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ON THE COMPLEX GINZBURG–LANDAU EQUATION WITH A DELAYED FEEDBACK

A. C. CASAL

Dept. de Matemática Aplicada, E.T.S. Arquitectura,
Univ. Politécnica de Madrid, 28040 Madrid, Spain
acasal@aq.upm.es

J. I. DÍAZ

Dept. de Matemática Aplicada, Fac. de Matemáticas,
Univ. Complutense de Madrid, 28040 Madrid, Spain
ji_diaz@mat.ucm.es

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We show how to stabilize the *uniform oscillations* of the *complex Ginzburg–Landau equation* with periodic boundary conditions by means of some global delayed feedback. The proof is based on an abstract *pseudo-linearization principle* and a careful study of the spectrum of the linearized operator.

Keywords: Complex Ginzburg–Landau equation; non-local delayed control; principle of pseudo-linearized stability.

AMS Subject Classification: 35B35, 39B32, 47H06

1. Introduction. Main Result

We study the stabilization of the uniform oscillations for the *complex Ginzburg–Landau equation* by means of some global delayed feedback. The domain is given by $\Omega = (0, L_1) \times (0, L_2)$. We define the faces of the boundary

$$\Gamma_j = \partial\Omega \cap \{x_j = 0\}, \quad \Gamma_{j+2} = \partial\Omega \cap \{x_j = L_j\}, \qquad j = 1, 2,$$

on which we assume periodic boundary conditions and, so, the problem under study can be formulated as

$$(P_1) \quad \begin{cases} \dfrac{\partial \mathbf{u}}{\partial t} - (1 + i\varepsilon)\Delta\mathbf{u} = (1 - i\omega)\mathbf{u} - (1 + i\beta)\,|\mathbf{u}|^2\,\mathbf{u} \\ \qquad\qquad\qquad + \mu e^{i\chi_0}\mathbf{F}(\mathbf{u}, t, \tau) & \Omega \times (0, +\infty), \\ \left.\mathbf{u}\right|_{\Gamma_j} = \left.\mathbf{u}\right|_{\Gamma_{j+2}}, & \\ \left(-\left.\dfrac{\partial \mathbf{u}}{\partial \mathbf{n}}\right|_{\Gamma_j} =\right) \left.\dfrac{\partial \mathbf{u}}{\partial x_j}\right|_{\Gamma_j} = \left.\dfrac{\partial \mathbf{u}}{\partial x_j}\right|_{\Gamma_{j+2}} \left(= \left.\dfrac{\partial \mathbf{u}}{\partial \mathbf{n}}\right|_{\Gamma_{j+2}}\right) & \left.\right\} \partial\Omega \times (0, +\infty), \\ \mathbf{u}(x, s) = \mathbf{u}_0(x, s) & \Omega \times [-\tau, 0], \end{cases}$$

where $\mathbf{n}$ is the outgoing normal unit vector, and

$$\mathbf{F}(\mathbf{u},t,\tau)=[m_1\mathbf{u}(t)+m_2\bar{\mathbf{u}}(t)+m_3\mathbf{u}(t-\tau,x)+m_4\bar{\mathbf{u}}(t-\tau)]$$

with

$$\bar{\mathbf{u}}(s)=\frac{1}{|\Omega|}\int_\Omega \mathbf{u}(s,x)dx.$$

Here the parameters $\varepsilon,\beta,\omega,\mu,\chi_0,m_i$ and τ are real numbers, in contrast with the solution $u(x,t)=u_1(x,t)+iu_2(x,t)$. We point out that most of our results remain true for N-dimensional domains (with $N>2$) as well as for Neumann boundary conditions (a previous study dealing with the one-dimensional case was carried out in Ref. 14).

This type of equations (called as of Stuart–Landau in the absence of the diffusion term) arises in the study of the stability of reaction diffusion equations such as $\frac{\partial \mathbf{X}}{\partial t}-D\Delta X=f(X:\eta)$ where $X:\Omega\times(0,+\infty)\to\mathbb{R}^n$ and η is a real scalar parameter when the deviation $\mathbf{v}$ from the uniform state solution $\mathbf{X}_\infty$ is developed asymptotically in terms of some multiple scales (see Kuramoto[21]). Coefficient ε measures the degree to which the diffusion matrix $\mathbf{D}$ deviates from a scalar.

Notice that the presence of complex coefficients introduces important differences with the classical Ginzburg–Landau equations arising in superconductivity (Bethuel[10]).

With the basis of a sound experimental work, many recent studies of a more descriptive nature, but of a great originality and interest have been written. In those studies the delay term $\mathbf{F}(\mathbf{u},t,\tau)$ has been taken corresponding to $m_4=1,m_i=0$ for $i=1,2,3$ and introduced as a control mechanism (see Battogtokh,[7] Mertens[23]). Our main goal is to carry out a rigorous analysis of those studies. We also want to investigate the possibility of controlling the turbulence by using other terms (see Remark 4). In particular our treatement does not use the Fourier transform, apparently hard to be rigorously justified in this setting.

We focus our attention on the so-called *slowly varying complex amplitudes* defined by $\mathbf{u}(x,t)=\mathbf{v}(x,t)e^{-i\omega t}$. Thus, $\mathbf{v}$ satisfy (P_2):

$$(P_2)\quad\begin{cases}\dfrac{\partial\mathbf{v}}{\partial t}-(1+i\varepsilon)\Delta\mathbf{v}=\mathbf{v}-(1+i\beta)|\mathbf{v}|^{\mathbf{2}}\mathbf{v}\\ \qquad\qquad+\mu e^{i\chi_0}\left[m_1\mathbf{v}+m_2\bar{\mathbf{v}}\right.\\ \qquad\qquad\left.+e^{i\omega\tau}(m_3\mathbf{v}(t-\tau,x)+m_4\bar{\mathbf{v}}(t-\tau))\right] & \text{in } \Omega\times(0,+\infty),\\ \left.\begin{array}{l}\mathbf{v}|_{\Gamma_j}=\mathbf{v}|_{\Gamma_{j+2}},\\ \left(-\left.\dfrac{\partial\mathbf{v}}{\partial\mathbf{n}}\right|_{\Gamma_j}=\right)\left.\dfrac{\partial\mathbf{v}}{\partial x_j}\right|_{\Gamma_j}=\left.\dfrac{\partial\mathbf{v}}{\partial x_j}\right|_{\Gamma_{j+2}}\left(=\left.\dfrac{\partial\mathbf{v}}{\partial\mathbf{n}}\right|_{\Gamma_{j+2}}\right),\end{array}\right\} & \text{on } \partial\Omega\times(0,+\infty),\\ \mathbf{v}(x,s)=\mathbf{u}_0(x,s)e^{i\omega s} & \text{on } \Omega\times[-\tau,0].\end{cases}\tag{1.1}$$

We study the stability of *uniform oscillations*, i.e. special solutions of (P_2) of the form $\mathbf{v}_{\mathrm{uosc}}(x,t) = \rho_0 e^{-i\theta t}$ which determines completely ρ_0 and θ. As we shall see, the only effect of the delay τ is that it controls the effective phase shift $\chi(\tau)$.

In the absence of delay ($\tau = 0$), and for $|\Omega| = +\infty$ and $\mu = 0$, it is known (see Kuramoto[21] and Mertens[23]) that the *Benjamin–Feir condition* $\beta < -\frac{1}{\varepsilon}$ implies the instability of such uniform oscillations. Here we shall assume merely that

$$\beta \leq 0 \quad \text{and} \quad \varepsilon \geq 0 \tag{1.2}$$

and we shall prove that this instability holds, in the absence of delay, for $L < +\infty$ once $\chi_0 \in (\frac{\pi}{2}, \frac{3\pi}{2})$ and $\mu > \frac{1}{|\cos\chi_0|}$. Moreover, we shall also prove that when $\tau > 0$ is suitably chosen then the uniform oscillation becomes linearly stable. We point out that the above stabilization phenomenon requires a nonzero complex component perturbation (notice that χ_0 cannot be zero) and that it applies to the case of $\mu > 0$ and $\varepsilon = \beta = \omega = 0$.

