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Existence of weak solutions to a system of nonlinear partial differential equations modelling ice streams ☆

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Abstract

This paper deals with the mathematical analysis of a nonlinear system of three differential equations of mixed type. It describes the generation of fast ice streams in ice sheets flowing along soft and deformable beds. The system involves a nonlinear parabolic PDE with a multivalued term in order to deal properly with a free boundary which is naturally associated to the problem of determining the basal water flux in a drainage system. The other two equations in the system are an ODE with a nonlocal (integral) term for the ice thickness, which accounts for mass conservation and a first order PDE describing the ice velocity of the system. We first consider an iterative decoupling procedure to the system equations to obtain the existence and uniqueness of solutions for the uncoupled problems. Then we prove the convergence of the iterative decoupling scheme to a bounded weak solution for the original system.

MSC: 22E46; 53C35; 57S20

Keywords: Ice sheet models; Nonlinear partial differential equations system of mixed type; Free boundaries

1. Introduction

It is well known that the relationships between the ice sheets, the atmosphere and the ocean dynamics have a major effect on the climate of the Earth (see for example [19,18]). This fact motivates the scientific community to pursue a better knowledge of the large scale behavior of ice sheets by means of modelling their complex nonlinear dynamics, which in time constitutes an important application of mathematics to the field of geophysical fluids mechanics and more generally to continuum mechanics. In particular, the applied mathematics community is specially interested in looking for instability mechanisms that could explain the detected oscillations in the ice flow regime in the West Antarctic Ice Sheet (WAIS). Two crucial phenomena associated to oscillations in the ice flow regime are the *ice surging*, defined as the fast and sudden advance of ice masses, and the *ice streaming*, associated to the spontaneous generation of fast ice streams when compared with the slow surrounding ice.

Ice streaming is a phenomenon concerning the development of lateral instabilities in the ice flow regime mainly due to the basal sliding over a underlying deformable layer of sediments. The complexity of the physical processes

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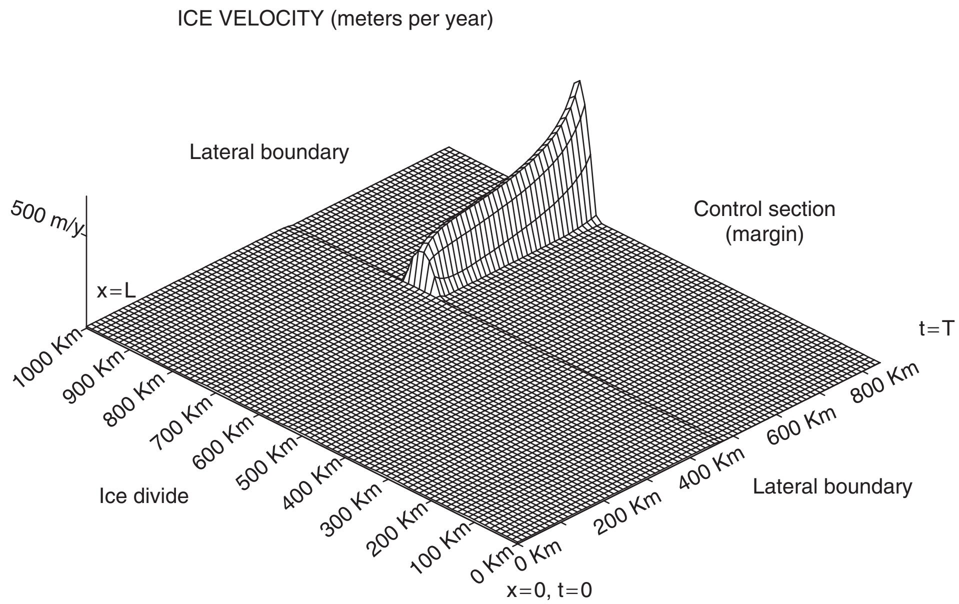

Fig. 1. Results obtained for the ice velocity in [6], where a numerical resolution of the model studied in this paper, is presented. The main flow takes place in the direction parallel to the t-axis. This figure illustrates the development of a region where the ice velocity takes considerable greater values than in the rest of the domain, i.e., the ice streaming generation.

involved in this phenomenon makes that its mathematical modelling results in models consisting of nonlinear systems of differential equations involving multivalued terms, which account for the free boundary nature of the problem.

The system of equations we deal with in this paper make up a model, referred to as the multivalued model (proposed in [16]), is related to a parameterized model derived by Fowler and Johnson (for the physics of the problem we refer to Fowler [9]) to generate ice streams similar to those detected in the Siple Coast (WAIS). Note that our aim is not to justify the physics of the ice streaming model, not to provide qualitative information about the behavior of solution. Our aim is to show the existence of bounded weak solutions to the system of equations describing the model. In fact, the main result of the paper which is stated in Section 3 concerns the existence of a bounded weak solution (b.w.s) to the multivalued model. This result justifies, theoretically, the numerical treatment of the model carried out in [6], where a finite element algorithm combined with a duality technique were employed in order to cope with the free boundary nature of the model. The uniqueness of b.w.s result appears in [17], where existence was assumed.

The rest of the paper is devoted to the proof of the existence theorem, developed throughout Sections 4 and 5 as it splits into two main parts. The first part is detailed in Section 4, where we consider an iterative decoupling procedure to the system equations. This strategy leads to the analysis of uncoupled systems, denoted by (S_j), $j \in \mathbb{N}$ (parameter which denotes the iterative decoupling step). Each system (S_j) comprises three decoupled problems, one for each of the variables which are studied separately. We end this section with the result stating the existence and uniqueness of b.w.s. to the systems (S_j). The second part of the proof is given in Section 5 and consists of the convergence of the iterative decoupling procedure, i.e., we prove that a sequence of solutions relative to the decoupled systems converges to a b.w.s. of the multivalued model (Fig. 1).

2. Some notation and preliminaries

In this section, we shall briefly introduce the *strong formulation* of the Fowler and Johnson's model, as the multivalued model constitutes a generalization of it in a sense specified later on. Fowler and Johnson's model is a stationary two-dimensional model in which the variables do not depend on the vertical coordinate and is derived from conservation equations complemented with suitable constitutive laws. Hence, let T, $L > 0$ be two positive constants. We shall consider the rectangular domain $\Omega_T = (0, T) \times \Omega$, $(t, x) \in \Omega_T$, $x \in \Omega = (0, L)$, resembling the Siple Coast ice flow area. Note that the first coordinate t is considered with respect to the direction parallel to the main flow and the second coordinate x is considered with respect to the perpendicular direction to the previous one, i.e., with respect to the cross stream direction. Let us define the subset $\Omega_T^+ \subset \Omega_T$,

$$\Omega_T^+ := \{(t,x) \in \Omega_T,\ Q(t,x) > 0\} \subset \Omega_T,$$

which is a priori unknown. Simple algebraical manipulations of the original Fowler and Johnson's model lead to an equivalent formulation, referred to as the *strong formulation* of Fowler and Johnson's model, in terms of a system of three equations for the variables: water flux, $Q(t,x)$, ice thickness, $h(t)$ and accumulated velocity, $\xi(t,x)$. Let ∂_x and ∂_t denote the partial differentiation operators. The strong formulation consists of the following system of coupled equations for the unknowns Q, h and ξ, complemented with suitable initial and boundary conditions.

Given $Q_0(x) > 0$ and the positive constants h_0 and ξ_0, find three functions Q, h and ξ satisfying:

$$\begin{cases} \partial_t Q - \dfrac{1}{n}\partial_x[(Q+\bar{Q})^{-1/n}\partial_x Q] = f(\xi, h, \partial_t h, Q) & \text{in } \Omega_T^+, \quad (1.1)\\ \partial_t h = -M^r h^{-(1+r)}(\int_0^L (Q+\bar{Q})^{s/nr}\,\mathrm{d}x)^{-r} & \text{in } \Omega_T^+, \quad (1.2)\\ \partial_t \xi = (h|\partial_t h|)^{1/r}(Q+\bar{Q})^{s/nr} & \text{in } \Omega_T^+, \quad (1.3)\\ \partial_x Q(t,0) = \partial_x Q(t,L) = 0, & t \in (0,T), \quad (1.4)\\ Q(0,x) = Q_0(x), \quad h(0,x) = h_0, \quad \xi(0,x) = \xi_0, & x \in \Omega, \quad (1.5) \end{cases}$$

with $f(\xi, h, \partial_t h, Q)$ given by

$$f(\xi, h, \partial_t h, Q) = (h|\partial_t h|)^{1/r}(h|\partial_t h| - \xi^{-1/2})(Q+\bar{Q})^{s/nr} + \gamma - \delta h^{-1}.$$

The constants that appear in the model, Eqs. (1.1)–(1.5), are the following: $0 < \bar{Q} \ll 1$ which stands for a residual basal water flux (see [11]) and M, the prescribed value for the ice flux at the ice divide. The parameters of the model are $\gamma \approx 0.2$, which represents the geothermal water flux and $\delta \approx 0.4$ which measures the importance attributed to the conductive cooling. The exponents r and s are the ones relative to the Boulton and Hindmarsh's ice rheology (see [8]), where $r, s \in (0,1)$ and n is the exponent considered in Glen's flow law (see [14]), usually taken to be $n = 3$. Note that, according to Fowler's assumptions, the variable h does not depend on the lateral cross stream coordinate x, i.e., $h = h(t)$, so hereafter we will use the notation h' to denote $\partial_t h$. Eq. (1.1) stands for the basal water flux conservation equation and is a parabolic equation, where the longitudinal downstream coordinate t would be the *time-like* coordinate, with nonlinear diffusion and a nonlinearity in the forcing term $f(\xi, h, h', Q)$ (see [10] for physical interpretations). We prescribe Q at the *ice divide* and the lateral boundary conditions given by (1.4), which is of homogeneous Neumann type and whose physical meaning is no water flux through condition. Eq. (1.2) models the ice mass conservation and is an ODE, in which stands out the presence of a nonlocal integral term. This Eq. (1.2) is complemented with a prescribed value at the ice divide (initial condition (1.5)). Finally, (1.3) is a first order partial differential equation complemented with the condition (1.5) and represents a constitutive law of the Boulton and Hindmarsh type, which relates the ice velocity $(\partial_t \xi)$ to the shear (given by $h|h'|$) and to the effective pressure in the drainage system (modelled by the term $(Q+\bar{Q})^{-1/n}$).

Note that the domain of application of (1.1)–(1.5), Ω_T^+, is a priori unknown and therefore its determination is part of the problem. So, in order to deal properly with this free boundary problem, Muñoz et al. [16] proposed a new formulation in the framework of an obstacle problem with a multivalued operator. This weak formulation of the problem, referred to as the multivalued model, is presented in Section 3 and generalizes the previous Fowler and Johnson's strong formulation.

