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Similarity solutions of an equation describing ice sheet dynamics

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Abstract

This paper focuses upon the derivation of the similarity solutions of a nonlinear equation associated with a free boundary problem arising in glaciology. We present a potential symmetry analysis of this second order nonlinear degenerate parabolic equation related to non-Newtonian ice sheet dynamics in the isothermal case. A full classical and also a non-classical symmetry analysis are presented. After obtaining a general result connecting the thickness function of the ice sheet and the solution of the nonlinear equation (without any unilateral formulation), a particular example of a similarity solution to a problem formulated with Cauchy boundary conditions is described. This allows us to obtain several qualitative properties of the free moving boundary in the presence of an accumulation–ablation function with realistic physical properties.

Keywords: Nonlinear degenerate equations; Ice flow dynamics; Potential symmetries

1. Introduction

In recent years there has been much interest in modelling ice sheet dynamics especially because of its importance in the understanding of global climate change, global energy balance and circulation models. Although various physical theories for large ice sheet motion have been presented there exist still many open questions related to its mathematical treatment. In this paper we consider an obstacle formulation of slow, isothermal, one dimensional ice flow on a rigid bed due to Fowler [1].

The model describing the ice sheet dynamics is formulated in terms of an obstacle problem associated with a one dimensional nonlinear degenerate diffusion equation (see [2]). The original *strong* formulation can be stated in the following terms: let $T > 0$, $L > 0$ be positive fixed real numbers and let $\Omega = (-L, L)$ be an open bounded interval of $\mathbb{R}$ (a sufficiently large, fixed spatial domain). Given an accumulation/ablation rate function $a = a(x, t)$ and a function $f(x, t)$ (a sliding velocity, eventually zero) defined on $Q = (0, T) \times (-L, L)$ (a large, fixed, parabolic domain) and an initial thickness $h_0 = h_0(x) \geq 0$ (bounded and with $h_0(x) > 0$ on its support $I(0) \subset \Omega$), find two curves $S_+, S_- \in C^0([0, T])$, with $S_-(t) \leq S_+(t)$, $I(t) := (S_-(t), S_+(t)) \subset \Omega$ for any $t \in [0, T]$, and a sufficiently smooth function $h(x, t)$ defined on the set $Q_T := \bigcup_{t\in(0,T)} I(t)$ such that

$$(SF) := \begin{cases} h_t = \left[\dfrac{h^{n+2}}{n+2}|h_x|^{n-1}h_x - fh\right]_x + a \quad \text{in } Q_T, \\ h = \left(\dfrac{h^{n+2}}{n+2}|h_x|^{n-1}h_x - fh\right) = 0, \\ \qquad \text{on } \{S_-(t)\} \cup \{S_+(t)\},\ t \in (0, T), \\ h = h_0 \quad \text{on } I(0), \end{cases}$$

and $h(x, t) > 0$ on Q_T. We recall that n denotes the, so called, Glen exponent, and that several constitutive assumptions are admitted, the most relevant case corresponds to $n = 3$ (see, for example Fowler [1]).

Notice that, for each fixed $t \in [0, T]$, $I(t) = (S_-(t), S_+(t)) = \{x \in \Omega : h(x, t) > 0\}$ denotes the ice covered region. The curves $S_\pm(t)$ are called the interface curves or free boundaries associated with the problem and are defined by:

$$S_-(t) = \mathrm{Inf}\{x \in \Omega : h(x, t) > 0\},$$
$$S_+(t) = \mathrm{Sup}\{x \in \Omega : h(x, t) > 0\}.$$

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These curves define the interface separating the regions in which $h(x,t) > 0$ (i.e., ice regions) from those where $h(x,t) = 0$ (i.e. ice free regions). In the physical context they represent the propagation fronts of the ice sheet.

The qualitative description of solutions of this problem is quite difficult due to the doubly nonlinear terms appearing in the differential operator and, especially, to its formulation involving the unknown fronts $S_{\pm}(t)$ (the free boundaries). Nevertheless, some mathematical and numerical results are already available in the literature. So, for instance, the physical problem may be characterized by the following properties as has recently been discussed by Calvo et al. [2]:

1. Given an initial ice sheet initial $h(x,0)$, and known $a(x,t)$, $f(x,t)$ the nonlinear partial differential equation determines $h(x,t)$ over its parabolic positivity set.
2. The ice free region (melt zone) $h(x,t) = 0$ always exists (from the assumptions on $h(x,0)$) and one defines the two free boundaries $S_-(t)$ and $S_+(t)$ which are extended to the interval $[0,T]$ if, for $t \in [0,T]$, $a(x,t) > 0$ on some subinterval of Ω.
3. The more realistic solutions (from a physical point of view) are non-negative solutions $h(x,t) \geq 0$ corresponding to ablation data functions such that $a > 0$ except in a region near the two free boundaries where $a < 0$.