We start by pointing out that the existence and uniqueness of a solution of (P_1) can be proven once we assume that $\mathbf{u}_0 \in \mathbf{C}([-\tau, 0] : \mathbf{L}^2(\Omega))$ (see Ref. 18).

We are interested in the stability analysis of the time-periodical function $\mathbf{v}_{\mathrm{uosc}}(x,t) = \rho_0 e^{-i\theta t}$. In order to avoid the application of techniques for the study of the stability of periodic solutions we can reduce the study to the stability of stationary solutions of some auxiliary problem by introducing the change of unknown $\mathbf{z}(x,t) = \mathbf{v}(x,t)e^{i\theta t}$ where $\mathbf{v}(x,t)$ is a solution of (P_2). Thus $\mathbf{z}(x,t)$ satisfies

$$(P_3)\quad \begin{cases} \dfrac{\partial \mathbf{z}}{\partial t} - (1+i\varepsilon)\Delta\mathbf{z} = (1+i\theta)\mathbf{z} - (1+i\beta)|\mathbf{z}|^2\mathbf{z} \\ \qquad\qquad + \mu e^{i\chi_0}\big[m_1\mathbf{z} + m_2\bar{\mathbf{z}} + e^{i(\omega+\theta)\tau}(m_3\mathbf{z}(t-\tau,x) \\ \qquad\qquad + m_4\bar{\mathbf{z}}(t-\tau))\big] & \text{in } \Omega\times(0,+\infty), \\ \left.\begin{aligned} &\mathbf{z}|_{\Gamma_j} = \mathbf{z}|_{\Gamma_{j+2}}, \\ &\left(-\left.\frac{\partial\mathbf{z}}{\partial\mathbf{n}}\right|_{\Gamma_j} =\right)\left.\frac{\partial\mathbf{z}}{\partial x_j}\right|_{\Gamma_j} = \left.\frac{\partial\mathbf{z}}{\partial x_j}\right|_{\Gamma_{j+2}} \left(= \left.\frac{\partial\mathbf{z}}{\partial\mathbf{n}}\right|_{\Gamma_{j+2}}\right), \end{aligned}\right\} & \text{on } \partial\Omega\times(0,+\infty), \\ \mathbf{z}(x,s) = \mathbf{u}_0(x,s)e^{i(\omega-\theta)s} & \text{on } \Omega\times[-\tau,0]. \end{cases} \tag{1.3}$$

Now, $\mathbf{v}_{\mathrm{uosc}}(x,t) = \rho_0 e^{-i\theta t}$ is a uniform oscillation if and only if $\mathbf{z}(x,t) = \mathbf{v}_{\mathrm{uosc}}(x,t)e^{i\theta t} = \mathbf{z}_\infty = \rho_0$ is a stationary solution of (P_3), i.e.

$$\begin{aligned} \mathbf{0} &= (1+i\theta)\mathbf{z}_\infty - (1+i\beta)|\mathbf{z}_\infty|^2\mathbf{z}_\infty \\ &\quad + \mu e^{i\chi_0}\big[m_1 + m_2 + e^{i(\omega+\theta)\tau}(m_3+m_4)\big]\mathbf{z}_\infty. \end{aligned} \tag{1.4}$$

In order to keep some resemblance with Battogtokh[7] we shall assume that

$$m_1 + m_2 = 0 \quad \text{and} \quad m_3 + m_4 = 1. \tag{1.5}$$

Then we get the expressions $\rho_0(\tau) = (1 + \mu\cos\chi(\tau))^{1/2}$, where $\chi(\tau) = \chi_0 + (\omega + \theta(\tau))\tau$ and with $\theta(\tau)$ given as the solution of the implicit equation

$$\theta = \beta - \mu(\sin(\chi_0 + (\omega+\theta)\tau) - \beta\cos(\chi_0 + (\omega+\theta)\tau)). \tag{1.6}$$

Notice that if $\mu = 0$ we deduce that $\rho_0(\tau) = 1$ and that $\theta(\tau) = \beta$ for any τ and that $\rho_0(0) = (1 + \mu\cos\chi_0)^{1/2}, \theta(0) = \beta - \mu(\sin\chi_0 - \beta\cos\chi_0)$. It is not difficult to prove (see Sec. 3 below) the existence and uniqueness of such a function $\theta(\tau)$ and that $\theta \in C^1$.

Our main stabilization result is the following:

Theorem 1.1. *Assume* (1.2), (1.5), $\chi_0 \in (\pi, \frac{3\pi}{2})$,

$$3 - m_1 - 2m_3 \geq 0, \quad m_1 + m_3 \geq 0, \quad 3 + 2m_3 > 0, \tag{1.7}$$

$$\mu > \max\left\{ \frac{1}{|\cos\chi_0|}, \frac{3\beta - \omega + 3(\omega + \beta)\sin\chi_0 + \cos\chi_0}{5(-\beta)\sin\chi_0\cos\chi_0 + 1}, \right.$$
$$\left. \frac{m_3(3\beta - \omega - \varepsilon\frac{\pi^2}{L^2}) + 3(\omega + \beta)\sin\chi_0 + (m_1 + m_3)\cos\chi_0}{(3 - m_1 - 2m_3)\sin^2\chi_0 + (m_1 + m_3)\cos^2\chi_0 + (-\beta)(3 + 2m_3)\sin\chi_0\cos\chi_0} \right\}.$$

Then there exists some $\tau_0 \in (0, 1)$ *such that if we assume* $\tau \in (\tau_0, 1)$ *we get that*

$$|\mathbf{v}(x, t) - \rho_0| \leq Me^{-\alpha t}\|\mathbf{u}_0(\cdot, \cdot)e^{i\omega\cdot} - \rho_0\|.$$

For the proof we shall first introduce a new and quite general *pseudo-linearization principle.* Then, we shall show the applicability of it to the delayed problem and, at the end, we shall study the eigenvalues of the linear part to find the range of parameters for the stability of the linear part.

2. Some Abstract Results: Pseudo-Linearization Principle

We are interested in the study of the stabilization, as $t \to \infty$, of the solutions of the nonlinear abstract functional differential equation

$$\begin{cases} \dfrac{du}{dt}(t) + Au(t) + Bu(t) \ni F(u_t(.)) & \text{in } X, \\ u(s) = u_0(s) & s \in [-\tau, 0], \end{cases} \tag{2.1}$$

on a Banach space X, where

$$u_t(\theta) = u(t + \theta), \quad \theta \in [-\tau, 0],$$

to the associated equilibria: $w \in D(A) \subset D(B) \subset X$ such that

$$Aw + Bw \ni F(\hat{w}(.)),$$

where $\hat{w} \in C := C([-\tau, 0] : X)$ is the function which takes constant values equal to w. Our main goal is to extend, to a broad class of nonlinear operators A, the usual linearized stability principle saying, roughly speaking, that for the special case of A linear (single valued) and B and F are differentiable, the asymptotic stability of the zero solution of the linearized equation,

$$\begin{cases} \dfrac{dv}{dt}(t) + Av(t) + DB(w)v(t) = DF(\hat{w})v_t(.) & \text{in } X, \\ v(s) = u_0(s) & s \in [-\tau, 0], \end{cases} \tag{2.2}$$

implies that $u(t : u_0) \to w$ as $t \to \infty$, at least if $u_0(.)$ is close enough to $\hat{w}$. We point out that our results seem to be new even without the delayed and nonlocal term (i.e. for $F \equiv 0$).

