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Partial Differential Equations

On a nonlinear Schrödinger equation with a localizing effect

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Abstract

We consider the nonlinear Schrödinger equation associated to a singular potential of the form $a|u|^{-(1-m)}u + bu$, for some $m \in (0,1)$, on a possible unbounded domain. We use some suitable energy methods to prove that if $\mathrm{Re}(a) + \mathrm{Im}(a) > 0$ and if the initial and right hand side data have compact support then any possible solution must also have a compact support for any $t > 0$. This property contrasts with the behavior of solutions associated to regular potentials ($m \geqslant 1$). Related results are proved also for the associated stationary problem and for self-similar solution on the whole space and potential $a|u|^{-(1-m)}u$. The existence of solutions is obtained by some compactness methods under additional conditions. ***To cite this article: P. Bégout, J.I. Díaz, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Sur une équation de Schrödinger non-linéaire avec un effet localisant. Nous considérons l'équation de Schrödinger non-linéaire associée à un potentiel singulier de la forme $a|u|^{-(1-m)}u + bu$, avec $m \in (0,1)$, sur un domaine éventuellement non borné. Nous employons des méthodes d'énergie appropriées pour montrer que si $\mathrm{Re}(a) + \mathrm{Im}(a) > 0$ et si les données (initiale et source) ont un support compact alors toute solution doit également avoir un support compact pour tout $t > 0$. Cette propriété contraste avec le comportement des solutions associées aux potentiels réguliers ($m \geqslant 1$). Des résultats similaires sont également établis pour le problème stationnaire associé et pour les solutions auto-similaires sur l'espace entier et le potentiel $a|u|^{-(1-m)}u$. L'existence des solutions est obtenue par des méthodes de compacité sous certaines conditions. ***Pour citer cet article : P. Bégout, J.I. Díaz, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Version française abrégée

Une des modifications principales introduites par la Mécanique Quantique sur la Mécanique Classique est l'impossibilité de localiser l'état (position et vitesse) dans la dynamique d'une particule. Ce fait est relié à l'étude du support des solutions de l'équation de Schrödinger associée. Ce sujet a été en grande partie étudié pour l'équation de Schrödinger linéaire et aussi lorsque l'équation est perturbée avec un potentiel régulier non-linéaire (voir, par exemple, les

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livres de Sulem et Sulem [11] et Cazenave [6]). Le but principal de cette Note est de réaliser une première présentation de l'étude d'une classe des équations de Schrödinger non-linéaires avec un potentiel non-linéaire singulier, qui ne semble pas avoir été étudié auparavant, et qui permet d'obtenir la localisation du support de la solution (contrairement au cas des potentiels réguliers). Plus précisément, nous considérons le problème de Cauchy–Dirichlet (*PSE*) sur un ouvert Ω de $\mathbb{R}^N$ (non nécessairement borné) et son problème stationnaire associé (*SSE*) où, pour fixer les idées, on suppose que la non-linéarité est du type (3). Nous considérons également les solutions auto-similaires de l'équation de Schrödinger non-linéaire homogène (*HSE*). Nos résultats principaux sont les suivants.

Théorème 0.1. *Soient les hypothèses* (3) *avec* $\mathrm{Im}(a) > 0$ *et* $\mathrm{Im}(b) \geqslant 0$ (*respectivement,* $\mathrm{Re}(a)+\mathrm{Im}(a) > 0$ *et* $\mathrm{Re}(b) + \mathrm{Im}(b) \geqslant 0$), *et soit* $F \in L^{\frac{m+1}{m}}(\Omega;\mathbb{C})$ *à support compact. Soit* $u \in W^{2,\frac{m+1}{m}}_{\mathrm{loc}}(\Omega;\mathbb{C})$ *une solution faible de* (*SSE*)*. Alors pour presque tout* $x \in N(u)$, $u(x) = 0$, *où* $N(u)$ *est défini par* (5)*, pour un certain* $\varepsilon > 0$ *ne dépendant que de* N, $\mathrm{Im}(a)$ (*respectivement,* $\mathrm{Re}(a) + \mathrm{Im}(a)$)*, m et* $\|F\|_{L^{\frac{m+1}{m}}(\Omega)}$*. L'ensemble* $N(u)$ *n'est pas vide si, par exemple,* $\Omega = \mathbb{R}^N$ *ou si* $\|F\|_{L^{\frac{m+1}{m}}(\Omega)}$ *est suffisamment petite. Enfin, une telle solution u existe si l'on suppose que* $a \in \mathbb{C}$ *est quelconque,* $\mathrm{Re}(b) > 0$ *et* $\mathrm{Im}(b) = 0$.