In Section 5 we prove that it is possible to obtain estimates on the ice covered region $I(t)$ and the solution $h(x,t)$ (the thickness of the ice sheet) by means of the comparison with the solution $u(x,t)$ of the nonlinear equation

$$\Psi(x,t,u,u_t,u_x,u_{xx}) \equiv u_t - a - \left[\frac{u^{n+2}}{n+2}|u_x|^{n-1}u_x - fu\right]_x = 0. \tag{1}$$

So, any description of special solutions of Eq. (1) (which do not involve obstacle formulation) leads to useful estimates for the more complex formulation for $h(x,t)$. As a matter of fact, the study of the nonlinear equation (1) is of importance in its own right since the equation arises in many other different contexts (with different values of the exponent n), for instance, filtration in porous media with turbulent regimes, suitable non-Newtonian flow problems, and so on (see, e.g., the monograph [3] and its references).

We emphasize that very few explicit solutions of the ice sheet free boundary formulation are known in the literature. One of them corresponds to a stationary solution due to Paterson [4] and was used as a numerical test in the paper [2]. It corresponds to the special case of no sliding (i.e. $f = 0$) and the following piecewise constant accumulation–ablation function:

$$a(x) = \begin{cases} a_1 & \text{if } 0 \leq |x| < R \\ -a_2 & \text{if } R \leq |x| \leq L, \end{cases}$$

where $L > 1$, $a_1 > 0$, $a_2 > 0$ and $R \in (0,1)$. Moreover, it is assumed that $a_1 R = a_2(1-R)$. Thus, for the particular values $a_1 = 0.01$ and $a_2 = 0.03$, we have the steady state solution

$$h(x) = \begin{cases} H\left[1 - \left(1 + \dfrac{a_1}{a_2}\right)^{1/3} \left(\dfrac{|x|}{L}\right)^{4/3}\right]^{3/8} & \text{if } |x| \leq R \\ H\left(1 + \dfrac{a_2}{a_1}\right)^{1/8} \left(1 - \dfrac{|x|}{L}\right)^{1/2} & \\ \quad \text{if } R \leq |x| \leq 1 & \\ 0 \quad \text{if } 1 \leq |x| \leq L, & \end{cases} \tag{2}$$

where $H = (40\, a_1 R)^{1/8}$ represents the thickness at $x = 0$.

In Sections 2–4 of this paper we shall carry out the study of some special transient solutions of Eq. (1) of a similarity type which are compatible with the above statements. The similarity solutions will be obtained by conducting a Lie or classical symmetry analysis of (1). The method is described in the next section. In addition Bluman et al. [5,6] described how the range of symmetries may be extended whenever a Lie symmetry analysis is conducted on a partial differential equation that may be written in a conserved or a potential form. This is the case with (1) where the corresponding equivalent ice model may be described in terms of the first order potential system, $\boldsymbol{\Psi} \equiv (\Psi_1, \Psi_2) = \mathbf{0}$ where

$$\Psi_1 = v_x - u + \lambda = 0$$
$$\Psi_2 = v_t - \frac{u^{n+2}|u_x|^{n-1}u_x}{n+2} + fu = 0 \tag{3}$$

for a potential function $v = v(x,t)$ and with $\lambda = \lambda(x,t)$ chosen such that

$$a \equiv \lambda_t. \tag{4}$$

We recall that, as demonstrated in [5,6], the Lie point symmetries of the potential system induce non-Lie contact symmetries for the original partial differential equation. The treatment presented in Sections 3 and 4 is made independently of the positiveness subset of the solution and so it is carried out directly in terms of Eq. (1), without any other requirements on the solution (no study on any free boundaries is made in these sections). An application to the strong formulation of the free boundary problem, for some concrete data, is given in Section 5.

2. Potential symmetry analysis of the ice sheet equation

In the classical Lie group method, one-parameter infinitesimal point transformations, with group parameter ε are applied to the dependent and independent variables (x,t,u,v). In this case the transformations, including that of the potential variable are

$$\bar{x} = x + \varepsilon\eta_1(x,t,u,v) + O\left(\varepsilon^2\right)$$
$$\bar{t} = t + \varepsilon\eta_2(x,t,u,v) + O\left(\varepsilon^2\right)$$
$$\bar{u} = u + \varepsilon\phi_1(x,t,u,v) + O\left(\varepsilon^2\right)$$
$$\bar{v} = v + \varepsilon\phi_2(x,t,u,v) + O\left(\varepsilon^2\right) \tag{5}$$

and the Lie method requires form invariance of the solution set:

$$\Sigma \equiv \{u(x,t), v(x,t), \boldsymbol{\Psi} = 0\}. \tag{6}$$

This results in a system of overdetermined, linear equations for the infinitesimal η_1, η_2, ϕ_1 and ϕ_2. The corresponding Lie algebra of symmetries is the set of vector fields

$$\mathcal{X} = \eta_1 (x, t, u, v) \frac{\partial}{\partial x} + \eta_2 (x, t, u, v) \frac{\partial}{\partial t} + \phi_1 (x, t, u, v) \frac{\partial}{\partial u} + \phi_2 (x, t, u, v) \frac{\partial}{\partial v}. \quad (7)$$