The motivation to keep A nonlinear after the process of linearization (the reason why we used the term of *pseudo-linearization principle*) comes from the fact that if we use the representation for the unknown of the delayed nonlinear equation (P_3) as $\mathbf{z}(x,t) = \rho(x,t)e^{i\phi(x,t)}$, then we arrive at a coupled nonlinear system of delayed equations for ρ and ϕ which can be described in terms of the representation operator given by $\mathbf{P} : \mathbb{R}^2 \to \mathbb{C}$, $\mathbf{P}(\rho,\phi) = \rho e^{i\phi}$. Indeed, notice that $\mathbf{P}$ is nonlinear and that if $\mathbf{q} = (\rho,\ \phi)$ then $\mathbf{z}(x,t) = \mathbf{P}(\mathbf{q}(x,t))$ and the (P_3) can be formulated as $\frac{d\mathbf{P}(\mathbf{q}(\cdot,t))}{dt} + A\mathbf{P}(\mathbf{q}(\cdot,t)) + B\mathbf{P}(\mathbf{q}(\cdot,t)) = F(\mathbf{P}(\mathbf{q}(\cdot))_t)$. By using that the matrix $\mathbf{C}(\mathbf{q}(\cdot,t)) = \operatorname{grad}\mathbf{P}(\mathbf{q}(\cdot,t))$ is not singular, we can arrive to the simpler formulation

$$\frac{d\mathbf{q}}{dt}(\cdot,\ t) + \mathbf{C}(\mathbf{q}(\cdot,t))^{-1}[A\mathbf{P}(\mathbf{q}(\cdot,\ t)) + B\mathbf{P}(q(\cdot,\ t))] = \mathbf{C}(\mathbf{q}(\cdot,t))^{-1}F(\mathbf{P}(\mathbf{q}(\cdot))_t). \tag{2.3}$$

Notice that, although this delayed system can also be (formally) linearized (this is the procedure followed in Battogtokh and Mikhailov[7] and Mertens *et al.*[23] the above diffusion operator $\mathbf{C}(\mathbf{q}(\cdot,\ t))^{-1}A\mathbf{P}(\mathbf{q}(\cdot,\ t))$ becomes now quasilinear on $\mathbf{q}$ and thus the mathematical justification is much more delicate.

There are some other linearization principles in the literature. Their motivation is usually a particular problem, but its applicability is wider. Close to ours we can mention that of Ruess,[26] although the formulation, scope and proof are different. Besides its applicability to the problem in this work, ours can also be applied to the case in which A is nondifferentiable and nonlinear, among many others (see Casal and Díaz[13]).

We point out that some relevant examples of nonlinear functional equations arise in the most different contexts (see, for instance, Díaz and Hetzer[17] for one example in Climatology, Chukwu[15] for a family of examples dealing with the wealth of nations and the general exposition made in Hale[19]).

Coming back to the abstract formulation, the structural assumptions we shall assume in this paper are the following:

(H1): $A \in \mathcal{A}(\omega : X)$, for some $\omega \in \mathbb{C}$, with

$$\mathcal{A}(\omega : X) = \{A : D_X(A) \subset X \to \mathcal{P}(X), \\ \text{such that } A + \omega I \text{ is an } m\text{-accretive operator}\}.$$

(See Brezis[16] for the case of $X = H$ a Hilbert space and the works by Benilan, Crandall, Pazy[9] and others for the case of a general Banach space and Vrabie[28].)

(H2): The operator semigroup $T(t) : \overline{D_x(A)}^X \to X$, $t \geq 0$, generated by A, is compact (see Vrabie[28]).

(H3): $B \in \mathcal{A}(0 : X)$, B is single valued, Fréchet differentiable, and B is dominated by A; i.e.

$$D_X(A) \subset D_X(B) \quad \text{and} \quad |Bu| \leq k|A^0u| + \sigma(|u|) \tag{2.4}$$

for any $u \in D_X(A)$, some $k < 1$ and some continuous function $\sigma : \mathbb{R} \to \mathbb{R}$.

Here and in what follows, $|.|$ denotes the norm in the space X (in contrast with the norm in space C which will be denoted by $\|.\|$ if there is no ambiguity, when handling two spaces X and Y the corresponding norms will be indicated), $|A^0u| := \inf\{|\xi| : \xi \in Au\}$ for $u \in D_X(A)$.

(H4): $F : C \to X$ satisfies a local Lipschitz condition, i.e. for any $R > 0$, and $\phi, \psi \in C$, with $\|\phi\|, \|\psi\| \leq R$, there exists $L(R) > 0$ such that

$$|F(\phi) - F(\psi)| \leq L(R)\|\phi - \psi\|. \tag{2.5}$$

(H5): There exists $\delta^F > 0$ such that $F : B^X_{\delta^F}(\hat{w}) \to X$ is Fréchet differentiable with the Fréchet derivative $DF(\hat{w})$ given by $D(F(\hat{w}))\phi = \int_{-\tau}^0 d\eta(\theta)\phi(\theta), \phi \in C$, for $\eta : [-\tau, 0] \to B(X, X)$ of bounded variation and the Fréchet derivative is locally Lipschitz continuous, where $B^X_{\delta^F}(\hat{w}) = \{\phi \in C; \|\phi - \hat{x}\| < \delta^F\}$.

We further assume the main condition of our arguments:

(H6): The operator $y \to Ay + By - DF(\hat{w})(e^{\omega\cdot}y)$ belongs to $\mathcal{A}(\omega : X)$, for some $\omega \in \mathbb{C}$ with $\mathrm{Re}\ \omega = \gamma < 0$ where $e^{\omega\cdot}v \in C$ is defined by

$$(e^{\omega\cdot}v)(s) = e^{\omega s}\hat{v}(s), \quad \text{with } \hat{v}(s) = v, \text{ for any } s \in [-\tau, 0], \text{ for } v \in X. \tag{2.6}$$

In order to treat the case in which B is differentiable we introduce the conditions

(H7): There exists a Banach space Y and there exists $\delta^B > 0$ such that B is Fréchet differentiable as function from $B_{\delta^B}(w) = \{z \in D(B); |w - z| < \delta^B\}$ into Y, with the Fréchet derivative $DB(w)$ locally Lipschitz continuous,

and

(H8): The operator $y \to Ay + DB(w)y - DF(\hat{w})(e^{\omega^*}y)$ belongs to $\mathcal{A}(\omega^* : Y)$, for some $\omega^* \in \mathbb{C}$ with $\mathrm{Re}\ \omega^* = \gamma^* < 0$.