3. The multivalued model

The multivalued formulation corresponding to the Fowler and Johnson strong formulation (Eqs. (1.1)–(1.5)) allows for a mathematically correct description of the free boundary and also for considering only physically admissible solutions, i.e., those with nonnegative water flux. The multivalued formulation of the model is the following:

$$Q_t - \frac{1}{n}[(Q+\bar{Q})^{-1/n}Q_x]_x + \beta(Q) \ni f(\xi, h, h', Q) \quad \text{in } \Omega_T, \tag{1.1b}$$

$$h' = -M^r h^{-(1+r)} \left(\int_0^L (Q + \bar{Q})^{s/nr} \, dx \right)^{-r} \quad \text{in } \Omega_T, \tag{1.2b}$$

$$\xi_t = (h|h'|)^{1/r} (Q + \bar{Q})^{s/nr} \quad \text{in } \Omega_T, \tag{1.3b}$$

complemented with the boundary condition (1.4) and the initial conditions (1.5). The multivalued operator β is a maximal monotone graph of $\mathbb{R}^2$ which is defined as follows:

$$\beta(r) = \emptyset \quad \text{if } r < 0, \qquad \beta(0) = (-\infty, 0] \quad \text{and} \quad \beta(r) = 0 \quad \text{if } r > 0, \tag{1}$$

where the symbol $\emptyset$ denotes the empty set. In order to study the free boundary value problem consisting of Eqs. (1.1b)–(1.3b), complemented with (1.4) and (1.5), we shall introduce the new variable

$$w = \frac{1}{n-1} (Q + \bar{Q})^{(n-1)/n} \text{ and the function } b(w) = [(n-1)w]^{n/(n-1)}. \tag{2}$$

Note that $b(w)$ is Lipschitz continuous and that the change of variable given by (2) results in a shift of the obstacle in the sense that now it is $w = \Phi$ with $\Phi = \bar{Q}^{(n-1)/n}/(n-1) > 0$. Next we define the closed convex set $\mathbb{K}$ which is naturally associated to the definition of b.w.s to the multivalued model:

$$\mathbb{K} = \{v \in H^1(\Omega), \text{ such that } v(x) \geqslant \Phi, \text{ almost everywhere (a.e.) } x \in \Omega\}. \tag{3}$$

Definition 3.1. It will be said that the initial data (i.e., prescribed data at the ice divide) are admissible when $w_0 \in H^1(\Omega)$, $w_0 \in \mathbb{K}$ and, ξ_0 and h_0 are positive constants. And given h_0, the data (constants) T, L, M and Φ are considered to be admissible if

$$m_h = [h_0^{r+2} - (r+2)TM^r L^{-r}(2\Phi)]^{1/(r+2)} > 0. \tag{4}$$

Remark 3.1. The assumption $m_h > 0$ will allow us to assure that the ice thickness, h, takes only strictly positive values. Note also that a solution to the multivalued problem might be considered in fact a local b.w.s as here we prove its existence only for $t \in (0, T)$.

The multivalued formulation (1.1b)–(1.3b) written in terms of w is the following:

Given $w_0(y) > 0$, $\xi_0 > 0$ *and admissible data* (*in the sense of Remark 3.1*), *find three functions* w, h *and* ξ *such that the following is satisfied*:

$$\partial_t b(w) - w_{xx} + \beta(w - \Phi) \ni f(\xi, h, h', w) \quad \text{in } \Omega_T, \tag{5}$$

$$h' = -M^r h^{-(1+r)} \left(\int_0^L ((n-1)w)^{s/r(n-1)} \, dx \right)^{-r} \quad \text{in } \Omega_T, \tag{6}$$

$$\partial_x \xi = (h|h'|)^{1/r} ((n-1)w)^{s/r(n-1)} \quad \text{in } \Omega_T, \tag{7}$$

$$\partial_x w(0, t) = \partial_x w(L, t) = 0, \quad t \in (0, T), \tag{8}$$

$$w(x, 0) = w_0(x), \quad h(x, 0) = h_0, \quad \xi(0, x) = \xi_0, \quad x \in \Omega, \tag{9}$$

with

$$f(\xi, h, h', w) = (h|h'|)^{1/r} (h|h'| - \xi^{-1/2})[(n-1)w]^{s/r(n-1)} + \gamma - \delta h^{-1}. \tag{10}$$

From now on we shall assume the values for the exponents already considered in [12,15,16], i.e., $n = 3$ and $s = r = 1/2$. More general results, considering $r \in (0, 1)$ and $s/nr \in (0, 1)$, can be obtained with minor changes. As in [1,7], we shall consider the *complementarity formulation* associated to (5)–(9), which consists of the following system:

$$(S)\quad \begin{cases} w \geqslant \Phi & \text{in } \Omega_T,\\ \partial_t b(w) - w_{xx} - f(\xi, h, h', w) \geqslant 0 & \text{in } \Omega_T,\\ [\partial_t b(w) - w_{xx} - f(\xi, h, h', w)](w - \Phi) = 0 & \text{in } \Omega_T,\\ h' = -M^{1/2} h^{-3/2} [\int_\Omega (2w)^{1/2}\,\mathrm{d}x]^{-1/2} & \text{in } \Omega_T,\\ \partial_t \xi = (hh')^2 (2w)^{1/2} & \text{in } \Omega_T,\\ \partial_x w(0,t) = \partial_x w(L,t) = 0, & t \in (0,T),\\ w(x,0) = w_0(x), \quad h(x,0) = h_0, \quad \xi(x,0) = \xi_0, & x \in \Omega. \end{cases}$$

3.1. Bounded weak solution

In this section, we present the definition of bounded weak solution (b.w.s) to the system (S) and the theorem which states the existence of at least one b.w.s to (S).

Definition 3.2. Let V denote the functional space given by $V = V_w \times V_h \times V_\xi$, where

$$V_w := \{\eta : \eta \in L^2(0,T;\mathbb{K}) \cap L^\infty(\Omega_T), \partial_t b(\eta) \in L^2(\Omega_T)\},$$

$$V_h := \{\phi : \phi \in C([0,T]), \phi' \in L^\infty(0,T)\}$$

and

$$V_\xi := \{\psi : \psi \in W^{1,\infty}(0,T; L^\infty(\Omega)) \cap L^2(0,T; H^1(\Omega))\}.$$

It will be said that $(w, h, \xi) \in V_w \times V_h \times V_\xi$ is a bounded weak solution to the system (S), if the following conditions holds:

$$\int_0^T \int_\Omega \partial_t b(w) v + \int_0^T \int_\Omega [b(w) - b(w_0)] \partial_t v = 0, \tag{11}$$

$$\forall v \in L^2(0,T; H^1(\Omega)) \cap W^{1,1}(0,T; L^\infty(\Omega)) \quad \text{such that } v(\cdot, T) = 0,$$

$$\int_0^T \int_\Omega \partial_t b(w)(\eta - w) + \int_0^T \int_\Omega w_x (\eta - w)_x \geqslant \int_0^T \int_\Omega f(\xi, h, h', w)(\eta - w) \quad \forall \eta \in L^2(0,T;\mathbb{K}), \tag{12}$$

$$h(t) = \left[h_0^{5/2} - \frac{5M^{1/2}}{2} \int_0^t \left(\int_\Omega (2w(x,r))^{1/2}\,\mathrm{d}x \right)^{-1/2} \mathrm{d}r \right]^{2/5}, \quad t \in [0,T], \tag{13}$$

$$\xi(x,t) = \xi_0 + \int_0^t (2w(x,s))^{1/2} (h(s)h'(s))^2\,\mathrm{d}s \quad \text{a.e. } t \in (0,T) \text{ a.e. } x \in \Omega. \tag{14}$$

The main result of this paper states:

Theorem 3.1. *Let the function $w_0(x)$, the constants $\xi_0 > 0$ and $h_0 > 0$, and the positive constants L, M, T, Φ be admissible data in the sense given in Definition* 3.1, *then the system (S) has, at least, a* bounded weak solution $(w, h, \xi) \in V$.

Next, we shall prove the above theorem through several steps which will be developed in Sections 4 and 5.

4. Proof of Theorem 3.1: iterative decoupling

The first part of the proof consists in the application of an iterative decoupling procedure for the system (S). Let the parameter $j = 1, \ldots, J \to \infty$ denote the steps of the iterative decoupling scheme and let us define for each $j \in \mathbb{N}$ the system (S_j) as follows:

Definition 4.1. In the hypothesis of Theorem 3.1, for $j \in \mathbb{N}$ we shall consider the following decoupled system (S_j) for the variables $w_j \in V_w$, $h_j \in V_h$ and $\xi_j \in V_\xi$:

$$(S_j)\quad \begin{cases} w_j \geqslant \Phi & \text{in } \Omega_T,\\ \partial_t b(w_j) - (w_j)_{xx} - F_{j-1}(b(w_j))^{1/3} - D_{j-1} \geqslant 0 & \text{in } \Omega_T,\\ (\partial_t b(w_j) - (w_j)_{xx} - F_{j-1}(b(w_j))^{1/3} - D_{j-1})(w_j - \Phi) = 0 & \text{in } \Omega_T,\\ h_j' = -M^{1/2} h_j^{-3/2} \left(\int_\Omega A_j \,\mathrm{d}x\right)^{-1/2} & \text{in } \Omega_T,\\ \partial_t \xi_j = E_j A_j & \text{in } \Omega_T,\\ \partial_x w_j(0,t) = \partial_x w_j(L,t) = 0, & t \in (0,T),\\ w_j(x,0) = w_0(x), \quad h_j(x,0) = h_0, \quad \xi_j(x,0) = \xi_0, & x \in \Omega, \end{cases}$$

where the coefficient functions A_j, B_j, C_j, D_j, E_j and F_j are defined in terms of $w_j(x,t)$, $\xi_j(x,t)$ and $h_j(t)$, as follows:

$$A_j(x,t) = (2w_j(x,t))^{1/2}, \quad B_j(t) = (h_j(t)h_j'(t))^3, \quad C_j(x,t) = \frac{(h_j(t)h_j'(t))^2}{(\xi_j(x,t))^{1/2}},$$

$$D_j(t) = \gamma - \delta h_j^{-1}(t), \quad E_j(t) = (h_j(t)h_j'(t))^2, \quad F_j(x,t) = B_j(t) - C_j(x,t).$$

Note that the system (S_j) is composed of a problem for the variable w_j, given by $(S_j)_1 - (S_j)_3$ and $(S_j)_6 - (S_j)_7$, which we shall denote by $P(w_j)$ (detailed in Section 4.1). The coefficient functions in $P(w_j)$ do not depend on the variables h_j and ξ_j, but on h_{j-1} and ξ_{j-1} which are obtained in the previous step of the iterative scheme. So once we prove the existence of a solution w_j to $P(w_j)$, we can pass to solve the problem given by $(S_j)_4$ and $(S_j)_7$ for the ice thickness h_j (see Section 4.2), denoted by $P(h_j)$ and finally we tackle with the problem for the accumulated velocity ξ_j, consisting of $(S_j)_5$ and $(S_j)_7$, which will be denoted by $P(\xi_j)$ (treated in Section 4.3). Then the triple $\{(w_j(x,t), h_j(t), \xi_j(x,t))\}$, with $w_j(x,t)$, $h_j(t)$ and $\xi_j(x,t)$ being solutions to the problems $P(w_j)$, $P(h_j)$ and $P(\xi_j)$, respectively, results to be the unique b.w.s to the system (S_j) in the following sense:

Definition 4.2. It will be said that $(w_j, h_j, \xi_j) \in V$ is a *bounded weak solution* to (S_j) if the following conditions hold:

$$\int_0^T \int_\Omega \partial_t b(w_j) v + \int_0^T \int_\Omega [b(w_j) - b(w_0)] \partial_t v = 0,$$