The condition for invariance of (1) is the equation

$$\mathcal{X}_E^{(1)} (\boldsymbol{\Psi})|_{\Psi_1=0, \Psi_2=0} = 0 \quad (8)$$

where the first prolongation operator $\mathcal{X}_E^{(1)}$ is written in the form

$$\mathcal{X}_E^{(2)} = \mathcal{X} + \phi_1^{[t]} \frac{\partial}{\partial u_t} + \phi_1^{[x]} \frac{\partial}{\partial u_x} + \phi_2^{[t]} \frac{\partial}{\partial v_t} + \phi_2^{[x]} \frac{\partial}{\partial v_x} \quad (9)$$

where $\phi_1^{[t]}$, $\phi_1^{[x]}$ and $\phi_2^{[t]}$, $\phi_2^{[x]}$ are defined through the transformations of the partial derivatives of u and v. In particular to the first order in ε:

$$\bar{u}_{\bar{x}} = u_x + \varepsilon \phi_1^{[x]} (x, t, u, v) \qquad \bar{u}_{\bar{t}} = u_t + \varepsilon \phi_1^{[t]} (x, t, u, v)$$
$$\bar{v}_{\bar{x}} = v_x + \varepsilon \phi_2^{[x]} (x, t, u, v) \qquad \bar{v}_{\bar{t}} = v_t + \varepsilon \phi_2^{[t]} (x, t, u, v). \quad (10)$$

Once the infinitesimals are determined the symmetry variables may be found from the conditions for invariance of surfaces $u = u(x, t)$ and $v = v(x, t)$:

$$\Omega_1 = \phi_1 - \eta_1 u_x - \eta_2 u_t = 0$$
$$\Omega_2 = \phi_2 - \eta_1 v_x - \eta_2 v_t = 0. \quad (11)$$

In the following both Macsyma and Maple software have been used to calculate the determining equations. In the case of the ice equation (3) there are nine overdetermined linear determining equations. From these equations it may be shown that:

$$\eta_1 = \eta_1 (x, t) = (c_0 - z(t)) x + s \quad (12)$$
$$\eta_2 = \eta_2 (t) \quad (13)$$
$$\phi_1 = \phi_1 (t, u) = z(t) u \quad (14)$$
$$\phi_2 = \phi_2 (x, t, v) = g(x, t) + c_0 v \quad (15)$$

where c_0 is an arbitrary constant such that:

$$(3n + 2) z(t) + \eta_{2_t} - (n + 1) c_0 = 0 \quad (16)$$
$$(z(t) x - s(t) - c_0 x) \lambda_x - \eta_2 (t) \lambda_t + z(t) \lambda - g(t)(x, t)_x = 0 \quad (17)$$
$$x \lambda z(t)_t - \lambda s_t - g(x, t)_t = 0 \quad (18)$$
$$(z(t) x - s(t) - c_0 x) f_x - \eta_2 (t) f_t \quad (19)$$
$$= -f((3n + 1) z(t) - n c_0) + x z(t)_t - s(t)_t. \quad (20)$$

When it is assumed that $s(t)$ and $z(t)$ are known then Eq. (16) may be used to determine $\eta_2 (t)$ whilst (17)–(19) may be used to determine $g(x, t)$, $\lambda(x, t)$ and $f(x, t)$.

We observe that we have shown that the potential symmetries of the conserved form of the ice dynamics equations (3) are entirely equivalent to those of the single equation (1). This is so because according to [6] additional symmetries can only be induced by the potential system when:

$$\eta_{1_v}^2 + \eta_{2_v}^2 + \phi_{1_v}^2 \neq 0. \quad (21)$$

Clearly substitution of Eqs. (12)–(14) demonstrates that this is not the case.

In addition to that a differential consequence of Eqs. (17) and (18) incorporating the relation (4) is the differential equation for a, similar in form to (19), namely:

$$(z(t) x - s(t) - c_0 x) a_x - \eta_2 (t) a_t = -a(n + 1)(3z(t) - c_0). \quad (22)$$

Moreover, we note that Eq. (17) may be obtained directly by differentiating the second surface invariant condition (11) with respect to x and then applying (3) and (12)–(15) together with the first of (11).

In summary, the results (16)–(22) together with the first invariant condition of (11) may be simplified by eliminating $z(t)$ using (16) and combined to give three first order partial differential equations which $u(x, t)$, $a(x, t)$ and $f(x, t)$ must satisfy, namely:

$$\left(s(t) + \frac{((2n + 1) c_0 + r_t (t))}{3n + 2} x \right) u_x + r(t) u_t = \frac{(n + 1) c_0 - r_t (t)}{3n + 2} u \quad (23)$$

$$\left(s(t) + \frac{((2n + 1) c_0 + r_t (t))}{3n + 2} x \right) a_x + r(t) a_t = \frac{(n + 1)}{3n + 2} (c_0 - 3 r_t (t)) a \quad (24)$$

$$\left(s(t) + \frac{((2n + 1) c_0 + r(t)_t)}{3n + 2} x \right) f_x + r(t) f_t = \frac{((2n + 1) c_0 - (3n + 1) r_t (t))}{3n + 2} f + \frac{x r_{tt} (t)}{3n + 2} + s_t (t), \quad (25)$$

where $r(t) \equiv \eta_2 (t)$ has been used to simplify the notation.

3. Solutions for the case $n = 3$

As stated in Section 1 the exponent n which occurs in (1) is Glen's exponent and Fowler [1] suggests that $n \approx 3$ in physically realistic situations Thus in the following we will assume that $n = 3$ although the analysis is unchanged for any non-Newtonian values $n > 1$. The results presented assumed that each of the functions u, a and f explicitly depend on x and t.