A concrete statement of the *pseudo-linearization principle* is the following:

Theorem 2.1. *Assume* (H1)–(H6). *Then there exists* $\alpha > 0$, $\varepsilon > 0$ *and* $M \geq 1$ *such that if* $u_0 \in B^X_\varepsilon(\hat{w})$, $u_0(s) \in D_X(B)$ *for any* $s \in [-\tau, 0]$ *then the solution*

$u(\cdot : u_0)$ of (2.1) *exists on* $[-\tau, +\infty)$ *and*

$$|u(t : u_0) - w| \leq Me^{-\alpha t}\|u_0 - \hat{w}\|, \quad \textit{for any } t > 0. \tag{2.7}$$

Moreover, if we also assume (H7), *that* (H1)–(H5) *holds on the space Y and* (H8) *then there exist $\alpha^* > 0$, $\varepsilon^* \in (0, \varepsilon]$ and $M^* \geq 1$ such that if $u_0 \in B^{X \cap Y}_{\varepsilon^*}(\hat{w})$, $u_0(s) \in D_X(B) \cap D_Y(B)$ for any $s \in [-\tau, 0]$ then, for any $t > 0$,*

$$|u(t : u_0) - w|_X + |u(t : u_0) - w|_Y \leq M^* e^{-\alpha^* t}\big(\|u_0 - \hat{w}\|_X + \|u_0 - \hat{w}\|_Y\big). \tag{2.8}$$

Proof. From assumptions (H4) and (H5)

$$F(\phi) = F(\hat{w}) + DF(\hat{w})(\phi - \hat{w}) + G^F(\hat{w}, \phi), \quad \text{for any } \phi \in B^X_{\delta^F}(\hat{w}).$$

Moreover, since $DF(\hat{w})$ is locally Lipschitz continuous, there exists a continuous increasing function b^F_X such that

$$|G^F(\hat{w}, \phi)| \leq b^F_X(\|\phi - \hat{w}\|)\|\phi - \hat{w}\|, \quad \text{for any } \phi \in B^X_{\delta^F}(\hat{w}). \tag{2.9}$$

Then

$$\frac{du}{dt}(t) - \frac{dw}{dt} + Au(t) - Aw + Bu(t) - Bw - DF(\hat{w})(u_t - \hat{w}) \ni -G^F(\hat{w}, u_t). \tag{2.10}$$

We now use assumption (H6). We claim that we can find a constant $K \geq 1$ such that

$$\|u_t - \hat{w}\| \leq Ke^{\gamma t}\|u_0 - \hat{w}\| + \int_0^t Ke^{\gamma(t-s)}|G^F(\hat{w}, u_s)|\, ds. \tag{2.11}$$

Indeed, as $u(t)$ and w are "integral solutions" in the sense of Benilan (see, e.g., Benilan, Crandall, Pazy[9]), then, by (H6), if we multiply (2.10) by $u(t) - w$ (by using the usual semi inner-bracket $[,]$: see, for instance Benilan, Crandall and Pazy[9] or Vrabie[28] (Sec. 1.4)) we get that

$$|u(t) - w| \leq Ke^{\gamma(t-t_0)}|u(t_0) - w| + \int_{t_0}^t Ke^{\gamma(t-s)}|G^F(\hat{w}, u_s)|\, ds \tag{2.12}$$

for any $t \geq t_0 \geq 0$ (see, for instance, Benilan, Crandall and Pazy[9] or Vrabie[28] Theorem 1.7.5). Then,

$$|u(t) - w| \leq Ke^{\gamma t}\|u_0 - \hat{w}\| + \int_0^t Ke^{\gamma(t-s)}|G^F(\hat{w}, u_s)|\, ds \tag{2.13}$$

for any $t \geq 0$. Finally, since (2.13) holds trivially for $t \in [-\tau, 0]$ we get (2.11) by taking the maximum, in (2.12), on intervals of the form $[t - \tau, t]$ for any $t \geq 0$. Now, let $R \in (0, \delta^F)$ be chosen so that

$$b^F_X(R) < (-\gamma)/(4K). \tag{2.14}$$

Define $\varepsilon = \min\{R/(2K), \delta^F_X\}$. Let us show that if $u_0 \in B^X_\varepsilon(\hat{w})$ then the associated solution u of (2.1) exists and $\|u_t - \hat{w}\| < R$ for all $t \geq 0$. Thanks to assumption

(H2) we can apply some maximal continuation results (see, for instance, Chap. 3 of Vrabie[28], or Chap. 2 of Wu[29] when A is linear), it suffices to show that there exists no $t_1 > 0$ so that $\|u_{t_1}\| = R$ and $\|u_t\| < R$ for $t \in [0, t_1)$. By contradiction, if there exists such a t_1, then on $[0, t_1]$ we have

$$\begin{aligned}\|u_t - \hat{w}\| &\leq Ke^{\gamma t}\|u_0 - \hat{w}\| + \int_0^t Ke^{\gamma(t-s)}|G^F(\hat{w}, u_s)|\, ds \\ &\leq Ke^{\gamma t}\|u_0 - \hat{w}\| + 2Kb_X^F(R)\int_0^t e^{\gamma(t-s)}\|u_s - \hat{w}\|\, ds.\end{aligned}$$

In particular, at $t = t_1$ we have

$$\|u_{t_1} - \hat{w}\| \leq K\varepsilon + \frac{2Kb_X^F(R)}{(-\gamma)}R \leq R,$$

a contradiction to the choice of t_1. Finally, to end the proof, let $u_0 \in B_\varepsilon^X(\hat{w})$, $u_0(s) \in D_X(B)$ for any $s \in [-\tau, 0]$ and let u be the associated solution of (2.1). Since we have shown that $\|u_t - \hat{w}\| \leq R$ for all $t \geq 0$ we get that

$$\|u_t - \hat{w}\| \leq Ke^{\gamma t}\|u_0 - \hat{w}\| + Kb_X^F(R)\int_0^t e^{\gamma(t-s)}\|u_s - \hat{w}\|\, ds \tag{2.15}$$

holds for all $t \geq 0$. Thus, by using the Gronwall's inequality, we get

$$\begin{aligned}\|u_t - \hat{w}\| &\leq Ke^{[\gamma - Kb(R)]t}\|u_0 - \hat{w}\| \\ &\leq Ke^{(\gamma/2)t}\|u_0 - \hat{w}\|, \quad u_0 \in B_\varepsilon^X(\hat{w})\end{aligned}$$

which shows (2.7). In order to show the decay estimate (2.8), we repeat the same arguments as before but now on the space Y. Then, from assumptions (H3) on Y and (H7), there exist δ_Y^F and δ_X^B such that

$$\begin{aligned}B(z) &= B(w) + DB(w)(z - w) + G^B(w, z), \quad \text{for any } z \in B_{\delta_X^B}(w), \\ F(\phi) &= F(\hat{w}) + DF(\hat{w})(\phi - \hat{w}) + G^F(\hat{w}, \phi), \quad \text{for any } \phi \in B^Y_{\delta_Y^F}(\hat{w}).\end{aligned}$$

where now

$$\begin{aligned}B_{\delta_X^B}(w) &= \{z \in D_X(B) \cap D_Y(B); |w - z| < \delta_X^B\}, \\ B_{\delta_Y^F}(\hat{w}) &= \{\phi \in C; \|\phi - \hat{x}\|_Y < \delta_Y^F\},\end{aligned}$$

and, as before, $\|.\|_Y$ denotes the norm on the space $C_Y := C([-\tau, 0] : Y)$. Moreover, there exist two continuous increasing functions b_X^B and b_Y^F such that

$$|G^B(w, z)|_Y \leq b_X^B(|w - z|)|w - z|, \qquad \text{for any } z \in B_{\delta_X^B}(w), \tag{2.16}$$

$$|G^F(\hat{w}, \phi)|_Y \leq b_Y^F(\|\phi - \hat{w}\|_Y)\|\phi - \hat{w}\|_Y, \quad \text{for any } \phi \in B_{\delta_Y^F}(\hat{w}). \tag{2.17}$$

Now

$$\begin{aligned}\frac{du}{dt}(t) - \frac{dw}{dt} &+ Au(t) - Aw + DB(w)(u(t) - w) \\ &- DF(\hat{w})(u_t - \hat{w}) \ni G^B(w, u(t)) - G^F(\hat{w}, u_t).\end{aligned} \tag{2.18}$$