$\forall v \in L^2(0,T; H^1(\Omega)) \cap W^{1,1}(0,T; L^\infty(\Omega))$ such that $v(\cdot, T) = 0$,

$$\int_0^T \int_\Omega \partial_t b(w_j)(\eta - w_j) + \int_0^T \int_\Omega (w_j)_x (\eta - w_j)_x \geqslant \int_0^T \int_\Omega f_j(\eta - w_j) \quad \forall \eta \in L^2(0,T; \mathbb{K}),$$

where

$$f_j := [B_{j-1} - C_{j-1}] A_j + D_{j-1},$$

$$h_j(t) = \left[h_0^{5/2} - [5/2] M^{1/2} \int_0^t \left(\int_\Omega A_j(x,r)\,\mathrm{d}x \right)^{-1/2} \mathrm{d}r \right]^{2/5}, \quad t \in [0,T],$$

$$\xi_j(x,t) = \xi_0 + \int_0^t A_j(x,s) E_j(s)\,\mathrm{d}s \quad \text{a.e. } t \in (0,T) \text{ a.e. } x \in \Omega.$$

Properties 3.1. *The coefficient functions $A_j, \ldots, F_j$, $j \in \mathbb{N}$, satisfy the following regularity and monotonicity properties*:

(1) $A_j \in L^\infty(\Omega_T) \cap L^2(0,T;H^1(\Omega)) \cap C([0,T];L^2(\Omega))$ *and* $A_j \geqslant (2\Phi)^{1/2} > 0$, *a.e.* $t \in (0,T)$, *a.e.* $x \in \Omega$,
(2) $B_j \in L^\infty(0,T)$ *and* $B_j(t) > 0$, *a.e.* $t \in (0,T)$,
(3) $C_j \in L^\infty(\Omega_T)$, $C_j(x,t) > 0$, *a.e.* $(x,t) \in \Omega_T$,
(4) $D_j \in C([0,T])$, $\mathrm{d}D_j/\mathrm{d}t = D'_j \in L^\infty(0,T)$ *and* $D'_j > 0$, *a.e.* $t \in (0,T)$,
(5) $E_j \in L^\infty(0,T)$ *and* $E_j(t) > 0$, *a.e.* $t \in (0,T)$,
(6) $F_j \in L^\infty(\Omega_T)$.

Remark 4.1. Note that the properties are derived from the fact that $(w_j, h_j, \xi_j) \in V$. Moreover, we shall see that the norms in the space $L^\infty(0,T)$ of the functions B_j, C_j, D_j, D'_j and E_j are uniformly bounded with respect to $j \in \mathbb{N}$.

In order to initiate the iterative decoupling scheme, we shall consider, as usually made in these kind of strategies (see for instance [7]), the functions $h_0(t)$ and $\xi_0(x,t)$ obtained by means of extending continuously the data h_0 and ξ_0 to the whole domain, i.e., $h_0(t) = h_0$ and $\xi_0(x,t) = \xi_0$, $\forall t \in (0,T)$. As a consequence we obtain the coefficient functions B_0, C_0 (note that both functions result to be the null function as $h'_0(t) \equiv 0$) and D_0 to be included in system (S_1). Then, once we have defined the system (S_1) as starting point of the iterative process, we proceed to study the existence of solution to (S_j) for a *j*th arbitrary step, assuming that the coefficient functions that come from the previous step have some regularity and monotonicity properties. Such assumptions turn out to be real facts as the coefficient functions entering in system (S_j) are defined in terms of the solution to (S_{j-1}) which belong, as we shall prove, to the functional space V. To be precise, the main goal of this section will be to prove the following theorem:

Theorem 4.1. *For each* $j \in \mathbb{N}$, *let* $w_0(x)$, $\xi_0 > 0$ *and* $h_0 > 0$, *and the positive constants* L, M, T, Φ, *be admissible data in the sense of Definition* 3.1. *Then there exists a unique bounded weak solution* $(w_j, h_j, \xi_j) \in V$ *to the system* S_j.

The proof of Theorem 4.1 will amount to proving the existence of solution to each one of the problems $P(w_j)$, $P(h_j)$ and $P(\xi_j)$. This is done throughout the next three sections. Therefore, we apply an inductive reasoning to build a sequence $\{(w_j(x,t), h_j(t), \xi_j(x,t))\}$ consisting of the unique b.w.s. to systems (S_j), $j \in \mathbb{N}$.

4.1. Decoupled problem related to the water flux

In this section, we prove the existence and uniqueness of a solution to $(S_j)_1 - (S_j)_3$ and $(S_j)_6 - (S_j)_7$, i.e., $P(w_j)$, departing from the assumption that we have already proved the existence and uniqueness of solution to the system S_{j-1}.

Definition 4.3. For $j \in \mathbb{N}$, let the function $w_0(x)$ and the positive constants L, M, T, Φ be admissible data, let the coefficient functions D_{j-1} and F_{j-1} satisfy Properties 3.1. We define the unilateral obstacle problem $P(w_j)$ related to the equations $(S_j)_1 - (S_j)_4$ and to the conditions $(S_j)_6 - (S_j)_7$, as follows:

$$P(w_j) \quad \begin{cases} w_j \geqslant \Phi & \text{in } \Omega_T, \\ \partial_t b(w_j) - (w_j)_{xx} - F_{j-1}(b(w_j))^{1/3} - D_{j-1} \geqslant 0 & \text{in } \Omega_T, \\ [\partial_t b(w_j) - (w_j)_{xx} - F_{j-1}(b(w_j))^{1/3} - D_{j-1}](w - \Phi) = 0 & \text{in } \Omega_T, \\ \partial_x w_j(0,t) = \partial_x w_j(L,t) = 0, & t \in (0,T), \\ w_j(x,0) = w_0(x), & x \in \Omega. \end{cases}$$

It will be said that $w_j \in V_w$ is a *bounded weak solution* to the problem $P(w_j)$ (see the definition given in [1]) if the following conditions hold:

$$\int_0^T \int_\Omega \partial_t b(w_j) v + \int_0^T \int_\Omega [b(w_j) - b(w_0)] \partial_t v = 0, \tag{15}$$

$\forall v \in L^2(0, T; H^1(\Omega)) \cap W^{1,1}(0, T; L^\infty(\Omega))$ such that $v(\cdot, T) = 0$,

$$\int_0^T \int_\Omega \partial_t b(w_j)(\eta - w_j) + \int_0^T \int_\Omega (w_j)_x (\eta - w_j)_x$$

$$\geqslant \int_0^T \int_\Omega [F_{j-1}(b(w_j))^{1/3} + D_{j-1}](\eta - w_j) \quad \forall \eta \in L^2(0, T; \mathbb{K}). \tag{16}$$

Lemma 4.1. *There exists a unique bounded weak solution to the problem* $P(w_j)$.

Proof (*Existence part*). First of all, we note that if w_j is a solution to the problem $P(w_j)$ in the sense of Definition 4.2 then applying well known results (see for instance [3]) we get the following estimate:

$$\|b(w_j)\|_{L^p(\Omega_T)} \leqslant e^{C^* T} \left\{ \|b(w_0)\|_{L^p(\Omega)} + L^{1/p} \int_0^T e^{-C^* s} |D_{j-1}(s)| \, ds \right\}, \tag{17}$$

for $1 \leqslant p \leqslant \infty$. In particular, the estimate (17) implies the existence of a positive constant W such that $\|w_j\|_{L^\infty(\Omega_T)} < W$. Later, it will be proved that W can be chosen uniformly in $j \in \mathbb{N}$. Next, we start considering a sequence of regularized problems, denoted by $P_{n,j}(w)$, $n \in \mathbb{N}$, which approximate the problem $P(w_j)$ by means of replacing $F_{j-1}(x, t)(b(w))^{1/3}$ by

$$Y_n(b(w)) := Y_n(x, t, b(w(x, t))) = F_{j-1}(x, t) p_n(b(w)), \tag{18}$$

where $\{p_n\}$ is a sequence of Yosida approximations (and therefore, Lipschitz continuous) for the function $p(s) = s^{1/3}$.

Definition 4.4. Under the hypothesis of Definition 4.3 and Y_n as in (18), we consider the unilateral obstacle problem $P_{n,j}$ given by

$$P_{n,j} \quad \begin{cases} w \geqslant \Phi & \text{in } \Omega_T, \\ \partial_t b(w_{nj}) - (w_{nj})_{xx} - Y_n(b(w_{nj})) - D_{j-1} \geqslant 0 & \text{in } \Omega_T, \\ [\partial_t b(w_{nj}) - (w_{nj})_{xx} - Y_n(b(w_{nj})) - D_{j-1}](w_{nj} - \Phi) = 0 & \text{in } \Omega_T, \\ \partial_x w_{nj}(0, t) = \partial_x w_{nj}(L, t) = 0, & t \in (0, T), \\ w_{nj}(x, 0) = w_0(x), & x \in \Omega. \end{cases}$$

It will be said that $w_{nj} \in V_w$ is a *bounded weak solution* to $P_{n,j}$, if the following conditions hold:

$$\int_0^T \int_\Omega \partial_t b(w_{nj}) v + \int_0^T \int_\Omega [b(w_{nj}) - b(w_0)] \partial_t v = 0, \tag{19}$$

$\forall v \in L^2(0, T; H^1(\Omega)) \cap W^{1,1}(0, T; L^\infty(\Omega))$ such that $v(\cdot, T) = 0$,

$$\int_0^T \int_\Omega \partial_t b(w_{nj})(\eta - w_{nj}) + \int_0^T \int_\Omega (w_{nj})_x (\eta - w_{nj})_x \tag{20}$$

$$\geqslant \int_0^T \int_\Omega (Y_n(b(w_{nj})) + D_{j-1})(\eta - w_{nj}) \quad \forall \eta \in L^2(0, T; \mathbb{K}). \tag{21}$$

Let $j \in \mathbb{N}$ be fixed, let w_{nj} +ion of the problem $P_{n,j}$, then $w_{nj}(x, t) \geqslant \Phi$, a.e. $(x, t) \in \Omega_T$, hence, we only care the properties of the functions p_n and p in the interval $[(2\Phi)^{3/2}, \infty)$. Note that $p_n(b)$ (and $p(b)$) is Lipschitz continuous for $b \geqslant (2\Phi)^{3/2}$. Then, we can apply well known results (see [3]) to obtain that $\forall n \in \mathbb{N}$,

$$\|b(w_{nj})\|_{L^p(\Omega_T)} \leqslant \mathrm{e}^{C^*T}\left\{\|b(w_0)\|_{L^p(\Omega)} + L^{1/p}\int_0^T \mathrm{e}^{-C^*s}|D_{j-1}(s)|\,\mathrm{d}s\right\},$$

for $1 \leqslant p \leqslant \infty$. Therefore, due to the monotonicity of b, we get that there exists a constant W such that

$$\|w_{nj}\|_{L^\infty(\Omega_T)} \leqslant W.$$

In order to prove the result of existence of b.w.s. to the problem $P_{n,j}$ we shall resort to a penalization technique, arguing in a similar way to that performed in [1]. In fact, for $n, j \in \mathbb{N}$, we shall consider the following regularized problems P_{rnj}, which approximate the problem $P_{n,j}$.