3.1. The case $f(x, t) = 0$

Firstly, substitution of $f(x, t) = 0$ into Eq. (25) gives $r(t) = c_1 t + c_2$ and $s(t) = c_3$.

3.1.1. The subcase $7c_0 + c_1 \neq 0$, $c_1 \neq 0$

The solution of (23) and (24) may be expressed in terms of the similarity variable $\omega = \omega(x, t)$ for which:

$$\omega(x, t) = (x + c_3)(c_1 t + c_2)^{-\frac{7c_0 + c_1}{11 c_1}} \quad \text{when } 7c_0 + c_1 \neq 0 \quad (26)$$

with:

$$u(x,t) = \psi(\omega(x,t))(c_1 t + c_2)^{\frac{4c_0 - c_1}{11c_1}} \quad (27)$$

$$a(x,t) = A(\omega(x,t))(c_1 t + c_2)^{\frac{4c_0 - 12c_1}{11c_1}}. \quad (28)$$

Substituting the relationships into Eq. (1) with $n = 3$ gives rise to the ordinary differential equation:

$$\frac{d}{d\omega}\left\{\frac{\psi^5\psi_\omega^3}{5} + \frac{(c_1 + 7c_0)\,\omega\psi}{11}\right\} - c_0\psi - A = 0. \quad (29)$$

3.1.2. The subcase $7c_0 + c_1 = 0$, $c_1 \neq 0$

For this subcase it may be shown that:

$$\omega(x,t) = x + c_3 \ln(c_1 t + c_2) \quad \text{when } 7c_0 + c_1 = 0 \quad (30)$$

with

$$u(x,t) = \psi(\omega(x,t))(c_1 t + c_2)^{-\frac{1}{7}} \quad (31)$$

$$a(x,t) = A(\omega(x,t))(c_1 t + c_2)^{-\frac{8}{7}}. \quad (32)$$

Substituting the relationships into Eq. (1) with $n = 3$ gives rise to the ordinary differential equation:

$$\frac{d}{d\omega}\left\{\frac{\psi^5\psi_\omega^3}{5} + 7c_0c_3\psi\right\} - c_0\psi - A = 0. \quad (33)$$

3.1.3. The subcase $c_1 = 0$

Without loss of generality consider the case $c_2 = 1$. The solution of (23) and (24) may be expressed in terms of the similarity variable $\omega = \omega(x,t)$ for which:

$$\omega(x,t) = (x + c_3)\,e^{-\frac{7c_0 t}{11}} \quad (34)$$

with:

$$u(x,t) = \psi(\omega(x,t))\,e^{\frac{4c_0 t}{11}} \quad (35)$$

$$a(x,t) = A(\omega(x,t))\,e^{\frac{4c_0 t}{11}}. \quad (36)$$

Substituting the relationships into Eq. (1) with $n = 3$ gives rise to the ordinary differential equation:

$$\frac{d}{d\omega}\left\{\frac{\psi^5\psi_\omega^3}{5} + \frac{7c_0\omega\psi}{11}\right\} - c_0\psi - A = 0. \quad (37)$$

3.2. The case $s(t) = 0$, $r(t) \neq 0$, $f(x,t) \neq 0$

In this case Eqs. (23)–(25) may be integrated immediately to give solutions in terms of the similarity variable $\omega = \omega(x,t)$ for which:

$$\omega(x,t) = x r(t)^{-\frac{1}{11}} \exp\left(-\frac{7c_0}{11}\int \frac{dt}{r(t)}\right) \quad (38)$$

with:

$$u(x,t) = \psi(\omega(x,t))\, r(t)^{-\frac{1}{11}} \exp\left(\frac{4c_0}{11}\int \frac{dt}{r(t)}\right) \quad (39)$$

$$a(x,t) = A(\omega(x,t))\, r(t)^{-\frac{12}{11}} \exp\left(\frac{4c_0}{11}\int \frac{dt}{r(t)}\right) \quad (40)$$

$$f(x,t) = \left[\frac{\omega(x,t) r_t(t)}{11} + F(\omega(x,t))\right] r(t)^{-\frac{10}{11}} \times \exp\left(\frac{7c_0}{11}\int \frac{dt}{r(t)}\right). \quad (41)$$

Substituting the relationships into Eq. (1) with $n = 3$ gives rise to the ordinary differential equation:

$$\frac{3\psi^5\psi_\omega^2\psi_{\omega\omega}}{5} + \psi^4\psi_\omega^4 + \frac{7c_0\omega\psi_\omega}{11} - \frac{4c_0\psi}{11} - \psi F_\omega - \psi_\omega F - A = 0. \quad (42)$$

That is:

$$\frac{d}{d\omega}\left\{\frac{\psi^5\psi_\omega^3}{5} + \frac{7c_0\omega\psi}{11} - \psi F\right\} - c_0\psi - A = 0. \quad (43)$$

3.3. The case $s(t) \neq 0$, $r(t) \neq 0$, $f(x,t) \neq 0$

In this case the similarity variable has the form:

$$\omega(x,t) = x r(t)^{-\frac{1}{11}} \exp\left(-\frac{7c_0}{11}\int \frac{dt}{r(t)}\right) - b(t) \quad (44)$$

where

$$b(t) = \int \left\{\frac{s(t)}{r(t)^{\frac{12}{11}}} \exp\left(-\frac{7c_0}{11}\int \frac{dt}{r(t)}\right)\right\} dt \quad (45)$$

and the solutions (39) and (40) for $u(x,t)$ and $a(x,t)$ still apply. However the solution for $f(x,t)$ now becomes:

$$f(x,t) = \left[\frac{\omega(x,t) r_t(t)}{11} + F(\omega(x,t)) + h(t)\right] r(t)^{-\frac{10}{11}} \times \exp\left(\frac{7c_0}{11}\int \frac{dt}{r(t)}\right) \quad (46)$$

where

$$h(t) = \frac{\left(r(t)_t + 7c_0\right)}{11} b + r(t) b_t. \quad (47)$$

The resulting ordinary differential equation is once again (43).