Thus, by using (H8) and arguing as in the first part we get that there exists a constant $K^* \geq 1$ such that

$$\|u_t - \hat{w}\|_Y \leq K^* e^{\gamma^* t} \|u_0 - \hat{w}\|_Y + \int_0^t K^* e^{\gamma^*(t-s)} (|G^B(w, u(s))|_Y + |G^F(\hat{w}, u_s)|_Y) ds, \tag{2.19}$$

and then, by taking $\delta = \min\left(\delta_X^B, \delta_Y^F\right)$ and $R^* \in (0, \delta)$ such that

$$\max\left(b_X^B(R^*), b_Y^F(R^*)\right) < (-\gamma)/(4K), \tag{2.20}$$

we obtain that

$$\|u_t - \hat{w}\|_Y \leq K^* e^{\gamma^* t} \|u_0 - \hat{w}\|_Y + K^* \int_0^t e^{\gamma^*(t-s)} \left(b_X^B(R^*) \|u_s - \hat{w}\|_X + b_Y^F(R^*) \|u_s - \hat{w}\|_Y\right) ds. \tag{2.21}$$

We define $\tilde{R} = \min(R, R^*)$, $\tilde{K} = \max(K, K^*)$, $\tilde{\gamma} = \max(\gamma, \gamma^*) < 0$ and $\varepsilon^* = \min\{\tilde{R}/(2\tilde{K}), \delta\}$. Then, if $u_0 \in B_{\varepsilon^*}^{X \cap Y}(\hat{w})$, $u_0(s) \in D_X(B) \cap D_Y(B)$ for any $s \in [-\tau, 0]$ and we assume, for instance, that $\tilde{\gamma} = \gamma$, by adding (2.15) and (2.21) we deduce that

$$\begin{aligned} \|u_t - \hat{w}\|_X + \|u_t - \hat{w}\|_Y \leq \tilde{K} e^{\tilde{\gamma} t} \left(\|u_0 - \hat{w}\|_X + e^{(\gamma^* - \gamma)t} \|u_0 - \hat{w}\|_Y\right) \\ + \tilde{K} \int_0^t e^{\tilde{\gamma}(t-s)} \left[\left(b_X^F(\tilde{R}) + b_X^B(\tilde{R}) e^{(\gamma^* - \gamma)t}\right) \|u_s - \hat{w}\|_X \right. \\ \left. + b_Y^F(R^*) e^{(\gamma^* - \gamma)t} \|u_s - \hat{w}\|_Y\right] ds \end{aligned}$$

and the estimate (2.8) follows, again, by Gronwall's inequality. □

Remark 2.1. It is not difficult to show that the assumption (H8) is implied (when A is linear) by the condition: "if $\lambda \in \mathbb{C}$ is given so that there exists $y \in D(B)\backslash\{0\}$ such that $Ay + DB(w)y - \lambda y \ni DF(\hat{w})(e^{\lambda \cdot} y)$, then $\mathrm{Re}\, \lambda > 0$". This allows one to see Theorem 4.1 of Wu[29] (see also Parrot[24] and its references) as a special case of our abstract result with $B = 0$. In that case the "variation of the constants formula" can be used to get a different proof of the theorem since A is linear. Notice that if $B \neq 0$ and $D(B) \subsetneq X$ then the arguments of the proof of Wu[29] do not work (in spite of what is claimed in Example 4.8 given there).

Remark 2.2. When A is linear, as in the case without delay, assumption (H7) implies that the zero solution of the linearized problem $\frac{dU}{dt}(t) + AU(t) + DB(w)U(t) - DF(\hat{w})U_t(.) = 0$ in X, is locally asymptotically stable (Wu[29]).

Remark 2.3. It is possible to prove the existence of global solutions for a general class of initial data (not necessarily near $\hat{w}$) by using that $A + B \in \mathcal{A}(\omega : X)$, for some $\omega \in \mathbb{C}$, some truncation of the nonlocal term $F(u_t)$ and passing to the limit by the compactness of the semigroup generated by A (see Vrabie[28] for some related results).

An easy adaptation of the above proof leads to the following linearization result (now on a possibly smaller neighborhood of w) when A is differentiable.

Theorem 2.2. *The conclusion of the above result remains true if we assume, in addition, that condition* (H7) *also holds for A and we replace condition* (H8) *by*

(H9): *the operator $y \to DA(w)y + DB(w)y - DF(\hat{w})(e^{\omega \cdot} y)$ belongs to $\mathcal{A}(\omega)$, for some $\omega \in \mathbb{C}$ with* $\mathrm{Re}\,\omega = \gamma < 0$.

Remark 2.4. We claim that our arguments keeping A nonlinear after linearizing the rest of the terms (and in particular the way in which we apply Gronwall inequality) allow to extend, to the case of quasilinear equations, the so-called "method of quasilinearization" which, introduced by Bellman and Kalaba,[8] we used to find solutions of a parabolic semilinear problem through the iteration of solutions of the linearized equation when starting in a super- and a sub-solution of the original semilinear problem (see, e.g., Lakshmikantham and Leela,[22] Carl and Lakshmikantham[12] and their references therein). This will be the subject of a future work by us.

3. The Complex Ginzburg–Landau Equation

3.1. *Applications of the abstract results*

Motivated by the special form of the nonlinear term of the equation in (P_3) we shall take $X = \mathbf{L}^4(\Omega)$ and $Y = \mathbf{L}^{4/3}(\Omega)$ (notice that, in contrast with the case of scalar equations (see Parrot[24]), the space $\mathbf{L}^\infty(\Omega)$ is not suitable space to check assumption (H1): see Auscher, Barthelemy and Bénilan.[6] A detailed analysis of the associated diffusion operator is a consequence of some previous results in the literature: see, for instance, Amann.[4] Notice that the operator $A\mathbf{u}$ can be formulated matricially as

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \to \begin{pmatrix} \Delta & -\varepsilon\Delta \\ \varepsilon\Delta & \Delta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}.$$

So, if $\varepsilon \neq 0$ the diffusion matrix has a nonzero antisymmetric part. In particular, A is the generator of a semigroup of contractions $\{T(t)\}_{t\geq 0}$ on X and the compactness of the semigroup is a consequence of the compactness of the inclusion $D(A) \subset X$ (notice that, since $N = 2$, $\mathbf{W}^{1,4}(\Omega) \subset \mathbf{W}^{1,4/3}(\Omega) \subset \mathbf{C}(\bar{\Omega})$ with compact imbedding) and some regularity results for nonsymmetric systems.

Concerning the rest of the terms of the equation in (P_3), we define $B\mathbf{u} = (1+i\beta)|\mathbf{u}|^2\mathbf{u}$ with $D(B) = \mathbf{L}^{12}(\Omega)$. By using the characterization of the semi-inner-braket $[,]$ for the spaces $L^p(\Omega)$ (see, for instance Benilan, Crandall and Pazy[9] it is easy to see that $\mathbf{B}$ verifies (H3). Moreover, by the results on the Fréchet differentiability of Nemitsky operators (see Theorem 2.6 (with $p = 4$) of Ambrosetti and Prodi[5] we get that (H7) holds, with $DB(\mathbf{y})\mathbf{v} = 3(1 + i\beta)|\mathbf{y}|^2\mathbf{v}$, if we take $Y = \mathbf{L}^{4/3}(\Omega)$. It can be found in the above-mentioned reference that assumption (H7) does not hold if we take $X = Y = \mathbf{L}^2(\Omega)$.

