $|\mathbf{p}_t|^2 \geqslant g^2(x)$ and $|\mathbf{p}'_t|^2 \leqslant g^2(x')$, then for some constant K we have

$$\begin{aligned}|\zeta'^2 - \zeta^2| &= (g(x') - |\mathbf{p}'_t|)(g(x') + |\mathbf{p}'_t|) \leqslant 2\|g\|_\infty (g(x') - |\mathbf{p}'_t|) \\ &\leqslant 2\|g\|_\infty (g(x') - |\mathbf{p}'_t| + |\mathbf{p}_t| - g(x)) \\ &= 2\|g\|_\infty (g(x') - g(x)) + 2\|g\|_\infty (|\mathbf{p}_t| - |\mathbf{p}'_t|) \\ &\leqslant K(|x - x'| + |\mathbf{p}_t - \mathbf{p}'_t|) \leqslant K(|x - x'| + |\mathbf{p} - \mathbf{p}'|).\end{aligned}$$

The same estimate is obtained if $|\mathbf{p}_t|^2 \leqslant g^2(x)$ and $|\mathbf{p}_t'|^2 \geqslant g^2(x')$. Finally, if $|\mathbf{p}_t|^2 \leqslant g^2(x)$ and $|\mathbf{p}_t'|^2 \leqslant g^2(x')$ we get

$$\begin{aligned}\zeta'^2 - \zeta^2 &= (g^2(x') - g^2(x)) + (|\mathbf{p}_t|^2 - |\mathbf{p}_t'|^2)\\ &\leqslant K|x-x'| + (|\mathbf{p}_t'| + |\mathbf{p}_t|)(|\mathbf{p}_t'| - |\mathbf{p}_t|)\\ &\leqslant K(|x-x'| + |\mathbf{p}'-\mathbf{p}|).\end{aligned}$$

We also have the same estimate for $(\zeta^2 - \zeta'^2)$, and the proof is complete simply observing that $\zeta + \zeta' \geqslant \varepsilon$. $\square$

Assuming that g has been appropriately extended so that $G_\varepsilon \in C^{0,1}(\overline{\Omega} \times \mathbb{R} \times \mathbb{R}^3)$, the other sufficient conditions in [19, Corollary 7.7; conditions G2 and G3] are also satisfied. In fact we have the following technical result.

Lemma 3.11. *There are constants C and C' such that*:

(a) $|G_\varepsilon(x, z, \mathbf{p}_t)| \leqslant C\varphi(|z|)$ *where* $\varphi(t) = 1$ *if* $t \leqslant 1$ *and* $\varphi(t) = t$ *if* $t > 1$;
(b) $|G_{\varepsilon,\mathbf{p}}|, |G_{\varepsilon,z}|, |G_{\varepsilon,x}| \leqslant C'$.

(Here $G_{\varepsilon,\mathbf{p}}$, $G_{\varepsilon,z}$ and $G_{\varepsilon,x}$ denote the (weak) partial derivatives of G_ε with respect to $\mathbf{p}$, z and x, respectively.)

Proof. Since

$$G_\varepsilon(x, z, \mathbf{p}_t) = z - \sqrt{\varepsilon + (g^2(x) - |\mathbf{p}_t|^2)_+}$$

then

$$|G_\varepsilon(x, z, \mathbf{p}_t)| \leqslant |z| + \sqrt{\varepsilon + \|g\|_\infty^2}$$

and we have proved (a), where $C = 1 + \sqrt{\varepsilon + \|g\|_\infty^2}$.