Definition 4.5. For $r \in \mathbb{N}$, let us consider the problem P_{rnj} given by

$$P_{rnj} \quad \begin{cases} \partial_t b(w_{rnj}) - (w_{rnj})_{xx} + r\mathbf{j}(w_{rnj} - \mathbf{P}w_{rnj}) = Y_n(b(w_{rnj})) + D_{j-1} & \text{in } \Omega_T, \\ \partial_x w_{rnj}(0,t) = \partial_x w_{rnj}(L,t) = 0, & t \in (0,T), \\ w_{rnj}(x,0) = w_0(x), & x \in \Omega, \end{cases}$$

where $\mathbf{j}$ is a monotone and convex, duality operator $\mathbf{j} : H^1(\Omega) \to (H^1(\Omega))^*$ defined as follows:

$$\langle \mathbf{j}(w), \psi\rangle = \int_\Omega (w\psi + w_x\psi_x) \quad \forall w, \psi \in H^1(\Omega)$$

and $\mathbf{P}$ is projection operator over the convex set $\mathbb{K}$, $\mathbf{P} : H^1(\Omega) \to \mathbb{K}$ and

$$\langle \mathbf{j}(w - \mathbf{P}w), \mathbf{P}w - v\rangle \geqslant 0 \quad \text{for } v \in \mathbb{K}.$$

It will be said that $w_{rnj} \in V_w$ is a *bounded weak solution* to the problem P_{rnj}, if the following conditions hold:

$$\int_0^T\int_\Omega \partial_t b(w_{rnj})v + \int_0^T\int_\Omega [b(w_{rnj}) - b(w_0)]\partial_t v = 0, \tag{22}$$

$$\forall v \in L^2(0,T;H^1(\Omega)) \cap W^{1,1}(0,T;L^\infty(\Omega)) \quad \text{such that } v(\cdot,T) = 0.$$

$$\begin{aligned} &\int_0^T\int_\Omega \partial_t b(w_{rnj})\psi + \int_0^T\int_\Omega (w_{rnj})_x\psi_x + r\int_0^T \langle \mathbf{j}(w_{rnj} - \mathbf{P}w_{rnj}), \psi\rangle \\ &\quad = \int_0^T\int_\Omega (Y_n(b(w_{rnj})) + D_{j-1})\psi \quad \forall \psi \in L^2(0,T;H^1(\Omega)). \end{aligned} \tag{23}$$

Note that $w_{rnj}(\cdot,t) \in L^\infty(\Omega)$, a.e. $t \in (0,T)$ and hence $S(w_{rnj}(\cdot,t)) = [6\sqrt{2}/5](w_{rnj}(\cdot,t))^{5/2} \in L^\infty(\Omega)$. Using the integration by parts formula (see [1]), we obtain that

$$\begin{aligned} \int_0^t\int_\Omega \partial_t b(w_{rnj})w_{rnj} &= \int_0^t\int_\Omega \partial_t[b(w_{rnj})w_{rnj}] - \int_0^t\int_\Omega b(w_{rnj})\partial_t(w_{rnj}) \\ &= b(w_{rnj}(t))w_{rnj}(t) - b(w_0)w_0 + \frac{2}{3}[S(w_{rnj}(t)) - S(w_0)] \\ &= S(w_{rnj}(t)) - S(w_0). \end{aligned}$$

Then,

$$\int_0^t\int_\Omega \partial_t b(w_{rnj})w_{rnj} = \int_\Omega S(w_{rnj}(t)) - \int_\Omega S(w_0) \quad \text{a.e. } t \in (0,T).$$

Concerning the existence of solution for the problem P_{rnj}, it is a result of a well known technique (see for instance [2]) that resides in using a time semidiscretization. In fact, we shall replace the term $\partial_t b(w_{rnj})$ in P_{rnj} by the backward difference quotient $\partial_t^{-h} b(w_{rnj})$, defined by

$$\partial_t^{-h} b(w_{rnj}(\tau)) = (b(w_{rnj}(\tau)) - b(w_{rnj}(\tau - h)))/h, \quad \tau \in (0, T),$$

assuming that $w_{rnj}(\tau) = w_{rnj}^0 = w_0$, a.e. $\tau \in (-h, 0)$. This leads to a family of elliptic problems for which the existence of solution is a well known classical result (note the fundamental fact that $Y_n(t, x, b)$ are Lipschitz continuous with respect to the third variable), moreover those solutions converge to the solution of the problem P_{rnj} (see [2,4]). So, we have already proved the existence of a unique solution to the problem P_{rnj}.

Lemma 4.2. *There exists at least one solution to the problem $P_{n,j}$.*

Proof. Once we know about the existence of solution to the problem P_{rnj}, we shall obtain some a priori estimates of the energy of the solutions w_{rnj} in order to obtain a subsequence of $\{w_{rnj}\}$ ($n, j \in \mathbb{N}$ fixed and $r \to \infty$) that will converge to a function w_{nj}, which in turn will be a solution to $P_{n,j}$. To be precise, we shall get two estimates: from the first one, we shall deduce the existence of a subsequence of $\{w_{rnj}\}$ which will converge to a function w_{nj} in the weak topology of $L^2((0, T); H^1(\Omega))$, and from the second we will deduce the regularity of the parabolic term, to precise, we shall obtain that $\partial_t b(w_{rnj}) \in L^2(\Omega_T)$. For the sake of clarity, we opt for the notation $w = w_{rnj}$ and employ the expression $w(t)$, $t \in [0, T]$ to denote the function $w(t) : \Omega \to \mathbb{R}$ such that $w(t)(x) = w(x, t)$.

First estimate: We consider $(P_{rnj})_1$ and we multiply it by $\psi = w - w_0$, where w_0 is the prescribed data at $t = 0$ and we integrate over $(0, t) \times \Omega$,

$$\int_0^t \int_\Omega [\partial_t b(w)\psi + w_x(\psi)_x] + r \int_0^t \langle \mathbf{j}(w - \mathbf{P}w), \psi \rangle = \int_0^t \int_\Omega (Y_n(b(w)) + D_{j-1})\psi.$$

Then we have the following estimates for each of the integral terms:

$$\int_0^t \int_\Omega \partial_t b(w)\psi \geqslant (1-\delta) \int_\Omega S(w(t)) - (1 + C_\delta) \int_\Omega S(w_0),$$

$$\int_0^t \int_\Omega w_x \psi_x = \int_0^t \int_\Omega |w_x|^2 - \int_0^t \int_\Omega w_x (w_0)_x \geqslant (1-\delta) \int_0^t \int_\Omega |w_x|^2 - C_\delta \int_0^t \int_\Omega |(w_0)_x|^2,$$

$$r \int_0^t \langle \mathbf{j}(w - \mathbf{P}w), \psi \rangle \geqslant r \int_0^t \|w - \mathbf{P}w\|_{H^1(\Omega)}^2$$

and

$$\int_0^t \int_\Omega [Y_n(b(w)) + D_{j-1}]\psi \leqslant \int_0^t \int_\Omega C_1 S(w) + C_2 \int_\Omega [1 + S(w_0) + S^2(w_0)],$$

for some constants C_1 and C_2, and $\forall n, j, r \in \mathbb{N}$. From the above estimates we deduce that

$$\begin{aligned}(1-\delta) \int_\Omega S(w(t)) - (1 + C_\delta) \int_0^t \int_\Omega S(w_0) + (1-\delta) \int_0^t \int_\Omega |w_x|^2 \\ - C_\delta \int_0^t \int_\Omega |(w_0)_x|^2 + r \int_0^t \|w - \mathbf{P}w\|_{H^1(\Omega)}^2 \\ \leqslant \int_0^t \int_\Omega C_1 S(w) + C_2 \int_\Omega [1 + S(w_0) + S^2(w_0)],\end{aligned}$$

where C_1, C_2 and C_δ are positive constants. Ordering the above terms, we get that

$$(1-\delta) \left(\int_\Omega S(w(t)) + \int_0^t \int_\Omega |w_x|^2 \right) + r \int_0^t \|w - \mathbf{P}w\|_{H^1(\Omega)}^2 \leqslant C_1 \int_0^t \int_\Omega S(w) + C,$$

a.e. $t \in (0, T)$ where $0 < \delta \ll 1$ and C a positive constant. The non-negativeness of the second and third terms that appear in the left-hand side of the above inequality leads to the following inequality:

$$(1-\delta)\int_\Omega S(w(t)) \leqslant C_1 \int_0^t \int_\Omega S(w) + C.$$

Applying Gronwall's inequality and taking $\delta \to 0$, we get that there exists a positive constant C_{E1}, such that

$$\operatorname*{sup\,ess}_{\{0<t<T\}} \int_\Omega S(w(t)) + \int_0^t \int_\Omega |w_x|^2 + r\int_0^T \|w - \mathbf{P}w\|^2_{H^1(\Omega)} \leqslant C_{E1}. \tag{24}$$

Note that the constant C_{E1} can be chosen in a way that it does not depend on the parameters r and n, allowing us to deduce the existence of a subsequence of $\{w_{rnj}\}$ for n and j fixed, that we shall labelled by $\{w_{rnj}\}$, converging to a function w_{nj} in the weak topology of the space $L^2(0, T; H^1(\Omega))$. Later on it will be shown that we can find a constant C_{E1} valid $\forall j \in \mathbb{N}$.

Second estimate: In this case, we multiply $(P_{rnj})_1$ by $\partial_t w$ and we integrate over $(0, t) \times \Omega$ to obtain that

$$\int_0^t \int_\Omega [\partial_t b(w)\partial_t w + w_x(\partial_t w)_x] + r\int_0^t \langle \mathbf{j}(w - \mathbf{P}w), \partial_t w\rangle = \int_0^t \int_\Omega (Y_n(b(w)) + D_{j-1})\partial_t w.$$

Several simple calculations lead to the following estimates:

$$\int_0^t \int_\Omega \partial_t b(w)\partial_t w \geqslant C_b \int_0^t \int_\Omega |\partial_t b(w)|^2 \quad \text{with } C_b \text{ a positive constant,} \tag{25}$$

$$\int_0^t \int_\Omega w_x(\partial_t w)_x = \frac{1}{2}\int_0^t \int_\Omega \partial_t[(w_x)^2] = \frac{1}{2}\int_\Omega |w_x|^2(t) - \frac{1}{2}\int_\Omega |(w_0)_x|^2, \tag{26}$$

$$r\int_0^t \langle \mathbf{j}(w - \mathbf{P}w), \partial_t w\rangle \geqslant (r/2)\|w(t) - \mathbf{P}w(t)\|^2_{H^1(\Omega)}. \tag{27}$$

Regarding the obtention of (27), we approximate the partial derivative $\partial_t w$ by the backward difference quotient, $\partial_t^{-h} w = (w(t) - w(t-h))/h$, where we assume that $w(x, t) = w_0(x)$ for $t \in (-h, 0)$ and we estimate the left-hand side integral term of (27) considering $\partial_t^{-h} w$ instead of $\partial_t w$. Let us take h satisfying $0 < h \ll 1$, then

$$\begin{aligned} r\int_0^t \langle \mathbf{j}(w - \mathbf{P}w), \partial_t^{-h} w\rangle\,\mathrm{d}s &= \frac{r}{h}\int_0^t \langle \mathbf{j}(w - \mathbf{P}w), w(s) - \mathbf{P}w(s)\rangle\,\mathrm{d}s \\ &+ \frac{r}{h}\int_0^t \langle \mathbf{j}(w - \mathbf{P}w), \mathbf{P}w(s) - \mathbf{P}w(s-h)\rangle\,\mathrm{d}s \\ &+ \frac{r}{h}\int_0^t \langle \mathbf{j}(w - \mathbf{P}w), (\mathbf{P}w - w)(s-h)\rangle\,\mathrm{d}s. \end{aligned}$$