The nonlocal term is defined by

$$F(\mathbf{u}_t) = (1+i\theta)\mathbf{u}(t) + \mu e^{i\chi_0}\big[m_1\mathbf{u}(t) + m_2\bar{\mathbf{u}}(t) \\ + e^{i(\omega+\theta)\tau}(m_3\mathbf{u}(t-\tau) + m_4\bar{\mathbf{u}}(t-\tau))\big],$$

is locally Lipschitz continuous and its Fréchet derivative is given by

$$DF(\hat{\mathbf{y}})\mathbf{v}(t) = -(1+i\theta)\mathbf{v}(t) - \mu e^{i\chi_0}\big[m_1\mathbf{v}(t) + m_2\bar{\mathbf{v}}(t) \\ - e^{i(\omega+\theta)\tau}(m_3\mathbf{v}(t-\tau) - m_4\bar{\mathbf{v}}(t-\tau))\big] \tag{3.1}$$

since for any $\phi \in C$, the nonlocal operator $\phi \to \frac{1}{|\Omega|}\int_\Omega \phi(s)dx$ is linear and we can write $DF(\hat{\mathbf{y}})\phi = \int_{-\tau}^{0} d\eta(s)\phi(s)$, with

$$d\eta(s)\mathbf{v}(s) = \delta_0(s)(1+i\theta)\mathbf{v}(s) + \mu e^{i\chi_0}\big[\delta_0(s)(m_1\mathbf{v}(s) + m_2\bar{\mathbf{v}}(s)) \\ + e^{i(\omega+\theta)\tau}\delta_{-\tau}(s)(m_3\mathbf{v}(s) + m_4\bar{\mathbf{v}}(s))\big] \tag{3.2}$$

for any $\mathbf{v} \in C([-\tau,\infty) : \mathbf{L}^4(\Omega))$ and any $s \in [-\tau,\infty)$, where $\delta_0(s), \delta_{-\tau}(s)$ denote the Dirac delta at the points $s=0$ and $s=-\tau$ respectively. By well-known results, we have that $\eta : [-\tau,0] \to B(X,X)$ has a bounded variation and so, conditions (H4) and (H5) hold (and analogously replacing X by Y).

Finally, assumption (H6) can be read as a condition on the stationary state $\mathbf{y}$ (a study of the eigenvalue of operator A can be found, for instance, in Temam[27]).

Remark 3.1. By introducing the representation operator $\mathbf{P} : \mathbb{R}^2 \to \mathbb{C}$, $\mathbf{P}(\rho,\phi) = \rho e^{i\phi}$ it is clear that the quasilinear operator $A\mathbf{P}(\mathbf{q})$ obtained from the operator $A\mathbf{u} = -(1+i\varepsilon)\Delta\mathbf{u}$ satisfies also condition $A \in \mathcal{A}(\omega)$ (since $\mathbf{P}$ is merely a change of variables). We point out that,

$$A\mathbf{P}(\mathbf{q}) = -(1+i\varepsilon)[\Delta\rho - \rho|\nabla\phi|^2 + i(2\nabla\rho\cdot\nabla\phi + \rho\Delta\phi)]e^{i\phi}.$$

Then, the "formal linearization" of the operator $\mathbf{E}(\mathbf{q}) := A\mathbf{P}(\mathbf{q})$ at $\mathbf{q}^*(x,y) := \mathbf{y} \equiv \rho_0$ becomes

$$D\mathbf{E}(\mathbf{q}^*)(\rho e^{i\phi}) = -(1+i\varepsilon)[\Delta\rho + i\rho_0\Delta\phi]e^{i\phi}.$$

Notice that the linearization of $\mathbf{C}(\mathbf{q})^{-1}A\mathbf{P}(\mathbf{q})$ needs a slight modification of the above linear expression. □

3.2. *Study of the eigenvalues of the linearized problem*

In this section we shall study the eigenvalues $\lambda \in \mathcal{C}$, $\lambda = a+ib$ of the linearized problem and, which is crucial, we look for

$$\begin{cases} \text{any } \lambda \in \mathcal{C} \text{ such that } \exists\ v \in D(A),\ v \neq 0, & \text{such that} \\ 0 = \lambda v + Av + DB(w)v - DF(\hat{w})(e^{\lambda\cdot}v), & \text{and } \operatorname{Re}\lambda < 0, \end{cases} \tag{3.3}$$

where $e^{\lambda\cdot}v \in C$ is defined by

$$(e^{\lambda\cdot}v)(s) = e^{\lambda s}\hat{v}(s), \quad \text{with } \hat{v}(s) = v, \text{ for any } s \in [-\tau,0]. \tag{3.4}$$

As in the case without delay, (3.3) implies that the zero solution of the linearized problem $\frac{dU}{dt}(t) + AU(t) + DB(w)U(t) - DF(\hat{w})U_t(.) = 0$ in X, is locally asymptotically stable (Wu[29]).

We go back now to the problems 1.1 and 1.3, and recall the expresions (1.4)–(1.6)

$$\theta = \beta - \mu(\sin(\chi_0 + (\omega+\theta)\tau) - \beta\cos(\chi_0 + (\omega+\theta)\tau)). \tag{3.5}$$

Notice that if $\mu = 0$ we deduce that $\rho_0(\tau) = 1$ and that $\theta(\tau) = \beta$ for any τ and that $\rho_0(0) = (1+\mu\cos\chi_0)^{1/2}, \theta(0) = \beta - \mu(\sin\chi_0 - \beta\cos\chi_0)$. It is not difficult to prove (see the following proposition) the existence and uniqueness of such a function $\theta(\tau)$ and that $\theta \in C^1$.

Proposition 3.1. *There exists a unique function $\theta(\tau)$ such that*

$$\theta(\tau) - \beta + \mu(\sin(\chi_0 + (\omega+\theta(\tau))\tau) - \beta\cos(\chi_0 + (\omega+\theta(\tau))\tau)) = 0$$

for any $\tau \in [0,1]$. Moreover, $\theta \in C^1$.

Proof. It is enough to see, by the implicit function theorem, that $\theta(\tau)$ is characterized as the (unique) solution of the Cauchy problem associated to the ODE

$$\frac{d\theta}{d\tau}(\tau) = \frac{-[\mu(\cos(\chi_0 + (\omega+\theta(\tau))\tau)(\omega+\theta) + \beta\sin(\chi_0 + (\omega+\theta(\tau))\tau))](\omega+\theta(\tau))}{1+\mu(\cos(\chi_0 + (\omega+\theta(\tau))\tau)\tau + \beta\sin(\chi_0 + (\omega+\theta(\tau))\tau))\tau}.$$

□

We recall that in our case, $\mathbf{z}_\infty = \rho_0$ and so we can arrive at the linear problem

$$(P_4)\begin{cases} -(1+i\varepsilon)\Delta\mathbf{z} = -(a+ib)\mathbf{z} + [(1+i\theta) - 3(1+i\beta)\rho_0^2)]\mathbf{z} \\ \qquad\qquad + \mu e^{i\chi_0}\left[m_1\mathbf{z} + m_2\bar{\mathbf{z}} + e^{-a\tau + i(\omega+\theta-b)\tau}(m_3\mathbf{z} + m_4\bar{\mathbf{z}})\right] & \text{in } \Omega, \\ \dfrac{\partial\mathbf{z}}{\partial\mathbf{n}} = \mathbf{0} & \text{on } \partial\Omega. \end{cases}$$

As usual, the linear structure of the equation leads to the search of nontrivial solutions $\mathbf{z}(x)$ of the form $\mathbf{A_k}w_{\mathbf{k}}^j(x)$, with $j = 1,2$, where $w_{\mathbf{k}}^j(x)$ are the eigenfunctions for the usual Laplacian operator Δ with periodic boundary conditions on $\Omega = (0,L_1)\times(0,L_2)$. We recall that the eigenvalues of this problem are given by

$$\lambda_0^0 = 0, \quad \lambda_{\mathbf{k}}^0 = 4\pi\left(\frac{k_1^2}{L_1^2} + \frac{k_2^2}{L_2^2}\right), \quad k_1, k_2 \in \mathbb{N}$$

with the associate eigenfunctions

$$w_0 = \frac{1}{\sqrt{|\Omega|}}, \quad w_{\mathbf{k}}^1 = \sqrt{\frac{2}{|\Omega|}}\cos 2\pi\mathbf{kx}, \quad w_{\mathbf{k}}^2 = \sqrt{\frac{2}{|\Omega|}}\sin 2\pi\mathbf{kx}, \qquad \text{with } |\Omega| = L_1L_2,$$

where we have written $\mathbf{kx} := \left(\frac{k_1}{L_1}x_1 + \frac{k_2}{L_2}x_2\right)$ (see, e.g., Temam[27]).