3.4. The case $r(t) = 0$, $f(x,t) \neq 0$

In the following only the non-trivial case $c_0 \neq 0$ is considered. Eqs. (23)–(25) may be integrated immediately to give the following solutions:

$$u(x,t) = m(11s + 7c_0 x)^{\frac{4}{7}} \quad (48)$$

$$a(x,t) = n(11s + 7c_0 x)^{\frac{4}{7}} \quad (49)$$

$$f(x,t) = p(11s + 7c_0 x) - \frac{x s_t}{7c_0} \quad (50)$$

where the relationship between the functions $m = m(t)$, $n = n(t)$ and $p = p(t)$ may be found upon substitution of Eqs. (48)–(50) into (1). The following equation holds:

$$m_t = -11c_0 m p - n + \frac{704 c_0^4 m^8}{5}. \quad (51)$$

4. Results of the non-classical analysis

In this section consideration is given to the non-classical approach which is a generalization of the classical Lie method due to Bluman and Cole [7] that incorporates the surface invariant condition.

In the following the ice sheet equation will be considered in the form (1) and the symmetry generator will now have the form:

$$\mathcal{X} = \eta_1(x,t,u)\frac{\partial}{\partial x} + \eta_2(x,t,u)\frac{\partial}{\partial t} + \phi(x,t,u)\frac{\partial}{\partial u} \tag{52}$$

and the condition for invariance of (1) is the equation

$$\mathcal{X}_E^{(2)}(\Psi)|_{\Psi=0,\Omega=0} = 0 \tag{53}$$

where the second prolongation operator $\mathcal{X}_E^{(2)}$ is written in the form

$$\mathcal{X}_E^{(2)} = \mathcal{X} + \phi^{[t]}\frac{\partial}{\partial u_t} + \phi^{[x]}\frac{\partial}{\partial u_x} + \phi^{[xx]}\frac{\partial}{\partial u_{xx}} \tag{54}$$

where $\phi^{[t]}$, $\phi^{[x]}$ and $\phi^{[xx]}$ are defined through the transformations of the partial derivatives of u. In particular to the first order in ε:

$$\bar{u}_{\bar{x}} = u_x + \varepsilon\phi^{[x]}(x,t,u) \quad \bar{u}_{\bar{t}} = u_t + \varepsilon\phi^{[t]}(x,t,u)$$
$$\bar{u}_{\bar{x}\bar{x}} = u_x + \varepsilon\phi^{[xx]}(x,t,u) \tag{55}$$

and the condition for invariance of surface $u = u(x,t)$ is:

$$\Omega = \phi - \eta_1 u_x - \eta_2 u_t = 0. \tag{56}$$

With the aid of the Macsyma program symmgrp.max adapted for non-classical analysis it may be shown that Eq. (1) has the following infinitesimal:

$$\eta_2(x,t,u) = 1 \tag{57}$$

$$\eta_1(x,t,u) = h(t) + x\frac{\left((2n+1)g(t)^2 - g_t(t)\right)}{(3n+2)g(t)} \tag{58}$$

$$\phi(x,t,u) = u\frac{\left((n+1)g(t)^2 + g_t(t)\right)}{(3n+2)g(t)}. \tag{59}$$

For this case the functions $u(x,t)$, $a(x,t)$ and $f(x,t)$ satisfy:

$$\left[(3n+2)g(t)h(t) + x\left((2n+1)g(t)^2 - g_t(t)\right)\right]u_x + (3n+2)g(t)u_t = u\left((n+1)g(t)^2 + g_t(t)\right) \tag{60}$$

$$\left[(3n+2)g(t)h(t) + x\left((2n+1)g(t)^2 - g_t(t)\right)\right]a_x + (3n+2)g(t)a_t = a(n+1)\left(g(t)t^2 + g_t(t)\right) \tag{61}$$

$$\begin{aligned}\left[(3n+2)g(t)h(t) + x\left((2n+1)g(t)^2 - g_t(t)\right)\right]f_x &+ (3n+2)g(t)f_t \\ &= f\left(g(t)^2(1+2n) + g_t(t)(1+3n)\right) \\ &+ (3n+2)(g(t)h_t(t) - h(t)g_t(t)) \\ &- xg(t)\left(\frac{g_t(t)}{g(t)}\right)_t.\end{aligned} \tag{62}$$

We observe that Eqs. (60)–(62) are essentially the same as (23)–(25) and so conclude that the non-classical symmetries are equivalent to the potential cases.

5. A comparison result and some particular examples

We start by showing a useful result connecting the solution of the obstacle problem and the solutions of the nonlinear equation (1).