Taking into consideration the convexity of $\mathbf{j}$, that $\mathbf{P}w(s-h) \in \mathbb{K}$ and that $\langle \mathbf{j}(w - \mathbf{P}w), \mathbf{P}w(s) - \mathbf{P}w(s-h)\rangle \geqslant 0$, we derive that

$$\begin{aligned} r\int_0^t \langle \mathbf{j}(w - \mathbf{P}w), \partial_t^{-h} w\rangle &\geqslant (r/2h)\int_0^t \|w(s) - \mathbf{P}w(s)\|^2_{H^1(\Omega)} - (r/2h)\int_0^t \|w(s-h) - \mathbf{P}w(s-h)\|^2_{H^1(\Omega)} \\ &= (r/2h)\int_{t-h}^t \|w(s) - \mathbf{P}w(s)\|^2_{H^1(\Omega)}. \end{aligned}$$

Then taking the limit $h \to 0$, we find (27). Finally, we present the relating estimate to the source term, which is

$$\begin{aligned} &\left|\int_0^t \int_\Omega [Y_n(b(w)) + D_{j-1}]\partial_t w\right| \\ &\quad \leqslant C_F\left(K_\delta + K_1\int_\Omega [2 + S(w(t)) + S(w_0)] + \delta\int_0^t \int_\Omega |\partial_t(b(w))|^2\right) + C_D, \end{aligned} \tag{28}$$

where C_F, C_D (referred to the bounds of F_{j-1} and D_{j-1} in the spaces $L^\infty(\Omega_T)$ and $L^\infty(0,T)$, respectively), K_1 and K_δ are positive constants. As a consequence of (25)–(28), we obtain

$$\begin{aligned}(1-\delta C_F)\int_0^t\int_\Omega |\partial_t b(w)|^2 + (r/2)\|w(t)-\mathbf{P}w(t)\|^2_{H^1(\Omega)} + (1/2)\int_\Omega |w_x|^2(t)\\ \leqslant (1/2)\int_\Omega |(w_0)_x|^2 + C_F\left(K_\delta + K_1\int_\Omega [2+S(w(t))+S(w_0)]\right) + C_d.\end{aligned} \tag{29}$$

The regularity of the function $S(w)$ allows us to deduce from the inequality (29) that $\partial_t(b(w)) \in L^2(\Omega_T)$, $\forall n, r \in \mathbb{N}$. And finally, we have

$$\int_0^t\int_\Omega |\partial_t b(w)|^2\,\mathrm{d}x\,\mathrm{d}s + \frac{r}{2}\|w(t)-\mathbf{P}w(t)\|^2_{H^1(\Omega)} + \frac{1}{2}\int_\Omega |w_x|^2(t)\,\mathrm{d}x \leqslant C_{E2}, \tag{30}$$

a.e. $t\in(0,T)$, $\forall n, j, r\in\mathbb{N}$, with C_{E2} a positive constant C_{E2} which does not depend on the subindexes. Recall that we have been using the notation $w_{rnj}=w$.

Thanks to the uniform bound in (30) we deduce the existence of a subsequence of $\{\partial_t(b(w_{rnj}))\}$, that we shall label by $\{\partial_t(b(w_{rnj}))\}$, weakly convergent to a function $\partial_t(b(w_{nj}))$. Note that w_{nj} is in fact the weak limit of $\{w_{rnj}\}$ in $L^2(0,T;H^1(\Omega))$. Moreover

$$\int_0^T\int_\Omega \partial_t(b(w_{nj}))v = -\int_0^T\int_\Omega [b(w_{nj})-b(w_0)]\partial_t v \quad \forall v\in\mathcal{M},$$

where $\mathcal{M}=\{\eta:\ \eta\in L^2(0,T;H^1(\Omega))\cap W^{1,1}(0,T;L^\infty(\Omega)),\ \eta(\cdot,T)=0\}$. Note that w_{rnj} is a solution of the penalization problem, so we have that

$$\int_0^T\int_\Omega \partial_t(b(w_{rnj}))v = -\int_0^T\int_\Omega [b(w_{rnj})-b(w_0)]\partial_t v \quad \forall r\in\mathbb{N}\ \ \forall v\in\mathcal{M}.$$

Due to the fact that the sequence $\{\partial_t(b(w_{rnj}))\}$ is uniformly bounded with respect to r in $L^2(\Omega_T)$ by (30), there exists a subsequence, which will be labelled in the same way, that converges weakly to a function λ. As λ is the weak limit of $\{\partial_t(b(w_{rnj}))\}$ and $\mathcal{M}\subset L^2(\Omega_T)$, then

$$\lim_{r\to\infty}\int_0^T\int_\Omega \partial_t(b(w_{rnj}))v = \int_0^T\int_\Omega \lambda v \quad \forall v\in\mathcal{M}.$$

On the other hand, as w_{nj} is the weak limit of $\{w_{rnj}\}$ in $L^2(0,T;H^1(\Omega))$, we have that $w_{rnj}\to w_{nj}$, a.e. $(x,t)\in\Omega_T$ and therefore $\{b(w_{rnj})\}$ converges almost everywhere to $b(w_{nj})$. Then we have that for each $v\in\mathcal{M}$:

$$-\lim_{r\to\infty}\int_0^T\int_\Omega [b(w_{rnj})-b(w_0)]\partial_t v = -\int_0^T\int_\Omega [b(w_{nj})-b(w_0)]\partial_t v.$$

Then, by the uniqueness property of the limit,

$$\int_0^T\int_\Omega \lambda v = -\int_0^T\int_\Omega [b(w_{nj})-b(w_0)]\partial_t v \quad \forall v\in\mathcal{M},$$

that is to say $\lambda=\partial_t b(w_{nj})$ and, therefore, w_{nj} satisfies (19). Again, we resort to well known results (see for instance Benilan [3]) to obtain that

$$\|b(w_{rnj})\|_{L^p(\Omega_T)} \leqslant \mathrm{e}^{c_n^* T}\left\{\|b(w_0)\|_{L^p(\Omega)} + L^{1/p}\int_0^T \mathrm{e}^{-c_n^* s}|D_{j-1}(s)|\,\mathrm{d}s\right\}$$

and

$$\|w_{rnj}\|_{L^\infty(\Omega_T)} \leqslant W \quad \forall r\in\mathbb{N}. \tag{31}$$

Previous estimates allow us to apply well known convergence results so that to obtain a subsequence of $\{w_{rnj}\}$ that will be labelled in the same way and a function w_{nj}, such that

$$w_{rnj} \rightharpoonup w_{nj} \quad \text{in } L^2(0,T;H^1(\Omega)) \quad \text{and} \quad w_{rnj} \to w_{nj} \quad \text{in } L^p(\Omega_T)\ \forall p \in [1,\infty),$$

$$\partial_t b(w_{rnj}) \rightharpoonup \partial_t b(w_{nj}) \text{ in } L^2(\Omega_T), \quad S(w_{rnj}) \to S(w_{nj}) \text{ in } L^p(\Omega_T),\ p \in [1,\infty),$$

$$Y_n(x,t,b(w_{rnj})) \to Y_n(x,t,b(w_{nj})) \quad \text{in } L^p(\Omega_T), \quad p \in [1,\infty).$$

Note that $w_{rnj} \to w_{nj}$, a.e. $(x,t) \in \Omega_T$ and

$$\|w_{rnj}(t) - \mathbf{P}w_{rnj}(t)\|^2_{H^1(\Omega)} \leqslant (2/r)C_{E2} \to 0 \quad \text{as } r \to \infty. \tag{32}$$

Therefore, $w_{nj} = \mathbf{P}w_{nj}$, a.e. $(x,t) \in \Omega_T$ i.e., $w_{nj} \in \mathbb{K}$.

End of Proof of Lemma 4.2. Now, let w_{nj} be the function just obtained. Let us take limit with respect to r in the estimates (24) and (30), where the constants that appeared there, i.e., C_{E1} and C_{E2}, do not depend on the subindexes $\{r,n\}$. Then we find that

$$\sup_{\{0<t<T\}} ess \int_\Omega S(w_{nj}(t)) + \int_0^t \int_\Omega |(w_{nj})_x|^2 \leqslant C_{E1}, \tag{33}$$

$$\int_0^t \int_\Omega |\partial_t b(w_{nj})|^2 + \frac{1}{2}\int_\Omega |(w_{nj})_x|^2(t) \leqslant C_{E2} \quad \text{a.e. } t \in (0,T)\ \forall n, j \in \mathbb{N}, \tag{34}$$

where C_{E1} and C_{E2} are two suitable positive constants. Now, we can take limits in a weak sense in conditions (22) and (23) to get that w_{nj} satisfies conditions (19), (21) and therefore is a bounded weak solution to the problem $P_{n,j}(w)$. □

Next, we present a result of convergence regarding the sequence of solution $\{w_{nj}\}_n$, j fixed and $n \in \mathbb{N}$, whose proof resides in an application of a weak version of *Ascoli–Arzelá*'s theorem (see [21]).

Lemma 4.3. *Let w_{nj} be a bounded weak solution to the problem $P_{n,j}(w)$, then there exists a positive constant C such that*

$$\|Y_n(\cdot,t,b(w_{nj}))\|_{L^\infty(\Omega)} \leqslant C \quad \textit{a.e. } t \in (0,T)\ \forall n \in \mathbb{N}. \tag{35}$$

Moreover, the family $\{b(w_{nj})\}$ is equi-continuous in the following sense:

$$\|b(w_{nj}(t)) - b(w_{mj}(t))\|^2_{L^1(\Omega)} \leqslant M\left(\frac{1}{n} + \frac{1}{m}\right) \quad \forall t \in [0,T],$$

where M is a positive constant that only depends on the constants W and C (see (31) *and* (35)).

Proof. Let $\{w_{nj}\}$ be the sequence of solutions associated to the problems $P_{n,j}(w)$. As a consequence of (31), we have that

$$\|b(w_{nj}(\cdot,t))\|_{L^\infty(\Omega)} \leqslant b(W) \quad \forall t \in [0,T]\ \forall n, j \in \mathbb{N}. \tag{36}$$

Taking into account that $Y_n(b(w)) = F_{j-1}p_n(b(w))$, where p_n is the Yosida approximation of p, that $Y_n(b(w)) \leqslant F_{j-1}p(b(w))$, $\forall n \in \mathbb{N}$ and (36), we deduce the existence of a positive constant C, such that (35) is verified.