Concernant le problème parabolique, nous avons le

Théorème 0.2. *Soient les hypothèses* (3) *avec* $\mathrm{Re}(a) + \mathrm{Im}(a) > 0$ *et* $\mathrm{Re}(b) + \mathrm{Im}(b) \geqslant 0$*. Soient* $f \in L^\infty((0,\infty); L^{\frac{m+1}{m}}(\Omega;\mathbb{C}))$ *et* $u_0 \in L^2(\Omega;\mathbb{C})$ *vérifiant* (6)*. Soit* $u \in C([0,\infty); L^2(\Omega;\mathbb{C}))$ *une solution faible de* (*PSE*)*. Alors pour tout* $t > 0$, $u(t,x) = 0$ *pour presque tout* $x \in N(u)$*, où* $N(u)$ *est défini par* (7)*, pour un certain* $\varepsilon > 0$ *ne dépendant que de* N, $\mathrm{Re}(a) + \mathrm{Im}(a)$*,* m*,* $\|f\|_{L^\infty((0,\infty);L^{\frac{m+1}{m}}(\Omega))}$ *et* $\|u_0\|_{L^2(\Omega)}$*. L'ensemble* $N(u)$ *n'est pas vide si, par exemple,* $\Omega = \mathbb{R}^N$ *ou si* $\|f\|_{L^\infty((0,\infty);L^{\frac{m+1}{m}}(\Omega))} + \|u_0\|_{L^2(\Omega)}$ *est suffisamment petite. Enfin, une telle solution u existe pour tout* $(a,b) \in \mathbb{C}^2$.

Afin de décrire la dynamique de $\operatorname{supp} u(t)$, nous cherchons des solutions *auto-similaires* de (*HSE*), i.e., telles que pour tout $\lambda > 0$, $u(t,x) = \lambda^p u(\lambda^2 t, \lambda x)$, où $p \in \mathbb{C}$. Il est facile de voir que sous l'hypothèse $\mathrm{Re}(p) = -\frac{2}{1-m}$, la transformation $u(t,x) \longmapsto \lambda^p u(\lambda^2 t, \lambda x)$ laisse invariant l'espace $\mathcal{S}'(\mathbb{R}^N;\mathbb{C})$ des distributions tempérées de $\mathbb{R}^N$. De plus, on montre aisément que si $u \in \mathcal{S}'((0,\infty) \times \mathbb{R}^N)$ est une solution de (*HSE*) alors u est auto-similaire si et seulement si u vérifie l'identité (8). Son profil v défini par $v(x) = u(1,x)$ est alors solution de l'équation (*PSSE*).

Nous obtenons le résultat suivant.

Théorème 0.3. *Soit* $0 < m < 1$, *soit* $a \in \mathbb{C}$ *tel que* $\mathrm{Re}(a) > 0$ *et* $\mathrm{Im}(a) > 0$, *et soit* $p \in \mathbb{C}$ *tel que* $\mathrm{Re}(p) = -\frac{2}{1-m}$ *et* $\mathrm{Im}(p) < 0$*. Alors il existe une solution v de* (*PSSE*) *à support compact dans* $\mathbb{R}^N$.

Remarque 1. Dans Rosenau et Schochet [10], les auteurs proposent une équation de Schrödinger quasi-linéaire (et unidimensionnelle) afin d'obtenir des solutions à support compact pour chaque t fixé. Noter que leur équation et les techniques sont très différentes, bien que cette propriété qualitative soit obtenue dans les deux cas.

Remarque 2. Les versions détaillées des démonstrations des Théorèmes 0.1 et 0.2, y compris certaines généralisations, sont l'objet de l'article Bégout et Díaz [3]. L'étude de la solution auto-similaire pour l'équation non-linéaire homogène est réalisée dans Bégout et Díaz [4].

L'existence des solutions dans les Théorèmes 0.1 et 0.2 est obtenue pour les domaines bornés en utilisant la compacité de l'opérateur $(A + b\,\mathrm{Id})^{-1}$, où $(A, D(A))$ est défini par (10), et le fait que l'opérateur défini en (3) est un opérateur sous-linéaire (voir Vrabie [12]). La démonstration de l'existence du profil v satisfaisant (*PSSE*) est réalisée en ramenant le problème à une équation ordinaire d'une manière semblable à celle de l'article de Kavian et Weissler [7].

En ce qui concerne les propriétés localisantes des supports, elles sont obtenues par l'adaptation et la généralisation de différentes méthodes d'énergie comme celles utilisées dans le livre Antontsev, Díaz et Shmarev [2].