Theorem 1. *Let $a \in L^\infty(Q)$, $f \in L^\infty(Q)$ and take a compactly supported initial data $h_0 \in L^\infty(\Omega)$. Let $h(x,t)$ be the unique solution of the obstacle problem (SF). Also let $u(x,t)$ be any continuous solution of Eq. (1) corresponding to an ablation function $\widetilde{a} \in L^\infty(Q)$ and for which there exist two Lipschitz curves $x_\pm(t)$ such that*

$$u(x_\pm(t),t) = 0 \text{ and } u(x,t) > 0 \text{ for a.e. } x \in (x_-(t), x_+(t)) \text{ and any } t \in [0,T].$$

Assume that

$$\widetilde{a}(x,t) \le a(x,t) \quad \text{for a.e. } (x,t) \in Q,$$
$$u(x,0) \le h_0(x) \quad \text{for a.e. } x \in (x_-(0), x_+(0)).$$

Then, if $S_\pm(t)$ denotes the free boundaries generated by function $h(x,t)$ we have that

$$S_-(t) \le x_-(t) \le x_+(t) \le S_+(t) \quad \text{and any } t \in [0,T],$$

and

$$h(x,t) \ge u(x,t) \quad \text{for a.e. } x \in (x_-(t), x_+(t)) \text{ and any } t \in [0,T].$$

Moreover, if

$$\widetilde{a}(x,t) = a(x,t) \quad \text{a.e. } (x,t) \in Q,$$
$$u(x,0) = h_0(x) \quad \text{a.e. } x \in (x_-(0), x_+(0)), \tag{63}$$

and

$$\frac{u^{n+2}}{n+2}|u_x|^{n-1}u_x - fu = 0 \quad \text{on } \{(x_-(t),t)\} \cup \{(x_+(t),t)\}, \text{ for } t \in (0,T), \tag{64}$$

then $S_-(t) = x_-(t)$, $x_+(t) = S_+(t)$ and $h(x,t) = u(x,t)$ for a.e. $x \in (x_-(t), x_+(t))$ any for any $t \in [0,T]$.

Proof. We shall assume, additionally that $h_t, u_t \in L^1(Q)$ and that $f \equiv 0$. The general case, without this information, follows some technical arguments which can be found, for instance, in [8]. We take as a test function the approximation of the $\text{sign}_0^+(u^m - h^m)$ function (with $m = 2(n+1)/n$) given by $\Psi_\delta(\eta) := \min(1, \max(0, \frac{\eta}{\delta}))$, for $\delta > 0$ small. Then we define $v = \Psi_\delta(u^m - h^m)$. Notice that $v \in L^\infty\left(\cup_{t\in[0,T]}(x_-(t), x_+(t)) \times \{t\}\right)$ and that $v(.,t) \in W_0^{1,p}(x_-(t), x_+(t))$ for $p = n+1$ with

$$v_x = \begin{cases} \frac{1}{\delta}(u^m - h^m)_x & \text{if } 0 < u - h < \delta \\ 0 & \text{otherwise.} \end{cases}$$

Then, defining the set

$$A_\delta := \{(x,t), \text{ such that } t \in [0,T], x \in (x_-(t), x_+(t)) \text{ and } 0 < u(t,x) - h(t,x) < \delta\},$$

and multiplying by the difference of the solutions of the partial differential equations and integrating by parts (that is, by taking v as a test function) we find

$$\int_0^T \int_{(x_-(t),x_+(t))} (u_t - h_t)\, \Psi_\delta(u^m - h^m)\mathrm{d}x\mathrm{d}t + I(\delta) \leq 0$$

where

$$I(\delta) = \frac{1}{\delta}\int_0^T \int_{A_\delta} \left\{\phi((u^m)_x) - \phi((h^m)_x)\right\}((u^m)_x - (h^m)_x)\mathrm{d}x\mathrm{d}t,$$

with $\phi(r) = \mu\,|r|^{n-1}\,r$, $\mu = n^n/[2^n(n+1)^n(n+2)]$ and where we used the fact that $u(x_\pm(t),t) = 0 \leq h(x_\pm(t),t)$ for any $t \in [0,T]$. Then, from the monotonicity of $\phi(r)$ we can pass to the limit when $\delta \searrow 0$ and conclude that

$$\int_{(x_-(t),x_+(t))} \max\{u(t,x) - h(t,x), 0\}\mathrm{d}x\mathrm{d}t \leq 0.$$

which implies that $u \leq h$ on the set $(x_-(t), x_+(t))$.
In the special case of u satisfying (63) and (64) we find that the function $u^\#(x,t)$ defined as

$$u^\#(x,t) = \begin{cases} u(x,t) & \text{if } x \in (x_-(t), x_+(t)), \quad t \in [0,T], \\ 0 & \text{otherwise,} \end{cases}$$

satisfies all the conditions required for weak solutions of the obstacle problem and by the uniqueness of such solutions we also find that $h(x,t) = u^\#(x,t)$. □

Remark 2. We note that no information on the global boundary conditions satisfied by u on $\partial\Omega \times [0,T]$ is required in the above result.