Being $-w_{xx}+\beta(w-\Phi)$ a m-accretive operator in $L^1(\Omega)$ and $b(w_{nj}(0))=b(w_0)$, $\forall n\in\mathbb{N}$, and we have that (see [21]),

$$
\begin{aligned}
&\|b(w_{nj}(t))-b(w_{mj}(t))\|^2_{L^1(\Omega)}\\
&\quad\leqslant 2\int_0^t (b(w_{nj}(s))-b(w_{mj}(s)),\, F_{j-1}(s)[p_n(b(w_{nj}(s)))-p_m(b(w_{mj}(s)))])_+\,\mathrm{d}s\\
&\quad\leqslant 2\int_0^t (J_n(b(w_{nj}))-J_m(b(w_{mj})),\, F_{j-1}[p_n(b(w_{nj}))-p_m(b(w_{mj}))])_+\,\mathrm{d}s\\
&\qquad+2\int_0^t \left(\frac{1}{n}p_n(b(w_{nj}))-\frac{1}{m}p_m(b(w_{mj})),\, F_{j-1}[p_n(b(w_{nj}))-p_m(b(w_{mj}))]\right)_+\mathrm{d}s\\
&\quad\leqslant M\left(\frac{1}{n}+\frac{1}{m}\right),
\end{aligned}
\tag{37}
$$

where $(\cdot,\cdot)_+$ denotes the upper semi-interior product with respect to the usual norm of the space $L^1(\Omega)$ (see [21]), I is the identity operator and J_n is the resolvent. In the inequality (37) it has been used that $p_n(b)=n(I-J_n)(b)$ and therefore, $I=(1/n)p_n+J_n$. Using (35), (36) and the fact that J_n is a contraction (see [5]), we have that there exists a positive constant M such that (36) holds. □

Next, we observe that Lemma 4.3 allows for an application of a weak version of the Ascoli–Arzelá's theorem (see [21]) to the sequence $\{b(w_{nj})\}$, from which we derive the existence of a subsequence of $\{b(w_{nj})\}$, that we shall denote in the same way, and a function $\tilde{b}$, such that $\lim_{n\to\infty} b(w_{nj})=\tilde{b}$, strongly in the topology of $C([0,T];L^1(\Omega))$. Being the norms $\{\|w_{nj}\|_{L^\infty(\Omega)}\}$ uniformly bounded in $n\in\mathbb{N}$ and b strictly monotone, there exists a function w_j such that $\lim_{n\to\infty} w_{nj}=b^{-1}(\tilde{b})=w_j$ in $C([0,T];L^1(\Omega))$. Note that the constant M, in (37), at first seems to depend on the parameter $j\in\mathbb{N}$, however later on we shall prove that it can be chosen so that it does not. Next, we pass to consider the last part of the proof of Lemma 4.1, in which we shall check that the function w_j is a solution of the problem $P(w_j)$.

End of proof of the existence part of Lemma 4.1. From estimates (32)–(34), we derive that $w_{nj}\in\mathbb{K}$ and the uniform boundwith respect to $n\in\mathbb{N}$ of the norms $\|\partial_t b(w_{nj})\|_{L^2(\Omega_T)}$ and $\|w_{nj}\|_{L^2(0,T;H^1(\Omega))}$. These properties allow us to deduce the existence of a subsequence of $\{w_{nj}\}$, that converges to a function w_j in the topology of $C([0,T];L^1(\Omega))$ and weakly in $L^2(0,T;H^1(\Omega))$. Moreover,

$$
\begin{aligned}
&w_{nj}\to w_j,\quad b(w_{nj})\to b(w_j)\ \text{strongly in}\ L^p(\Omega_T),\quad 1\leqslant p<\infty,\\
&\partial_t b(w_{nj})\rightharpoonup \partial_t b(w_j)\ \text{weakly in}\ L^2(\Omega_T),\quad w_j(t)\in\mathbb{K}\quad \text{a.e. } t\in[0,T],\\
&Y_n(b(w_{nj}))=F_{j-1}p_n(b(w_{nj}))\to F_{j-1}p(b(w_j))\ \text{strongly in}\ L^p(\Omega_T),\quad 1\leqslant p<\infty.
\end{aligned}
$$

So, if we pass to the limit in a weak sense with respect to n in (33) and (34) then (note that the positive constants C_{E1} and C_{E2} appearing there did not depend on the parameters $n\in\mathbb{N}$) it results that w_j satisfies the estimates

$$
\mathop{sup\,ess}_{\{0<t<T\}}\int_\Omega S(w_j(t))+\int_0^T\int_\Omega |(w_j)_x|^2\leqslant C_{E1},
\tag{38}
$$

$$
\int_0^t\int_\Omega |\partial_t b(w_j)|^2+(1/2)\int_\Omega |(w_j)_x|^2(t)\leqslant C_{E2}\quad \text{a.e. } t\in(0,T),\quad j\in\mathbb{N},
\tag{39}
$$

and as a consequence we deduce that the functions $w_j\in V_w$, $j\in\mathbb{N}$ and are uniformly bounded in the norms of the spaces considered in V_w. Note also that taking limits in (19) and (21) we obtain that w_j is a solution of $P(w_j)$ in the sense given in Definition 4.3. □

Proof of the uniqueness part of Lemma 4.1. Regarding the uniqueness of solution to problem $P(w_j)$, we start presenting a comparison result.

Definition 4.6. Under the hypothesis considered in Definitions 4.2 and 4.3, we will consider the obstacle problems P_i, defined as follows:

$$P_i \quad \begin{cases} \partial_t b(u_i) - (u_i)_{xx} + \beta(u_i - \Phi) - Fp(b(u_i)) \ni D_i & \text{in } \Omega_T, \\ \partial_x u_i(0,t) = \partial_x u_i(L,t) = 0, & t \in (0,T), \\ u_i(x,0) = u_{0,i}(x), & x \in \Omega. \end{cases}$$

It will be said that u_i is a weak solution to P_i if $u_i \in L^2(0,T;\mathbb{K}) \cap L^\infty(\Omega_T)$, $\partial_t b(u_i) \in L^2(\Omega_T)$, u_i satisfies (19) and there exists a function $\Gamma_i \in L^1(\Omega_T)$, $\Gamma_i(x,t) \in \beta(u_i(x,t) - \Phi)$, a.e. $(x,t) \in \Omega_T$, such that

$$\int_0^T \int_\Omega \partial_t b(u_i) v + \int_0^T \int_\Omega [b(u_i) - b(u_{0,i})] \partial_t v = 0,$$

$$\forall v \in L^2(0,T;H^1(\Omega)) \cap W^{1,1}(0,T;L^\infty(\Omega)) \quad \text{such that } v(\cdot,T) = 0,$$

$$\iint_{\Omega_T} (\partial_t b(u_i)\eta + \Gamma_i \eta)\,\mathrm{d}x\,\mathrm{d}t = \iint_{\Omega_T} (u_i)_x \eta_x \,\mathrm{d}x\,\mathrm{d}t + \iint_{\Omega_T} (Fp(b(u_i)) + D_i)\eta\,\mathrm{d}x\,\mathrm{d}t,$$
$$\forall \eta \in L^2(0,T;H^1(\Omega)) \cap L^\infty(\Omega). \tag{40}$$

Note that (40) is an equivalent condition to (16) (see [7]). Let $[f]_+$ denote the positive part of an arbitrary function f, i.e., $[f]_+ := \max\{0, f\}$.

Lemma 4.4. *Under the hypothesis considered in Definition* 4.6, *let u_1 and u_2 be solutions to the obstacle problems P_i, $i = 1, 2$, respectively, then there exists a positive constant C^* such that*

$$\int_\Omega [b(u_1)(t) - b(u_2)(t)]_+ \,\mathrm{d}x \leqslant \mathrm{e}^{C^* t} \int_\Omega [b(u_{0,1}) - b(u_{0,2})]_+ \,\mathrm{d}x$$
$$+ \mathrm{e}^{C^* t} \int_0^t \mathrm{e}^{-C^* s} \|D_1 - D_2\|_{L^1(\Omega)} \,\mathrm{d}s \quad \text{a.e. } t \in (0,T). \tag{41}$$

Proof. Let u_i be a solution to P_i, $i = 1, 2$, then $u_i(x,t) \geqslant \Phi$ and

$$|p(b(u_1)) - p(b(u_2))| \leqslant C_p |b(u_1) - b(u_2)|,$$

as $p(b(z)) = (2z)^{1/2}$, for a positive constant C_p. Note that $p(b(z)) = (2z)^{1/2}$ is locally Lipschitz continuous in $(0,\infty)$, and, therefore, Lipschitz continuous in every compact interval contained in $(0,\infty)$. Let sig_0^+ be the function defined as: $sig_0^+(s) = 0$ if $s \leqslant 0$ and $sig_0^+(s) = 1$ if $s > 0$. Let us define the functions T_n for $n \in \mathbb{N}$,

$$T_n(s) = \begin{cases} 0, & s \leqslant 0, \\ n^2 s^2/2, & 0 < s \leqslant 1/n, \\ 2ns - (n^2 s^2/2) - 1, & 1/n < s \leqslant 2/n, \\ 1, & s > 2/n, \end{cases}$$

which verify the following estimates and convergence properties:

$$0 \leqslant T_n'(s) \leqslant n, \quad \lim_{n\to\infty} sT_n'(s) = 0, \quad |T_n(s)| \leqslant 1,$$
$$\lim_{n\to\infty} T_n(s) = sig_0^+(s) \quad \text{and} \quad \lim_{n\to\infty} sT_n(s) = [s]^+.$$

Let $z = b(u_1) - b(u_2)$. Note that $u_1, u_2 \in L^2(0,T;H^1(\Omega)) \cap L^\infty(\Omega)$ and T_n is a continuous function, therefore $T_n(u_1 - u_2) \in L^2(0,T;H^1(\Omega)) \cap L^\infty(\Omega)$. Let us consider $\Gamma_i(x,t) \in \beta(u_i(x,t) - \Phi)$, a.e. $(x,t) \in \Omega_T$, $i = 1, 2$, such that

$$\partial_t b(u_i) - (u_i)_{xx} + \Gamma_i - Fp(b(u_i)) = D_i \quad \text{a.e. } (x,t) \in \Omega, \quad i = 1, 2, \tag{42}$$

and subtract (in the weak sense of Definition 4.6) identity (42) applied for $i=2$ to the one for $i=1$,

$$\partial_t(b(u_1)-b(u_2))+\Gamma_1-\Gamma_2-(u_1-u_2)_{xx}=F[p(b(u_1))-p(b(u_2))]+D_1-D_2, \tag{43}$$

a.e. $(x,t)\in\Omega_T$. Next, we multiply Eq. (43) by the function $T_n(u_1-u_2)$ and integrate in $(0,t)\times\Omega$ to obtain the following identity:

$$\begin{aligned}-\int_0^t\int_\Omega T_n(u_1-u_2)z_t\,\mathrm{d}x\,\mathrm{d}s = &-\int_0^t\int_\Omega(\Gamma_1-\Gamma_2)T_n(u_1-u_2)\,\mathrm{d}x\,\mathrm{d}s\\ &-\int_0^t\int_\Omega[(u_1)_x-(u_2)_x]_x T_n(u_1-u_2)\,\mathrm{d}x\,\mathrm{d}s\\ &-\int_0^t\int_\Omega F[p(b(u_1))-p(b(u_2))]T_n(u_1-u_2)\,\mathrm{d}x\,\mathrm{d}s\\ &-\int_0^t\int_\Omega(D_1-D_2)T_n(u_1-u_2)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}\tag{44}$$

Now we study the integral terms appearing in (44). Due to the monotonicity of the approximating functions T_n and to the prescription of homogeneous Neumann boundary conditions, the application of the integration by parts formula leads us to the following estimate:

$$-\int_0^t\int_\Omega[(u_1)_x-(u_2)_x]_x T_n(u_1-u_2)\,\mathrm{d}x\,\mathrm{d}s=\int_0^t\int_\Omega T_n'(u_1-u_2)|(u_1)_x-(u_2)_x|^2\,\mathrm{d}x\,\mathrm{d}s\geqslant 0,$$

a.e. $t\in(0,T)$. Note that $\partial_t b(u_1)$ and $\partial_t b(u_2)\in L^2(\Omega_T)$, so $z_t\in L^2(\Omega_T)$. Moreover, $T_n(u_1-u_2)\in L^2(0,T;H^1(\Omega))$ and $T_n'(u_1-u_2)\in L^2(\Omega_T)$. On the other hand, by the monotonicity of the function b, we have that

$$\lim_{n\to\infty}T_n(u_1-u_2)=sig_0^+(u_1-u_2)=sig_0^+(b(u_1)-b(u_2))\quad\text{a.e. }(x,t)\in\Omega_T.$$