Since

$$G_\varepsilon(x, z, \mathbf{p}) = z + \langle \mathbf{p}, \mathbf{n}\rangle - \sqrt{\varepsilon}$$

if $|\mathbf{p}_t| \geqslant g(x)$, and

$$G_\varepsilon(x, z, \mathbf{p}) = z + \langle \mathbf{p}, \mathbf{n}\rangle - \sqrt{\varepsilon + g^2(x) - |\mathbf{p}_t|^2}$$

if $|\mathbf{p}_t| < g(x)$, then we have

$$G_{\varepsilon,\mathbf{p}} = \begin{cases}\mathbf{n} & \text{if } |\mathbf{p}_t| \geqslant g(x),\\ \mathbf{n} - \dfrac{\mathbf{p}_t}{\sqrt{\varepsilon + g^2(x) - |\mathbf{p}_t|^2}} & \text{if } |\mathbf{p}_t| < g(x).\end{cases}$$

Hence we get the estimate

$$|G_{\varepsilon,\mathbf{p}}| \leqslant 1 + \frac{\|g\|_\infty}{\sqrt{\varepsilon}}.$$

Differentiating G_ε with respect to z we get $G_{\varepsilon,z} = -1$. Finally,

$$G_{\varepsilon,x} = \begin{cases} 0 & \text{if } |\mathbf{p}_t| \geqslant g(x), \\ \dfrac{g\nabla g}{\sqrt{\varepsilon + g^2(x) - |\mathbf{p}_t|^2}} & \text{if } |\mathbf{p}_t| < g(x), \end{cases}$$

so

$$|G_{\varepsilon,x}| \leqslant \frac{1}{\sqrt{\varepsilon}} \|g\|_\infty \|\nabla g\|_\infty. \qquad \square$$

Summing up, from Lemmas 3.7, 3.9, 3.10, 3.11 and [19, Corollary 7.7] we can conclude the following.

Theorem 3.12. *Let* $\varepsilon > 0$ *and* $g \in C^{0,1}(\partial\Omega)$, $g \geqslant 0$. *Then problem* (3.16) *has a unique solution in* $C^{1,\beta}(\bar{\Omega}) \cap C^2(\Omega)$ *for some* $\beta > 0$.

Now let $\{\varepsilon_n\}$ be a sequence of nonnegative real numbers going to 0 as $n \to \infty$, and let v_{ε_n} be the unique classical solution of (3.16) for $\varepsilon = \varepsilon_n$.

Theorem 3.13. *The sequence* $\{v_{\varepsilon_n}\}$ *contains a subsequence converging uniformly on* $\bar{\Omega}$ *to a harmonic function* v. *In addition*, $v \in C^\alpha(\bar{\Omega})$ *for some* $\alpha \in (0, 1]$.

Proof. By Lemma 3.9, parts (a) and (b), the sequence $\{v_{\varepsilon_n}\}$ is uniformly bounded on $\bar{\Omega}$. From a theorem of Nadirashvili (see [26, Theorem 1.1]) there exist positive constants C and α (not depending on n) such that

$$\|v_{\varepsilon_n}\|_{C^\alpha(\bar{\Omega})} \leqslant C(\|v_{\varepsilon_n}\|_{C(\Omega)} + \|\partial v_{\varepsilon_n}/\partial \mathbf{n}\|_{L^\infty(\partial\Omega)}).$$

Therefore, by Lemma 3.9 and this estimate we can conclude that for all n there exist positive constants C and α such that

$$\|v_{\varepsilon_n}\|_{C^\alpha(\bar{\Omega})} \leqslant C$$

and this implies that the sequence $\{v_{\varepsilon_n}\}$ is equicontinuous at each point of $\bar{\Omega}$. By the Ascoli theorem and since the limit of a uniformly convergent sequence of harmonic functions is harmonic we then conclude the first part of this theorem. Finally, since

$$|v_{\varepsilon_n}(x) - v_{\varepsilon_n}(y)| \leqslant C\,|x - y|^\alpha$$

and since $v_{\varepsilon_n}(x) \to v(x)$ and $v_{\varepsilon_n}(y) \to v(y)$ as $n \to \infty$, then $v \in C^\alpha(\bar{\Omega})$. $\square$

Theorem 3.14. *The function* v *in Theorem* 3.13 *is a viscosity solution of* (3.1).