1. Introduction and main results

One of the main modifications to the Classical Mechanics introduced by Quantum Mechanics is the impossibility to localize the state (position and velocity) in the dynamics of a particle. This fact is connected with the study of the support of solutions of the associated Schrödinger equation. This subject has been largely studied for the linear Schrödinger equation and also when the equation is perturbed with a nonlinear regular potential (see, for instance, the monographs Sulem and Sulem [11] and Cazenave [6]). The main goal of this Note is to carry out a first presentation of the study of a class of nonlinear Schrödinger equations with a singular nonlinear potential which seems to be considered here by first time and which allows to show the localization of the support of the solution (in contrast to the case of regular potentials). More precisely, we consider

$$(PSE)\quad \begin{cases} \dfrac{\partial u}{\partial t} - \mathrm{i}\Delta u + \big(V_{\mathrm{Re}}(u) + \mathrm{i}V_{\mathrm{Im}}(u)\big)u = f(t,x), & \text{in } (0,\infty)\times\Omega, \\ u = 0, & \text{on } (0,\infty)\times\partial\Omega, \\ u(0,x) = u_0(x), & \text{in } \Omega, \end{cases} \tag{1}$$

and the associated stationary problem

$$(SSE)\quad \begin{cases} -\mathrm{i}\Delta u + \big(V_{\mathrm{Re}}(u) + \mathrm{i}V_{\mathrm{Im}}(u)\big)u = F(x), & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is an open (non necessarily bounded) set of $\mathbb{R}^N$ and, to fix ideas,

$$\begin{cases} V_{\mathrm{Re}}(u) + \mathrm{i}V_{\mathrm{Im}}(u) = a|u|^{-(1-m)} + b, & \text{for any } u \in \mathbb{C}, \\ \text{for some } (a,b) \in \mathbb{C}^2 \text{ and } 0 < m < 1, \end{cases} \tag{3}$$

and the data $f(t,x)$, $u_0(x)$ and $F(x)$ are given in some functional spaces which will be made precise later. We also consider the self-similar solution of the following homogeneous nonlinear Schrödinger equation in $\mathbb{R}^N$,

$$(HSE)\quad \frac{\partial u}{\partial t} - \mathrm{i}\Delta u + a|u|^{-(1-m)}u = 0, \quad \text{in } (0,\infty)\times\mathbb{R}^N, \tag{4}$$

with $a \in \mathbb{C}$ and $0 < m < 1$.

We point out that complex potentials presenting some singularities of different types arise in many different situations (see, for instance, Brezis and Kato [5], LeMesurier [8] and Liskevitch and Stollmann [9], and its references).

As a first statement of the localizing effects inherent to this type of singular potential, we have the following.

Theorem 1.1. *Assume* (3) *with* $\mathrm{Im}(a) > 0$ *and* $\mathrm{Im}(b) \geqslant 0$ (*respectively,* $\mathrm{Re}(a) + \mathrm{Im}(a) > 0$ *and* $\mathrm{Re}(b) + \mathrm{Im}(b) \geqslant 0$)*, and let* $F \in L^{\frac{m+1}{m}}(\Omega;\mathbb{C})$ *with compact support. Let* $u \in W^{2,\frac{m+1}{m}}_{\mathrm{loc}}(\Omega;\mathbb{C})$ *be any weak solution of* (*SSE*)*. Then for almost every* $x \in N(u)$, $u(x) = 0$*, where*

$$N(u) = \big\{x \in \Omega \setminus \operatorname{supp} F;\ \operatorname{dist}(x, \operatorname{supp} F) > \varepsilon\big\}, \tag{5}$$

for some $\varepsilon > 0$ *depending only on* N, $\mathrm{Im}(a)$ (*respectively,* $\mathrm{Re}(a) + \mathrm{Im}(a)$)*,* m *and* $\|F\|_{L^{\frac{m+1}{m}}(\Omega)}$*. The set* $N(u)$ *is not empty if, for instance,* $\Omega = \mathbb{R}^N$ *or if* $\|F\|_{L^{\frac{m+1}{m}}(\Omega)}$ *is small enough. Finally, such a u exists under the assumptions that* $\mathrm{Re}(b) > 0$ *and* $\mathrm{Im}(b) = 0$*, for any* $a \in \mathbb{C}$.