The following general lemma will be used in the study of $\mathbf{z}(x)$.

Lemma 3.1. *Let A be a self-adjoint operator on $L^2(\Omega)$ and let $\{\varphi_n\}$ be a family of eigenfunctions associated to the different eigenvalues $\left\{\lambda_n^0\right\}$. Assume that $\lambda_0^0 = 0$ is an eigenvalue and that $\varphi_0 = 1$ is an eigenfunction associated to λ_0. Then*

$$\int_\Omega \varphi_n = 0 \quad \textit{for any } n \neq 0.$$

Proof. It is enough to recall that $\int_\Omega \varphi_n \varphi_m = 0$ *for any $n \neq m$* since $\lambda_n^0 \neq \lambda_m^0$.

$$\lambda_n^0 \int_\Omega \varphi_n \varphi_m = \int_\Omega A\varphi_n \varphi_m = \int_\Omega \varphi_n A\varphi_m = \lambda_m^0 \int_\Omega \varphi_n \varphi_m.$$

Then taking $m = 0$ we get the conclusion. □

In order to keep a coherent notation with the one used in Battogtokh[7] we introduce the notation $\lambda_{\mathbf{k}} = a_{\mathbf{k}} + ib_{\mathbf{k}}$ for the real and imaginary parts of the eigenvalues of the problem stated in (H8). Notice that, by the previous lemma, $\int_\Omega w_{\mathbf{k}}^j = 0$ for any $\mathbf{k} \neq 0$ and $j = 1, 2$. Then we get that

$$\begin{aligned}(a_{\mathbf{k}} + ib_{\mathbf{k}}) - (1 + i\varepsilon)(-\lambda_{\mathbf{k}}) = (1 + i\theta) - 3(1 + i\beta)\rho_0^2 + \mu e^{i\chi_0}\big[m_1 + m_2\delta_{0\mathbf{k}} \\ + e^{-a\tau + i(\omega+\theta-b)\tau}(m_3 + m_4\delta_{0\mathbf{k}})\big],\end{aligned}$$

where $\delta_{0\mathbf{k}}$ denotes the Kronecker delta function. We arrive at

$$\begin{cases} a_{\mathbf{k}} = -\lambda_{\mathbf{k}}^0 - 2 - 3\mu\cos\chi(\tau) + \mu(m_1 + m_2\delta_{0k})\cos\chi_0 \\ \qquad + \mu e^{-a_{\mathbf{k}}\tau}(m_3 + m_4\delta_{0k})\cos(\chi_0 + (\omega + \theta - b_{\mathbf{k}})\tau), \\ b_{\mathbf{k}} = \theta - \varepsilon\lambda_{\mathbf{k}}^0 - 3\beta(1 + \mu\cos\chi) + \mu(m_1 + m_2\delta_{0k})\sin\chi_0 \\ \qquad + \mu e^{-a_{\mathbf{k}}\tau}(m_3 + m_4\delta_{0k})\sin(\chi_0 + (\omega + \theta - b_{\mathbf{k}})\tau). \end{cases} \tag{3.6}$$

The previous equations are transcendent and we cannot get an explicit expression for the real and imaginary part of the eigenvalues (for some similar transcendent equations arising in delayed ODEs see Hale[19]).

Now, we focus our attention on the dependence of $a_{\mathbf{k}}$ and $b_{\mathbf{k}}$ with respect to τ. So, by the regularity of the involved functions we can assume

$$a_{\mathbf{k}} = a_{\mathbf{k}0} + a_{\mathbf{k}1}\tau + o(\tau), \quad b_{\mathbf{k}} = b_{\mathbf{k}0} + b_{\mathbf{k}1}\tau + o(\tau),$$

as we get, for instance, by a “formal” series development in powers of τ argument. Here we used the Landau notation ($f(\tau) = o(\tau)$ means that $\frac{f(\tau)}{\tau} \to 0$ when $\tau \to 0$).

The terms of order zero in τ are obtained by making $\tau = 0$ in (3.6)

$$\begin{cases} a_{\mathbf{k}0} = -\left(2 + \lambda_{\mathbf{k}}^0\right) + \mu\cos\chi_0(m_1 + m_2\delta_{0\mathbf{k}} + m_3 + m_4\delta_{0\mathbf{k}}), \\ b_{\mathbf{k}0} = 4\beta - \varepsilon\lambda_{\mathbf{k}}^0 + 3\mu\beta\cos\chi_0 + \mu\sin\chi_0(m_1 + m_2\delta_{0\mathbf{k}} + m_3 + m_4\delta_{0\mathbf{k}}). \end{cases} \tag{3.7}$$

So, we can state a first result concerning the case without any delay:

Proposition 3.2. *Assume* $\tau = 0$, $\chi_0 \in \left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$, *and* $\mu > \frac{1}{|\cos\chi_0|}$. *Then the uniform oscillation* $v_{\rm uosc}(x,t) = \rho_0 e^{-i\theta t}$ *is linearly unstable.*

Proof. From (3.7) we see that $a_{00} > 0$ and since $\tau = 0$ we get the existence of at least one eigenvalue λ of the linearized problem with $\mathrm{Re}(\lambda) > 0$ which implies the result. $\square$

The first-order terms in τ are calculated below.

Lemma 3.2. *We have*

$$\begin{aligned} a_{\mathbf{k}1} = \left[\frac{da_{\mathbf{k}}}{d\tau}\right]_{\tau=0} &= \left(2+\lambda_{\mathbf{k}}^0\right) + \mu\big[3(\omega+\beta)\sin\chi_0 + (m_3+m_4\delta_{0\mathbf{k}})\left(3\beta - \varepsilon\lambda_{\mathbf{k}}^0 - \omega\right)\big] \\ &\quad + \mu^2\big\{-3\sin^2\chi_0 + 3\beta\sin\chi_0\cos\chi_0 \\ &\quad + (m_3+m_4\delta_{0\mathbf{k}})\big[\sin^2\chi_0 + 2\beta\sin\chi_0\cos\chi_0 \\ &\quad + (m_1+m_2\delta_{0\mathbf{k}}+m_3+m_4\delta_{0\mathbf{k}})\big]\left(\sin^2\chi_0 - \cos^2\chi_0\right)\big\}. \end{aligned} \tag{3.8}$$

Proof. Differentiating in (3.6) we get that

$$\begin{aligned} a_{\mathbf{k}1} &= \left[\frac{da_{\mathbf{k}}}{d\tau}\right]_{\tau=0} = \left[3\mu\sin\chi(\tau)\frac{d\chi}{d\tau}\right]_{\tau=0} \\ &\quad + \big[(-a_{\mathbf{k}})\mu e^{-a_{\mathbf{k}}\tau}(m_3+m_4\delta_{0\mathbf{k}})\cos(\chi_0 + (\omega+\theta-b_{\mathbf{k}})\tau)\big]_{\tau=0} \\ &\quad - \big[\mu e^{-a_{\mathbf{k}}\tau}(m_3+m_4\delta_{0\mathbf{k}})\sin(\chi_0+(\omega+\theta-b_{\mathbf{k}})\tau)\big]_{\tau=0}\left[\frac{d(\omega+\theta-b_{\mathbf{k}})\tau}{d\tau}\right]_{\tau=0} \\ &= (3\mu\sin\chi_0)(\omega+\beta-\mu(\sin\chi_0-\beta\cos\chi_0)) \\ &\quad - \left(-\left(2+\lambda_{\mathbf{k}}^0\right) + \mu\cos\chi_0(m_1+m_2\delta_{0\mathbf{k}}+m_3+m_4\delta_{0\mathbf{k}})\right)\mu(m_3+m_4\delta_{0\mathbf{k}})\cos\chi_0 \\ &\quad - \mu(m_3+m_4\delta_{0\mathbf{k}})(\omega+\beta-\mu(\sin\chi_0-\beta\cos\chi_0)-b_{\mathbf{k}})\sin\chi_0. \end{aligned}$$