Remark 3. Notice also that the conditions satisfied by $h(x,t)$ on the free boundary $S_\pm(t)$ indicate that the Cauchy problem on the curves $\cup_{t\in[0,T]}(S_\pm(t),t)$ does not satisfy the unique continuation property since h is identically zero to the left or the right sides of those curves. Some sharper information on the growth with t and the study of the differential equation satisfied by the free boundaries can be found by means of some arguments involving Lagrangian coordinates. This is the main object of the work [9] concerning a different simplified obstacle problem.

We consider now the particular example of a non-sliding ice sheet at the base so that $f(x,t) = 0$ and consider the values, $c_0 = -0.1$, $c_1 = 1$, $c_2 = 1$ and $c_3 = 0$ with the initial condition for the ice sheet profile:

$$u(x,0) = \psi(\omega(x,0)) = \frac{1}{2}\cos\left(\frac{\omega(x,0)}{4}\right). \tag{65}$$

Then according to he subcase $7c_0 + c_1 \neq 0$, $c_1 \neq 0$ and Eqs. (26) and (27) the similarity solution is

$$\omega(x,t) = \frac{x}{(1+t)^{0.0273}} \tag{66}$$

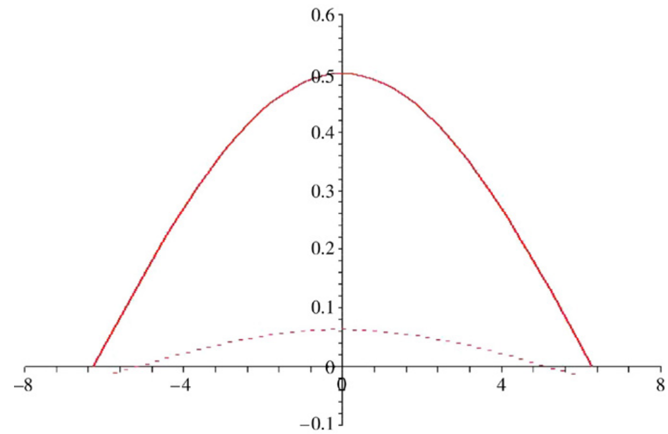

Fig. 1. Plots of the initial ice sheet profile, $u(x,0)$ (upper curve) and also the initial accumulation–ablation function, $a(x,0)$ (lower curve) versus x.

$$u(x,t) = \frac{\psi(\omega(x,t))}{(1+t)^{0.1272}} \tag{67}$$

with the accumulation–ablation function (which now is denoted by $\widetilde{a}(x,t)$) given by (28) and (29) so that:

$$\widetilde{a}(x,t) = A(\omega(x,t))(1+t)^{0.0727} \tag{68}$$

with

$$A(\omega) = 0.153 \times 10^{-4}\cos^4\left(\frac{\omega}{4}\right)\sin^4\left(\frac{\omega}{4}\right) - 0.916 \times 10^{-5}\cos^6\left(\frac{\omega}{4}\right)\sin^2\left(\frac{\omega}{4}\right) + 0.636 \times 10^{-1}\cos\left(\frac{\omega}{4}\right) - 0.341 \times 10^{-1}\omega\sin\left(\frac{\omega}{4}\right). \tag{69}$$

In this case the propagation fronts of the ice sheet region are found from:

$$\psi(x,t) = 0 \tag{70}$$

so

$$x_\pm(t) = \pm 2\pi\,(1+t)^{0.0273} \tag{71}$$

and the finite velocity is:

$$\frac{\mathrm{d}}{\mathrm{d}t}x_\pm(t) = \pm 0.0546\pi\,(1+t)^{-0.973}. \tag{72}$$

Figs. 1–3 illustrate the time evolution of the ice sheet $u(x,t)$ and also the accumulation–oblation function $\widetilde{a}(x,t)$.

As a consequence of Theorem 1 we have

Corollary 1. *Let $\Omega = (-L,L)$ with $L > 2\pi$ and $f(x,t) \equiv 0$ Let $a \in L^\infty(Q)$ with $\widetilde{a}(x,t) \leq a(x,t)$ for a.e. $(x,t) \in Q$, where $\widetilde{a}(x,t)$ is given by (68) and assume that*

$$h_0(x) \geq \begin{cases} \frac{1}{2}\cos\left(\frac{x}{4}\right) & \text{if } x \in (-2\pi, 2\pi), \\ 0 & \text{if } x \in (-L-2\pi) \cup (L, 2\pi). \end{cases}$$

Let $h(x,t)$ be the (unique) solution of the obstacle formulation (with $f(x,t) \equiv 0$) associated with the data a and h_0. Then

$$S_-(t) \leq -2\pi\,(1+t)^{0.0273} < 2\pi\,(1+t)^{0.0273} \leq S_+(t)$$

for any $t \in [0,T]$,

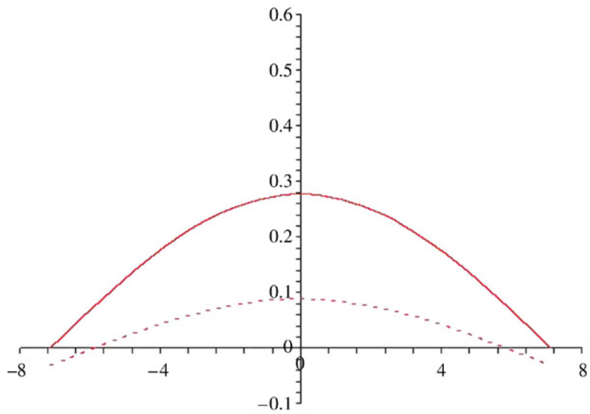

Fig. 2. Plots of the ice sheet profile, $u(x, 100)$ (upper curve) and also the accumulation–ablation function, $a(x, 100)$ (lower curve) at time $t = 100$ versus x.