Concerning the parabolic problem we have

Theorem 1.2. *Assume* (3) *with* $\mathrm{Re}(a) + \mathrm{Im}(a) > 0$ *and* $\mathrm{Re}(b) + \mathrm{Im}(b) \geqslant 0$*. Let* $f \in L^\infty((0,\infty); L^{\frac{m+1}{m}}(\Omega;\mathbb{C}))$ *and let* $u_0 \in L^2(\Omega;\mathbb{C})$ *be such that*

$$K \overset{not}{=} (\operatorname{supp} u_0) \cup \Big(\bigcup_{t>0} \operatorname{supp} f(t,\cdot)\Big) \text{ is a compact set of } \mathbb{R}^N. \tag{6}$$

Let $u \in C([0,\infty); L^2(\Omega;\mathbb{C}))$ *be any solution of* (*PSE*)*. Then for any* $t > 0$, $u(t,x) = 0$ *for almost every* $x \in N(u)$*, where*

$$N(u) = \big\{x \in \Omega \setminus K;\ \operatorname{dist}(x, K) > \varepsilon\big\}, \tag{7}$$

for some $\varepsilon > 0$ *depending on* N, $\mathrm{Re}(a) + \mathrm{Im}(a)$, m, $\|f\|_{L^\infty((0,\infty);L^{\frac{m+1}{m}}(\Omega))}$ *and* $\|u_0\|_{L^2(\Omega)}$. *The set* $N(u)$ *is not empty if, for instance,* $\Omega = \mathbb{R}^N$ *or if* $\|f\|_{L^\infty((0,\infty);L^{\frac{m+1}{m}}(\Omega))} + \|u_0\|_{L^2(\Omega)}$ *is small enough. Finally, such a* u *exists for any* $(a,b) \in \mathbb{C}^2$.

In order to describe the dynamics of $\operatorname{supp} u(t)$, we search for *self-similar* solutions of (*HSE*), i.e. such that for any $\lambda > 0$, $u(t,x) = \lambda^p u(\lambda^2 t, \lambda x)$, for some $p \in \mathbb{C}$. It is not difficult to show that the condition $\mathrm{Re}(p) = -\frac{2}{1-m}$ leads invariant (by the transformation $u(t,x) \mapsto \lambda^p u(\lambda^2 t, \lambda x)$) the space $\mathcal{S}'(\mathbb{R}^N;\mathbb{C})$ of tempered distributions on $\mathbb{R}^N$. We can prove that if $u \in \mathcal{S}'((0,\infty)\times\mathbb{R}^N)$ is a solution of (*HSE*) then u is self-similar if and only if

$$u(t,x) = t^{-\frac{p}{2}} v\left(\frac{x}{\sqrt{t}}\right), \tag{8}$$

where $v(x) = u(1,x)$, and its profile v satisfies the equation

$$(PSSE) \quad -\mathrm{i}\Delta v - \frac{p}{2}v - \frac{1}{2}x\cdot\nabla v + a|v|^{m-1}v = 0, \quad \text{in } \mathbb{R}^N. \tag{9}$$

We get

Theorem 1.3. *Let* $0 < m < 1$, *let* $a \in \mathbb{C}$ *be such that* $\mathrm{Re}(a) > 0$ *and* $\mathrm{Im}(a) > 0$, *and let* $p \in \mathbb{C}$ *be such that* $\mathrm{Re}(p) = -\frac{2}{1-m}$ *and* $\mathrm{Im}(p) < 0$. *Then there exists a solution* v *of* (*PSSE*) *with compact support in* $\mathbb{R}^N$.

Remark 1. It can be shown that problem (*SSE*) with assumption (3), for the case $\mathrm{Im}(a) < 0$, can be studied by considering a boundary value problem associated to a fourth-order elliptic equation like in Antontsev, Díaz and Oliveira [1].

Remark 2. In the recent paper of Rosenau and Schochet [10], the authors propose a (one-dimensional) quasilinear Schrödinger equation in order to get solutions with compact support for each t fixed. Notice that their equation and techniques are very different to the ones of this Note although this qualitative property is obtained in both cases.

Remark 3. The detailed proofs of Theorems 1.1 and 1.2, including some generalizations, is the object of the paper Bégout and Díaz [3]. The study of the self-similar solution for the homogeneous nonlinear Schrödinger equation in $\mathbb{R}^N$ is carried out in Bégout and Díaz [4].