Thus, by using the expression for $b_{\mathbf{k}}$ (see (3.6)) we obtain that

$$\begin{aligned} a_{\mathbf{k}1} &= (3\mu\sin\chi_0)(\omega+\beta-\mu(\sin\chi_0-\beta\cos\chi_0)) \\ &\quad - \left(-\left(2+\lambda_{\mathbf{k}}^0\right) + \mu\cos\chi_0(m_1+m_2\delta_{0\mathbf{k}}+m_3+m_4\delta_{0\mathbf{k}})\right)\mu(m_3+m_4\delta_{0\mathbf{k}})\cos\chi_0 \\ &\quad - \mu(m_3+m_4\delta_{0\mathbf{k}})(\omega+\beta-\mu(\sin\chi_0-\beta\cos\chi_0))\sin\chi_0 \\ &\quad + (3\mu\sin\chi_0)(\omega+\beta-\mu(\sin\chi_0-\beta\cos\chi_0)) \\ &\quad - \left(-\left(2+k^2\right) + \mu\cos\chi_0(m_1+m_2\delta_{0\mathbf{k}}+m_3+m_4\delta_{0\mathbf{k}})\right)\mu(m_3+m_4\delta_{0\mathbf{k}})\cos\chi_0 \\ &\quad - \mu(m_3+m_4\delta_{0\mathbf{k}})(\omega+\beta-\mu(\sin\chi_0-\beta\cos\chi_0))\sin\chi_0 \\ &\quad + \mu(m_3+m_4\delta_{0\mathbf{k}})(4\beta-\varepsilon\lambda_{\mathbf{k}}^0+3\mu\beta\cos\chi_0 \\ &\quad + \mu\sin\chi_0(m_1+m_2\delta_{0\mathbf{k}}+m_3+m_4\delta_{0\mathbf{k}}))\sin\chi_0. \end{aligned}$$

In consequence

$$\begin{aligned} a_{\mathbf{k}1} = &\big(2+\lambda^0_{\mathbf{k}}\big) + \mu\big(3(\omega+\beta)\sin\chi_0 \\ &- (m_3+m_4\delta_{0\mathbf{k}})(\omega+\beta) + (4\beta-\varepsilon\lambda^0_{\mathbf{k}})(m_3+m_4\delta_{0\mathbf{k}})\big) \\ &- \mu^2\big(3\sin\chi_0(\sin\chi_0-\beta\cos\chi_0) \\ &+ \cos^2\chi_0(m_1+m_2\delta_{0\mathbf{k}}+m_3+m_4\delta_{0\mathbf{k}})(m_3+m_4\delta_{0\mathbf{k}})\big) \\ &+ \mu^2(m_3+m_4\delta_{0\mathbf{k}})\big[(\sin\chi_0-\beta\cos\chi_0)\sin\chi_0 \\ &+ (3\beta\cos\chi_0+\sin\chi_0(m_1+m_2\delta_{0\mathbf{k}}+m_3+m_4\delta_{0\mathbf{k}}))\big]\sin\chi_0 \end{aligned}$$

which proves the result. $\square$

Proposition 3.3. *Assume* (1.2), $\chi_0 \in \big(\pi, \frac{3\pi}{2}\big)$, (1.5) *and*

$$\mu > \max\left\{0, \frac{3\beta-\omega+3(\omega+\beta)\sin\chi_0+\cos\chi_0}{5(-\beta)\sin\chi_0\cos\chi_0+1}\right\}.$$

Then $a_{00}+a_{01}<0$.

Proof. By using (3.7), (3.8) and (1.5) we get

$$a_{00}+a_{01} = \mu[(3\beta-\omega+3(\omega+\beta)\sin\chi_0+\cos\chi_0) - \mu(5(-\beta)\sin\chi_0\cos\chi_0+1)].$$

Then, the assumptions imply the positivity of the coefficient of μ^2 and the result holds. $\square$

Proposition 3.4. *Assume* (1.2), $\chi_0 \in (\pi, \frac{3\pi}{2})$, (1.7) *and*

$$\mu > \max\left\{0, \frac{m_3\left(3\beta-\omega-\varepsilon 4\pi\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}\right)\right)+3(\omega+\beta)\sin\chi_0+(m_1+m_3)\cos\chi_0}{(3-m_1-2m_3)\sin^2\chi_0+(m_1+m_3)\cos^2\chi_0+(-\beta)(3+2m_3)\sin\chi_0\cos\chi_0}\right\}.$$

Then, for any k, $a_{\mathbf{k}0}+a_{\mathbf{k}1}<0$. *Moreover, for any* $k\neq 0$ *and any* $\tau\in(0,1]$,

$$a_{k(n)0}+a_{k(n)1}\tau < a_{k(1)0}+a_{k(1)1}\tau.$$

Proof. By using (3.7), (3.8) and that $0<\lambda^0_{(1,1)}<\lambda^0_{\mathbf{k}}$ for any $\mathbf{k}\in\mathbb{N}^2$, $\mathbf{k}\neq(1,1)$, we obtain that

$$\begin{aligned} a_{\mathbf{k}0}+a_{\mathbf{k}1} = \mu\Bigg[\Bigg(&m_3\left(3\beta-\omega-\varepsilon 4\pi\left(\frac{1}{L_1^2}+\frac{1}{L_2^2}\right)\right) \\ &+3(\omega+\beta)\sin\chi_0+(m_1+m_3)\cos\chi_0\Bigg) \\ &-\mu((3-m_1-2m_3)\sin^2\chi_0+(m_1+m_3)\cos^2\chi_0 \\ &+(-\beta)(3+2m_3)\sin\chi_0\cos\chi_0)\Bigg]. \end{aligned}$$

Again, the assumptions made on the parameters imply the positivity of the coefficient of μ^2 and the result holds. Moreover, $a_{k(n)0}-a_{k(1)0}+(a_{k(n)1}-a_{k(1)1})\tau = -k(n)^2+k(1)^2-(m_3\varepsilon k(n)^2-m_3\varepsilon k(1)^2)\tau<0$. $\square$

The proof of Theorem 1 is now complete since from Propositions 3 and 4 we deduce the existence of some $\tau_0 \in (0,1)$ (independent of $\mathbf{k} \in \mathbb{N}^2$) such that for any $|\mathbf{k}| \geq 0$ we have $a_{\mathbf{k}0} + a_{\mathbf{k}1}\tau < 0$ for any $\tau \in (\tau_0, 1)$. This implies the hypothesis of the abstract result and the conclusion follows.

Remark 3.2. Notice that Theorem 1 applies to the case $m_1 = m_2 = m_3 = 0$ which corresponds to a formulation similar to the one of Ref. 7. Moreover, it also applies to the choice $m_1 = \kappa, m_2 = -1 - \kappa$, $m_3 = 0$ and $m_4 = 1$, for any $\kappa \in (0,1)$ which corresponds to a formulation quite close to the pioneering paper[25] (concerning chaotic ODEs).

Remark 3.3. Since the eigenvalue $\lambda_0^0 = 0$, using lemma it is possible to obtain the same result for Neumann conditions.

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