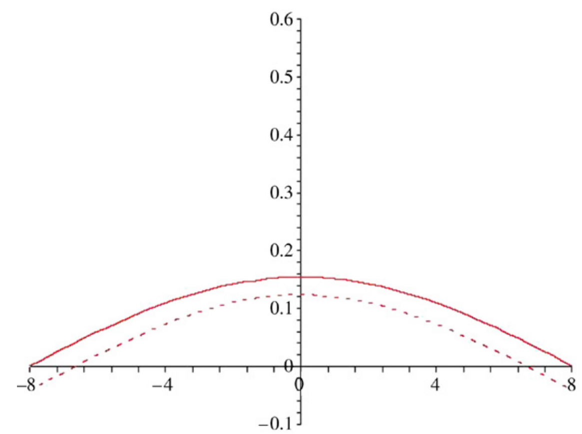

Fig. 3. Plots of the ice sheet profile, $u(x, 10000)$ (upper curve) and also the accumulation–ablation function, $a(x, 10000)$ (lower curve) at time $t = 10000$ versus x.

and

$$h(x,t) \geq \frac{\psi\left(\frac{x}{(1+t)^{0.0273}}\right)}{(1+t)^{0.1272}}$$

for a.e. $x \in (-2\pi\,(1+t)^{0.0273}, 2\pi\,(1+t)^{0.0273})$
and any $t \in [0, T]$,

where $\psi(\omega)$ *satisfies* (29).

This example clearly demonstrates the useful properties of the closed form solutions of (1) for an accumulation–ablation function which changes sign and is negative near the propagation fronts.

6. Comments and future work

In this paper we have concentrated on the problem of determining closed form similarity solutions of Eq. (1) (using potential symmetries) and its connections with the thickness function $h(x,t)$ of ice sheets as solution of the associated obstacle problem. The main aim has been to demonstrate that classes of such solutions exist and that they contain physically realistic properties. We observe that Eq. (1) contains certain modelling deficiencies (with respect the obstacle problem formulation) because inadmissible solutions for which $u(x,t) < 0$ (in some subset) are possible. Certainly the similarity solution approach presented here demonstrates the possibility of such unrealistic solutions for Eq. (1) and so we obtain only some estimates for the physical relevant function $h(x,t)$. It is for this reason that this research is continuing and in the next phase we are seeking similarity solutions corresponding to the strong formulation of the problem, when it is written (in other equivalent terms, as indicated in Díaz and Schiavi [10,11]) by using the multivalued maximal monotone graph $\beta(u)$ of $\mathbb{R}^2$ given by $\beta(r) = \phi$ (the empty set) if $r < 0$, $\beta(0) = (-\infty, 0]$ and $\beta(r) = \{0\}$ if $r > 0$. Then, the formulation is

$$\begin{cases} \Psi(x,t,u,u_t,u_x,u_{xx}) \\ \quad \equiv u_t - a - \left[\dfrac{u^{n+2}}{n+2}|u_x|^{n-1}u_x - fu\right]_x + \varphi = 0 \\ \text{with } \varphi(x,t) \in \beta(u(x,t)) \text{ a.e. } (x,t) \in \Omega \times (0,T). \end{cases}$$

In this case the focus is on both a classical and a non-classical symmetry reduction of the equation. It is expected that use of the non-classical method on this occasion will extend the range of possible solutions.

In a further development we consider a more general framework to encompass a wider range of physical applicability for the mathematical analysis. As before, this is achieved by defining two functions:

$$\phi(r) = |r|^{n-1} r \qquad \psi(s) = s^n \tag{73}$$

and defining the new functions $U(x,t)$ and $b(s)$ so that:

$$U = u^m = \psi(u) \Rightarrow U^{\frac{1}{m}} = u = \psi^{-1}(U) = b(u) \tag{74}$$

so that:

$$\phi\left(\psi(u)_x\right) = \phi(U_x) = |U_x|^{p-2} U_x \tag{75}$$

but now we may be interested in general values of $p > 1$ and $m > 0$ relevant also in other physical contexts (and so, not necessarily leading to $p = n+1$). In this way the mathematical framework may be taken to be

$$b(U)_t - [k\phi(W) - fb(U)]_x + \beta(U) - a(x,t) \ni 0 \tag{76}$$

where k is a constant and

$$W = U_x. \tag{77}$$

This may also be written in a conserved or potential form by writing

$$V_x + \lambda - b = 0 \tag{78}$$

$$V_t - k\psi + bf = 0 \tag{79}$$

$$W = U_x \tag{80}$$

where $b = b(U)$, $\psi = \psi(U)$, $\lambda = \lambda(x,t,U)$, $f = f(x,t)$ provided that

$$a(x,t) - \beta(U(x,t)) \ni \lambda_t(x,t,U). \tag{81}$$

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