2. Sketch of the proofs

The existence of solutions in Theorems 1.1 and 1.2 is previously obtained for bounded domains Ω by using the compactness of the operator $(A + b\,\mathrm{Id})^{-1}$, where A is defined by

$$Au = -\Delta u, \quad \forall u \in D(A) \text{ and } D(A) = \left\{u \in H_0^1(\Omega;\mathbb{C}),\ \Delta u \in L^2(\Omega;\mathbb{C})\right\}, \tag{10}$$

and the fact that $(V_{\mathrm{Re}}(u) + \mathrm{i}V_{\mathrm{Im}}(u))u$ is a sublinear operator (Vrabie [12]). Moreover, we prove that there exists $M > 0$ (independent on u and Ω) such that for any solution $u \in H^1(\Omega;\mathbb{C}) \cap L^{m+1}(\Omega;\mathbb{C})$, we have

$$\|\nabla u\|^2_{L^2(\Omega)} + \|u\|^{m+1}_{L^{m+1}(\Omega)} \leqslant M\|F\|^{\frac{m+1}{m}}_{L^{\frac{m+1}{m}}(\Omega)} \tag{11}$$

(multiply the equation by $\bar{u}$ and use Young's inequality). We also get that $u \in W^{2,\frac{m+1}{m}}(\Omega;\mathbb{C})$ and, in particular, $u \in C^1(\Omega;\mathbb{C})$.

Proof of the existence part of Theorem 1.2. Let $(A, D(A))$ be the operator defined by (10) and let for any $u \in L^2(\Omega;\mathbb{C})$, $\mathrm{C}(u) = a|u|^{-(1-m)}u + bu$. Then $\mathrm{i}A$ generates a compact semigroup of contractions on $L^2(\Omega)$. Moreover $\mathrm{C} \in C(L^2(\Omega);L^2(\Omega))$ and C is bounded on bounded sets. So C satisfies the *local integrability property* and is a *weakly* ∞*-Carathéodory function*. Then, by well-known results (Vrabie [12]), there exist $\tau \in (0,\infty]$ and $u \in C([0,\tau);L^2(\Omega))$ *mild solution* of (*PSE*) on $(0,\tau)\times\Omega$. From the growth of $\mathrm{C}(u)$, it is proved (Theorem 3.2.3 of Vrabie [12]) that, in fact, $\tau = \infty$.

The proof of the existence of the profile v satisfying $(PSSE)$ is carried out by reducing the problem to some ordinary differential equation in a similar way to the paper of Kavian and Weissler [7]. □

Proof of the localization properties of Theorems 1.1, 1.2 and 1.3. Concerning the localizing properties of the statements, they are obtained by the application of different energy methods (Antontsev, Díaz and Shmarev [2]). Then, for the proof of the estimate (5), we take $x_0 \in \Omega$ and $\rho_0 > 0$ such that $\Omega \cap B(x_0,\rho_0) \subset \Omega \setminus \operatorname{supp} F$ so that $F_{|B(x_0,\rho_0)\cap\Omega} \equiv 0$. If $\mathrm{Im}(a) > 0$ and $\mathrm{Im}(b) \geqslant 0$, by multiplying the equation by u (with some truncating and localizing weight), we prove that

$$E(\rho_0) \leqslant \left(\frac{(1-m)\rho_0}{2^\nu C_0}\right)^{\frac{k}{1-m}}, \quad \|u\|_{L^{m+1}(B(x_0,\rho_0))} \leqslant 1$$

or

$$E(\rho_0) \leqslant 1, \quad \|u\|_{L^{m+1}(B(x_0,\rho_0))}^{\frac{2\kappa(m+1)}{k}} \leqslant \frac{(1-m-2\kappa)\rho_0}{2^\nu C_0},$$

where $E(\rho) = \|\nabla u\|^2_{L^2(B(x_0,\rho))}$ is the local energy, $\kappa \in (0, \frac{1-m}{2})$ is small enough to have $\frac{4\kappa}{k} \leqslant \frac{1-m}{k} + \frac{1-m}{1+m}$, $C_0 = C_0(N, m, \mathrm{Im}(a)) > 0$ is the constant given by Theorem 5.1, [2], $k = N(1-m) + 2(1+m)$ and $\nu = \frac{m+1}{k}$. As in Theorem 5.1, Chapter 1 of [2], and using (11), we show that for a.e. $\rho \in (0, \rho_0)$,

$$E'(\rho) \geqslant \Lambda E(\rho)^\alpha, \quad \text{for some } \Lambda > 0 \text{ and } \alpha \in (0,1).$$

In consequence, we get that there exists $\varepsilon \in (0, \rho_0)$ such that $u \equiv 0$ on $\overline{B}(x_0, \varepsilon) \cap \Omega$. If $\mathrm{Re}(a) + \mathrm{Im}(a) > 0$, we multiply the equation by u, $\bar{u}$, and add the resulting inequalities. The proof of the compactness of the support of the profile v of the self-similar solution, in Theorem 1.3 follows from similar arguments.

For the proof of estimate (7), we multiply the equation by u and by its conjugate $\bar{u}$ (with some truncating and localizing weight) and adding the results we get a local energy similar to the one in Theorem 4.2, Chapter 3 of [2], which leads to the result. □

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