Hence,

$$\lim_{n\to\infty}\int_0^t\int_\Omega T_n(u_1-u_2)z_t\,\mathrm{d}x\,\mathrm{d}s=\int_0^t\int_\Omega sig_0^+(z)z_t\,\mathrm{d}x\,\mathrm{d}s\quad\text{a.e. }t\in(0,T),$$

and therefore

$$\int_0^t\int_\Omega T_n(z)z_t\,\mathrm{d}x\,\mathrm{d}s=\int_\Omega T_n(z(t))z(t)\,\mathrm{d}x-\int_\Omega T_n(z(0))z(0)\,\mathrm{d}x-\int_0^t\int_\Omega zT_n'(z)z_t\,\mathrm{d}x\,\mathrm{d}s,$$

a.e. $t\in(0,T)$, then, taking limit

$$\lim_{n\to\infty}\int_0^t\int_\Omega T_n(z)z_t\,\mathrm{d}x\,\mathrm{d}s=\int_\Omega sig_0^+(z(t))z(t)-\int_\Omega sig_0^+(z(0))z(0)\,\mathrm{d}x\,\mathrm{d}s,\ \text{ a.e. }t\in(0,T),$$

hence,

$$\begin{aligned}&\lim_{n\to\infty}\int_0^t\int_\Omega T_n(u_1-u_2)\partial_t(b(u_1)-b(u_2))\,\mathrm{d}x\,\mathrm{d}s\\ &\quad=\int_\Omega[b(u_1)-b(u_2)]_+(t)\,\mathrm{d}x-\int_\Omega[b(u_{0,1})-b(u_{0,2})]_+\,\mathrm{d}x\quad\text{a.e. }t\in(0,T).\end{aligned}$$

Taking into account that $\Gamma_i(x,t)\in\beta(u_i(x,t)-\Phi)$, a.e. $(x,t)\in\Omega_T$, $i=1,2$, we have $\int_0^t\int_\Omega(\Gamma_1-\Gamma_2)T_n(u_1-u_2)\,\mathrm{d}x\,\mathrm{d}s\geqslant 0$, a.e. $t\in(0,T)$. Hence, for a.e. $t\in(0,T)$,

$$\begin{aligned}\int_\Omega[b(u_1)-b(u_2)]_+(t)\,\mathrm{d}x\leqslant&\int_\Omega[b(w_{01})-b(w_{02})]_+\,\mathrm{d}x\\ &+\int_0^t\int_\Omega(F[p(b(u_1))-p(b(u_2))]+D_1-D_2)sig_0^+(u_1-u_2)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Let C^* be a positive constant such that $F \cdot |p(b(u_1)) - p(b(u_2))| \leqslant C^*|b(u_1) - b(u_2)|$. Let us define $v_1(x,t) = \mathrm{e}^{-C^*t} b(u_1(x,t))$ and $v_2(x,t) = \mathrm{e}^{-C^*t} b(u_2(x,t))$. Note that $sig_0^+(u_1 - u_2) = sig_0^+(v_1 - v_2)$ and $\partial_t v_i = -C^* v_i + \mathrm{e}^{-C^*t}\partial_t b(u_i) \in L^2(\Omega)$, a.e. $t \in (0,T)$, in a weak sense. Considering the above expression for $i = 1, 2$, and subtracting, we obtain that

$$\partial_t(v_1 - v_2) = -C^*(v_1 - v_2) + \mathrm{e}^{-C^*t}\partial_t(b(u_1) - b(u_2)). \tag{45}$$

Next, we multiply (45) by the function $T_n(v_1 - v_2)$ and argue in a analogous way as before to get

$$\begin{aligned}\int_\Omega [v_1(t) - v_2(t)]_+ \,\mathrm{d}x \leqslant{}& \int_\Omega [v_{0,1} - v_{0,2}]_+ \,\mathrm{d}x - C^* \int_0^t \int_\Omega [v_1 - v_2]_+ \,\mathrm{d}x\,\mathrm{d}s \\ &+ \int_0^t \int_\Omega [F(p(b(u_1)) - p(b(u_2))) + D_1 - D_2]\mathrm{e}^{-C^*s} sig_0^+(u_1 - u_2)\,\mathrm{d}x\,\mathrm{d}s,\end{aligned}$$

a.e. $t \in (0,T)$. Taking into account that

$$\begin{aligned}F[p(b(u_1)) - p(b(u_2))]\mathrm{e}^{-C^*s} sig_0^+(v_1 - v_2) &\leqslant C^*[b(u_1) - b(u_2)]\mathrm{e}^{-C^*s} sig_0^+(v_1 - v_2) \\ &= C^*[v_1 - v_2]_+ \quad \text{a.e. } s \in (0,T),\end{aligned}$$

we have that

$$\begin{aligned}\int_\Omega [v_1(t) - v_2(t)]_+ \,\mathrm{d}x \leqslant{}& \int_\Omega [v_{0,1} - v_{0,2}]_+ \,\mathrm{d}x - C^* \int_0^t \int_\Omega [v_1 - v_2]_+ \,\mathrm{d}x\,\mathrm{d}s \\ &+ C^* \int_0^t \int_\Omega [v_1 - v_2]_+ \,\mathrm{d}x\,\mathrm{d}s + \int_0^t \int_\Omega \mathrm{e}^{-C^*s}|D_1 - D_2|\,\mathrm{d}x\,\mathrm{d}s \quad \text{a.e. } t \in (0,T),\end{aligned}$$

from which we derive

$$\begin{aligned}\int_\Omega \mathrm{e}^{-C^*t}[b(u_1) - b(u_2)]_+(t)\,\mathrm{d}x \leqslant{}& \int_\Omega [b(u_{0,1}) - b(u_{0,2})]_+ \,\mathrm{d}x \\ &+ \int_0^t \int_\Omega \mathrm{e}^{-C^*s}|D_1 - D_2|\,\mathrm{d}x\,\mathrm{d}s \quad \text{a.e. } t \in (0,T),\end{aligned}$$

therefore

$$\int_\Omega [b(u_1) - b(u_2)]_+(t)\,\mathrm{d}x \leqslant \mathrm{e}^{C^*t}\left(\int_\Omega [b(u_{0,1}) - b(u_{0,2})]_+ \,\mathrm{d}x + \iint_{\Omega_t} \mathrm{e}^{-C^*s}|D_1 - D_2|\,\mathrm{d}x\,\mathrm{d}s\right),$$

a.e. $t \in (0,T)$ and finally we get (41). □

Corollary 4.1. *Let u_1 and u_2 be solutions to the problems P_1 and P_2, respectively. Let us assume that $u_{0,1} \leqslant u_{0,2}$, a.e. $x \in \Omega$, and that $D_1 = D_2$, then $u_1 \leqslant u_2$, a.e. $(x,t) \in \Omega_T$.*

Proof (*End of the proof of the uniqueness part of Lemma 4.1*). Let w_1 and w_2, be solutions to the problem $P(w_j)$ for the same data, then

$$w_1(x,t) = w_2(x,t) \quad \text{a.e. } (x,t) \in \Omega_T,$$

taking into consideration that $P(w_j)$ satisfies the generality mentioned in definition of P_i, and that, therefore, we can apply Lemma 4.4 to it, considering $w_1 = u_1$, $w_2 = u_2$, $u_{0,1} = u_{0,2}$ and $D_1 = D_2$. □

4.2. Decoupled problem for the ice thickness

In this section, we prove the existence and uniqueness of a solution to $(S_j)_4$ and $(S_j)_7$ departing from the assumption that we have already proved the existence and uniqueness of a triple $(w_{j-1}, h_{j-1}, \xi_{j-1}) \in V$ solution to the system S_{j-1} and the existence and uniqueness of solution to problem $P(w_j)$.

Definition 4.7. Let h_0 and the positive constants L, M, T be admissible data in the sense given in Definition 3.1 and let the coefficient function A_j satisfy Properties 3.1. For $j \in \mathbb{N}$, we shall consider the problem $P(h_j)$ which consists of $(S_j)_4$ and $(S_j)_7$:

$$P(h_j) \quad \begin{cases} h_j' = -M^{1/2} h_j^{-3/2} \left(\int_\Omega A_j(x,\cdot)\,\mathrm{d}x\right)^{-1/2} & \text{in } \Omega_T, \\ h_j(0) = h_0. \end{cases}$$

Then, we have the following result regarding the existence and uniqueness of solution to the problem $P(h_j)$.

Lemma 4.5. *There exists a unique strong solution* $h_j \in W^{1,\infty}(0,T) \subset C([0,T])$, *strictly positive and decreasing function* $\forall t \in (0,T)$, *to the problem* $P(h_j)$, *given by*

$$h_j(t) = \left[h_0^{5/2} - \frac{5}{2} M^{1/2} \int_0^t \left(\int_\Omega A_j(x,r)\,\mathrm{d}x \right)^{-1/2} \mathrm{d}r \right]^{2/5} \quad \forall t \in [0,T].$$

Proof. The equation of the problem $P(h_j)$ can be considered as one of separated variables because it can be expressed in the form

$$h_j'(t) = -M^{1/2} h_j^{-3/2}(t) X(t), \tag{46}$$

where $X(t) := (\int_\Omega A_j(x,t)\,\mathrm{d}x)^{-1/2}$ and $X \in C([0,T])$ as $A_j(x,t) > (2\Phi)^{1/2}$ a.e. $(x,t) \in \Omega_T$ and $A_j \in C([0,T]; L^2(\Omega))$. Integrating in (46) we obtain the explicit expression

$$h_j(t) = \left[h_0^{5/2} - \frac{5}{2} M^{1/2} \int_0^t X(s)\,\mathrm{d}s \right]^{2/5} \quad \text{a.e. } t \in (0,T).$$

Due to the fact that h_0 is assumed to be an admissible data it holds the following estimate:

$$h_j(t) = \left[h_0^{5/2} - \frac{5}{2} M^{1/2} \int_0^t \left(\int_\Omega A_j(x,r)\,\mathrm{d}x \right)^{-1/2} \mathrm{d}r \right]^{2/5} \geqslant m_h > 0 \quad \text{a.e. } t \in (0,T).$$

Note that $h_j' < 0$, a.e. $t \in (0,T)$ as $A_j > 0$, a.e. $(x,t) \in \Omega_T$. Hence, we deduce that

$$\|h_j\|_{L^\infty(0,T)} \leqslant h_0 \quad \text{and} \quad \|h_j'\|_{L^\infty(0,T)} \leqslant \frac{(M/L)^{1/2}}{m_h^{3/2}(2\Phi)^{1/4}} \quad \text{a.e. } t \in (0,T) \ \ \forall j \in \mathbb{N}.$$

From the previous estimates we have that $h_j \in W^{1,\infty}(0,T)$ and in particular, as $W^{1,\infty}(0,T)$ has compact embedding in $C([0,T])$ (see [5]), then $h_j \in C([0,T])$. As a consequence of being the function h_j continuous, we have that the estimates and expressions relative to h_j hold $\forall t \in [0,T]$, instead of a.e. $t \in (0,T)$. $\square$

Note that the family $\{\|h_j\|_{W^{1,\infty}(0,T)}\}$ is uniformly bounded in $j \in \mathbb{N}$. Therefore there will be a subsequence of $\{h_j\}$, that will be labelled in the same way, converging to a function $h \in C([0,T])$. It is enough to take into consideration that the sequence $\{\|h_j\|_{W^{1,\infty}(0,T)}\}$ is uniformly bounded with respect to $j \in \mathbb{N}$ and that $W^{1,\infty}(0,T)$ has compact embedding in $C([0,T])$. Besides, we get that the functions B_j, D_j, E_j belong to the space $L^\infty(0,T)$ and there exists a positive constant C, such that

$$\|B_j\|_{L^\infty(0,T)}, \quad \|D_j\|_{L^\infty(0,T)}, \quad \|E_j\|_{L^\infty(0,T)} \leqslant C \quad \forall j \in \mathbb{N}. \tag{47}$$

Remark 4.2. According to (17), being the functions D_j uniformly bounded with respect to the norm of the space $L^\infty(\Omega)$, then we obtain that there exists a positive constant W such that

$$\|w_j\|_{L^\infty(\Omega_T)}, \quad \|A_j\|_{L^\infty}(\Omega_T) \leqslant W \quad \forall j \in \mathbb{N}. \tag{48}$$

4.3. Decoupled problem for the accumulated velocity

In this section, we prove the existence and uniqueness of a solution to $(S_j)_5$ and $(S_j)_7$ departing from the assumption that we have already proved the existence and uniqueness of solutions to problems $P(w_j)$ and $P(h_j)$.