For convenience we here recall the definition of *viscosity solution* of a boundary value problem of the form

$$\begin{cases} F(x, u, \nabla u, D^2u) = 0 & \text{in } \Omega, \\ G(x, u, \nabla u) = 0 & \text{on } \partial\Omega. \end{cases} \tag{3.20}$$

(The following definition of viscosity solution is the one given by Barles in [5, Definition I.1].)

Definition 3.1. An upper-semicontinuous (resp. lower-semicontinuous) function u is said to be a viscosity subsolution of (3.20) (resp. a viscosity supersolution) if the following property holds: for all $\phi \in C^2(\bar{\Omega})$, at each maximum point $x_0 \in \bar{\Omega}$ of $u - \phi$, we have

$$\begin{aligned}
&F(x_0, u(x_0), \nabla\phi(x_0), D^2\phi(x_0)) \leqslant 0 \quad \text{if } x_0 \in \Omega,\\
&\min(F(x_0, u(x_0), \nabla\phi(x_0), D^2\phi(x_0)), G(x_0, u(x_0), \nabla\phi(x_0))) \leqslant 0\\
&\text{if } x_0 \in \partial\Omega.
\end{aligned} \tag{3.21}$$

(resp. for all $\phi \in C^2(\bar{\Omega})$, at each minimum point $x_0 \in \bar{\Omega}$ of $u - \phi$, we have

$$\begin{aligned}
&F(x_0, u(x_0), \nabla\phi(x_0), D^2\phi(x_0)) \geqslant 0 \quad \text{if } x_0 \in \Omega,\\
&\max(F(x_0, u(x_0), \nabla\phi(x_0), D^2\phi(x_0)), G(x_0, u(x_0), \nabla\phi(x_0))) \geqslant 0\\
&\text{if } x_0 \in \partial\Omega.)
\end{aligned} \tag{3.22}$$

A function u is said to be a viscosity solution of (3.20) iff its upper-semicontinuous envelope (i.e. the smallest upper-semicontinuous function $\geqslant u$) is a viscosity subsolution and its lower-semicontinuous envelope is a viscosity supersolution.

Remark 3.6. If $F \equiv -\Delta$, a function $u \in \mathscr{H}(\Omega)$ is automatically a viscosity solution inside Ω (that is to say, the first inequalities of (3.21) and (3.22) are satisfied). If, in addition, $u \in C(\bar{\Omega})$, then u is a viscosity solution of (3.20) (with $F \equiv -\Delta$) if:

for all $\phi \in C^2(\bar{\Omega})$, at each maximum (resp. minimum) point $x_0 \in \bar{\Omega}$ of $u - \phi$, we have

$$G(x_0, u(x_0), \nabla\phi(x_0)) \leqslant 0 \quad \text{if } x_0 \in \partial\Omega \text{ and } \Delta\phi(x_0) < 0,$$

(resp.

$$G(x_0, u(x_0), \nabla\phi(x_0)) \geqslant 0 \quad \text{if } x_0 \in \partial\Omega \text{ and } \Delta\phi(x_0) > 0.)$$

Proof of Theorem 3.14. Since $v \in \mathscr{H}(\Omega) \cap C(\bar{\Omega})$ we may take into account Remark 3.6, and then we shall prove that if $x_0 \in \partial\Omega$ is a maximum point of $v - \phi$ (where ϕ is an arbitrary $C^2(\bar{\Omega})$ function) then $G(x_0, u(x_0), \nabla\phi(x_0)) \leqslant 0$ if $\Delta\phi(x_0) < 0$; the proof of "v is a supersolution" being similar. (The proof that follows is based on ideas of Lions (see [20, Theorem 1]).) Let ψ be the function defined as follows:

$$\psi(x) = 1/2(1 - |x|^2), \quad x \in \bar{\Omega}.$$

It should be observed that $\psi \equiv 0$ on $\partial\Omega$, $\psi > 0$ in Ω, $\partial\psi/\partial\mathbf{n} = -1$ on $\partial\Omega$ and $\Delta\psi = -1$ in $\bar{\Omega}$. For any $\delta > 0$, the function $v - \phi - \delta\psi$ has as well a maximum point at $x_0 \in \partial\Omega$. Therefore, for sufficiently large n, the function $v_{\varepsilon_n} - \phi - \delta\psi$ has a maximum point at $x_n \in \bar{\Omega}$ and $x_n \to x_0$ as $n \to \infty$ (here $\{v_{\varepsilon_n}\}$ denotes the sequence converging uniformly on $\bar{\Omega}$ to v

according to Theorem 3.12). In addition, $x_n \in \partial\Omega$. In fact, if on the contrary $x_n \in \Omega$ then

$$0 \leqslant -\Delta(v_{\varepsilon_n} - \phi - \delta\psi)(x_n) = \Delta\phi(x_n) - \delta,$$

but this is not possible if $\Delta\phi(x_0) < 0$ since, for enough large n, $\Delta\phi(x_n) - \delta < \Delta\phi(x_0)$, and this proves our assertion. Therefore, since (at least for enough large n) $x_n \in \partial\Omega$, then

$$\begin{aligned} &G_{\varepsilon_n}(x_n, v_{\varepsilon_n}(x_n), \nabla\phi(x_n) + \delta\nabla\psi(x_n)) \\ &\quad = v_{\varepsilon_n}(x_n) + \frac{\partial\phi}{\partial\mathbf{n}}(x_n) - \delta - \sqrt{\varepsilon_n + (g^2(x_n) - |\nabla_s\phi(x_n)|^2)_+} \\ &\quad \leqslant v_{\varepsilon_n}(x_n) + \frac{\partial v_{\varepsilon_n}}{\partial\mathbf{n}}(x_n) - \sqrt{\varepsilon_n + (g^2(x_n) - |\nabla_s v_{\varepsilon_n}(x_n)|^2)_+} = 0, \end{aligned}$$

where we have used the following relations:

$$0 \leqslant \frac{\partial}{\partial\mathbf{n}}(v_{\varepsilon_n} - \phi - \delta\psi)(x_n) = \frac{\partial v_{\varepsilon_n}}{\partial\mathbf{n}}(x_n) - \left(\frac{\partial\phi}{\partial\mathbf{n}}(x_n) - \delta\right)$$

and

$$0 = \nabla_s(v_{\varepsilon_n} - \phi - \delta\psi)(x_n) = \nabla_s v_{\varepsilon_n}(x_n) - \nabla_s\phi(x_n).$$

The proof concludes by letting n go to ∞ and then letting δ go to 0. □

Concerning the uniqueness of solution of problem (3.1) we have.

Theorem 3.15. *Problem* (3.1) *has at most one solution* v.

Proof. Let us start with the case of classical solutions. Let v and w be two solutions of (3.1) and let $z = v - w$. Since $z \in \mathscr{H}(\Omega) \cap C^1(\bar{\Omega})$, it takes its maximum value at some point $x_0 \in \partial\Omega$ and its minimum value at some point $\tilde{x}_0 \in \partial\Omega$. Moreover, $\nabla_s z = 0$ at x_0 and $\tilde{x}_0$, so it follows that at x_0 and $\tilde{x}_0$ we have

$$z + \partial z/\partial\mathbf{n} = 0.$$

Now, we infer that $z(x_0) \leqslant 0$ and $z(\tilde{x}_0) \geqslant 0$, and this of course implies $z \equiv 0$. In fact, if $z(x_0) > 0$ then $\partial z/\partial\mathbf{n}(x_0) < 0$ but this is not possible at a maximum point; on the other hand, if $z(\tilde{x}_0) < 0$ then $\partial z/\partial\mathbf{n}(\tilde{x}_0) > 0$ which is not possible at a minimum point. This completes the proof of this lemma for classical solutions. Since the used arguments only use the maximum principle and the points where the functions attain their maximum the extension to viscosity solutions is standard. □

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