Definition 4.8. For $j \in \mathbb{N}$, let ξ_0 be an admissible data in the sense given in Definition 3.1 and let the coefficient functions A_j and E_j satisfy Properties 3.1. We consider the problem $P(\xi_j)$ consisting of $(S_j)_4$ and $(S_j)_7$:

$$P(\xi_j) \quad \begin{cases} \partial_t \xi_j = A_j(x,t)E_j(t) & \text{a.e. } t \in (0,T) \text{ a.e. } x \in \Omega, \\ \xi_j(x,0) = \xi_0 & \text{in } \Omega. \end{cases}$$

Then, regarding the existence and uniqueness of solution to the problem $P(\xi_j)$, we have the following result:

Lemma 4.6. *There is a unique strong solution to $P(\xi_j)$, $\xi_j \in W^{1,\infty}(0,T;L^\infty(\Omega)) \cap L^2(0,T;H^1(\Omega))$, given by*

$$\xi_j(x,t) = \xi_0 + \int_0^t A_j(x,s)E_j(s)\,\mathrm{d}s \quad \text{a.e. } t \in (0,T) \text{ a.e. } x \in \Omega. \tag{49}$$

Proof. Note that the product A_jE_j is integrable in Ω_T due to the fact that A_j and $E_j \in L^\infty(\Omega_T)$. In particular, $A_j(x,\cdot)E_j(\cdot)$ is integrable in $(0,t)$, a.e. $x \in \Omega$. Therefore by integrating in $(0,t)$ equation $P(\xi_j)_1$, we derive the explicit formula (49). The uniqueness is a consequence of the application of the fundamental theorem of the variational calculus. We also deduce from the fact that A_j and E_j are globally bounded that $\xi_j \in W^{1,\infty}(0,T;L^\infty(\Omega))$. Note that A_j and E_j are positive in a.e. $t \in (0,T)$ and a.e. $x \in \Omega$ and hence $\xi_j > 0$ and $\partial_t \xi_j > 0$, a.e. $t \in (0,T)$, a.e. $x \in \Omega$. □

Remark 4.3. As a consequence of (47) and (48), we can deduce the existence of a positive constant C, such that

$$\|C_j\|_{L^\infty(\Omega_T)}, \quad \|F_j\|_{L^\infty(\Omega_T)} \leqslant C \quad \forall j \in \mathbb{N}. \tag{50}$$

End of Proof of Theorem 4.1. Finally, we note that the triple (w_j, h_j, ξ_j), with w_j, h_j and ξ_j, solutions to $P(w_j)$, $P(h_j)$ and $P(\xi_j)$, respectively (see Lemmas 4.1, 4.5 and 4.6) is the unique solution to the decoupled system S_j in the sense given in Definition 4.2. □

5. Proof of Theorem 3.1: Convergence

In this section, we shall undertake the proof of the existence of a unique bounded weak solution of a bounded weak solution to the coupled system S. To be precise, we shall prove the existence result by proving that there is a subsequence of the sequence of solutions $\{(w_j, h_j, \xi_j)\}$ to the problems S_j, $j \in \mathbb{N}$, which converges to a weak solution to the system S.

We begin by observing that $\{\|h_j\|_{W^{1,\infty}(0,T)}\}$ is uniformly bounded and therefore we can find a subsequence of $\{h_j\}$ that converges to a function h in the space $C([0,T])$. Note also that thanks to (48), we have that for a subsequence

$$A_j = (2w_j)^{1/2} \to A = (2w)^{1/2} \text{ strongly in } L^p(\Omega_T), \quad 1 \leqslant p < \infty,$$

moreover $A \geqslant (2\Phi)^{1/2}$. As a consequence of this,

$$\lim_{j\to\infty} \int_\Omega A_j(x,t)\,\mathrm{d}x = \int_\Omega A(x,t)\,\mathrm{d}x \quad \text{a.e. } t \in (0,T).$$

To summarize, we have that

$$\lim_{j\to\infty} h_j(t) = \lim_{j\to\infty} \left[h_0^{5/2} - [5/2]M^{1/2} \int_0^t \left(\int_\Omega A(x,r)\,\mathrm{d}x \right)^{-1/2} \mathrm{d}r \right]^{2/5} \quad \text{a.e. } t \in (0,T),$$

and therefore

$$h(t) = \left[h_0^{5/2} - [5/2] M^{1/2} \int_0^t \left(\int_\Omega A(x,r)\,\mathrm{d}x \right)^{-1/2} \mathrm{d}r \right]^{2/5} \quad \forall t \in [0,T].$$

Hence, we deduce that $h \geqslant m_h > 0$, $\forall t \in [0,T]$, and from the estimates performed in Section 3.1 we get that $h'_j \to h'$ strongly in $L^p(0,T)$, $1 \leqslant p < \infty$,

$$h' = -M^{1/2} h^{-3/2} \left(\int_\Omega A(x,r)\,\mathrm{d}x \right)^{-1/2}, \quad h' \in L^\infty \quad \text{and} \quad h' < 0 \ \text{ a.e. } t \in (0,T).$$

Then $h \in V_h$. Note that the sequences $\{A_j\}$, $\{B_j\}$, $\{D_j\}$ and $\{E_j\}$, verify the following convergence results:

$$B_j \to B = (h|h'|)^3 \quad \text{and} \quad E_j \to E = (hh')^2 \ \text{strongly in } L^p(0,T), \quad 1 \leqslant p < \infty,$$

$$D_j \to D = \gamma - \frac{\delta}{h} \ \text{strongly in } C([0,T]), \quad D'_j \to D' \ \text{strongly in } L^p(0,T),$$

$1 \leqslant p < \infty$, and also that B, E and D satisfy Properties 3.1. Moreover, taking into account the estimates (47), (50) and (48), we deduce that the positive constants appearing in estimates (35)–(39), can be chosen in a way that they are valid $\forall j \in \mathbb{N}$. This fact implies that $\|w_j\|_{L^2(0,T;H^1(\Omega))}$ and $\|\partial_t b(w_j)\|_{L^2(\Omega_T)}$ are uniformly bounded with respect to j and that $w_j(t) \in \mathbb{K}$, a.e. $t \in [0,T]$, $\forall j \in \mathbb{N}$. Then, we can deduce the existence of a subsequence of $\{w_j\}$, that we label in the same way, such that

$$w_j \rightharpoonup w \ \text{ in } L^2(0,T;H^1(\Omega)) \quad \text{and} \quad \partial_t b(w_j) \rightharpoonup \partial_t b(w) \ \text{ in } L^2(\Omega_T).$$

Moreover, we can derive that $w \in V_w$. Since A_j converges strongly in $L^2(\Omega_T)$ to A and E_j converges strongly in $L^2(\Omega_T)$ to E, we have that $A_j E_j$ converges strongly in $L^2(\Omega_T)$ to AE. By Lemma 4.6, we have that $\xi_j \in L^\infty((0,T);W^{1,1}(\Omega))$, and that $\partial_t \xi_j \in L^\infty((0,T);L^1(\Omega))$ (recall that $\partial_t \xi_j \in L^\infty(\Omega_T)$) and the bounds of such a function in such a space are uniform in $j \in \mathbb{N}$. The uniform bound of ξ_j and $\partial_t \xi_j$ in the mentioned functional spaces allows us to apply a result presented in [20], and so, to deduce the existence of a subsequence of $\{\xi_j\}$, that will be labelled in the same way, that converges strongly to ξ in $C([0,T];L^p(\Omega))$, for $1 \leqslant p < \infty$ (note that $W^{1,1}(\Omega)$ has a compact embedding in the space $L^p(\Omega)$ for $1 \leqslant p < \infty$, ξ_j and $\partial_t \xi_j$ are inL^∞). Then, passing to the limit

$$\xi(x,t) = \xi_0 + \int_0^t A(x,s)E(s)\,\mathrm{d}s \quad \text{a.e. } t \in (0,T) \ \text{ a.e. } x \in \Omega.$$

Moreover, ξ is a positive, non-decreasing with respect to t function. Thanks to the regularity of the functions A and E, we deduce that $\xi \in V_\xi$. Moreover, we get that there exists a subsequence $\{C_j\}$, such that

$$C_j \to C = (h|h'|)^2 \xi^{-1/2} \ \text{strongly in } L^p(\Omega_T), \quad 1 \leqslant p < \infty,$$

and therefore, C, satisfies Properties 3.1. In addition,

$$F_j = B_j - C_j \to F = B - C \ \text{strongly in } L^p(\Omega_T), \quad 1 \leqslant p < \infty.$$

In particular, $F_j \to F$, strongly in $L^2(\Omega_T)$ and F satisfies Properties 3.1 and hence $F_j p(b(w_j)) \rightharpoonup Fp(b(w))$ in $L^2(\Omega_T)$.

To summarize, we have found a subsequence $\{w_j\}$, such that

$$w_j \rightharpoonup w \ \text{ in } L^2(0,T;H^1(\Omega)), \quad w_j \to w \ \text{ a.e. } (x,t) \in \Omega_T,$$

$$w(t) \in \mathbb{K} \ \text{ a.e. } t \in [0,T], \quad \partial_t b(w_j) \rightharpoonup \partial_t b(w) \ \text{ in } L^2(\Omega_T),$$

$$p(b(w_j)) \to p(b(w)) \ \text{ in } L^2(\Omega_T) \quad \text{and} \quad F_j p(b(w_j)) \rightharpoonup Fp(b(w)) \ \text{ in } L^2(\Omega_T).$$

The estimates and convergence results just obtained allow for passing to the limit (in a weak sense) with respect to the parameter j in (15) and (16), which leads to the fact that w also verifies (11) and (12). In conclusion, it results that the triple given by (w,h,ξ) is a weak solution to the system (S) as $(w,h,\xi) \in V$ and (11)–(14) are verified. □

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Further reading

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