

On an evolution problem associated to the modelling of incertitude into the environment

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Abstract

We consider a mathematical model, posed by J.E. Scheinkman, simulating that an industrial project takes place into the environment without destroying it. We introduce a change of variable leading the formulation to a nonlinear evolution problem which we study by means of L^∞ -accretive operator techniques. We prove that under suitable conditions there is extinction in finite time, which corresponds to some special behaviour of the solution of the original stochastic control problem.

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1. Introduction

In this paper we consider a mathematical model simulating that an industrial project takes place into the environment without destroying it. Following Scheinkman models [6], we assume that the benefit of the environment at the instant t changes according to a positive diffusion process such as

$$d\mathcal{X}(t) = \mu_1(\mathcal{X}(t)) dt + \sqrt{2}\sigma_1(\mathcal{X}(t)) d\mathcal{B}_1(t). \quad (1.1)$$

The alternative project also changes according to a positive diffusion process

$$d\mathcal{Y}(t) = \mu_2(\mathcal{Y}(t)) dt + \sqrt{2}\sigma_2(\mathcal{Y}(t)) d\mathcal{B}_2(t), \quad (1.2)$$

where \mathcal{B}_1 and \mathcal{B}_2 are standard Brownian motions defined in a probability space $(\Omega, \mathbb{F}, \mathbb{P})$ and with a coefficient of correlation $\varrho \in (-1, 1)$. Let us assume that, for $i = 1, 2$, the functions μ_i and σ_i are Lipschitz continuous and vanishing at the origin, i.e.

$$|\mu_i(z_1) - \mu_i(z_2)| + |\sigma_i(z_1) - \sigma_i(z_2)| \leq K |z_1 - z_2|, \quad (1.3)$$

$$\mu_i(0) = \sigma_i(0) = 0, \quad i = 1, 2. \quad (1.4)$$

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By $\theta(t)$ we denote the fraction of the environment transforming to the industrial project, then we will assume that the utility flow is as

$$U((1 - \theta(t))\mathcal{X}(t), \theta(t)\mathcal{Y}(t)), \tag{1.5}$$

where U is a continuous concave function and non-decreasing in their arguments and $U(0, 0) = 0$. Here the initial data are given by $\mathcal{X}(0) = x$, $\mathcal{Y}(0) = y$ and the environment transformation fraction by $\theta(0) = \theta$. We define the accumulate utility functional

$$J(x, y, \theta) = \mathbb{E} \left[\int_0^\infty e^{-\alpha s} U((1 - \theta(s))\mathcal{X}(s), \theta(s)\mathcal{Y}(s)) ds \right], \tag{1.6}$$

where $\mathcal{X}(t)$ and $\mathcal{Y}(t)$ are the solutions of (1.1) and (1.2), respectively, and

$$1 \geq \theta(t) = \theta + M(t). \tag{1.7}$$

Here $\alpha > 0$ is a discount coefficient and $M(t)$ an increasing process describing the irreversible development. Finally, \mathcal{A}_θ is the domain of process $M(t)$ where (1.7) holds. We point out that by hypothesis the cost of development is zero.

The main goal of this paper is to study the qualitative behaviour of the optimal value function v given by

$$v(x, y, \theta) = \sup_{\mathcal{A}_\theta} J(x, y, \theta). \tag{1.8}$$

We introduce the variable $t = 1 - \theta$ and study the associated evolution problem by means of L^∞ -accretive operators. We prove that under suitable conditions there is extinction in finite time, which corresponds to some special behaviour of the solution of the original stochastic control problem.

2. Modelling of the evolution problem

First of all, we are going to show that the optimal function v satisfies, in some sense to be detailed, the obstacle problem

$$\min\{-\mathcal{L}v + \alpha v - U((1 - \theta x, \theta y), -v_\theta)\} = 0 \quad \text{in } \Omega, \tag{2.1}$$

where \mathcal{L} is the differential operator associated to the diffusion $(\mathcal{X}(t), \mathcal{Y}(t))$, thus,

$$\mathcal{L}v \doteq \sigma_1^2(x)v_{xx} + \sigma_2^2(y)v_{yy} + 2\sigma_1(x)\sigma_2(y)\rho v_{xy} + \mu_1(x)v_x + \mu_2(y)v_y.$$

In order to simplify the exposition, we assume for the moment that v is smooth enough. At each instant t there are two possibilities: to protect the environment or to develop the industrial project. If the decision is to protect the environment in the period of time $[0, h]$, then $\theta(t) = \theta$. Therefore it follows

$$v(x, y, \theta) \geq \mathbb{E} \left[\int_0^h e^{-\alpha s} U((1 - \theta)\mathcal{X}(s), \theta\mathcal{Y}(s)) ds + e^{-\alpha h} v(\mathcal{X}(h), \mathcal{Y}(h), \theta(h)) \right], \tag{2.2}$$

with equality if the optimal decision is $\theta(t) = \theta$ in $[0, h]$.

Applying Ito's rule to $e^{-\alpha h} v(\mathcal{X}(h), \mathcal{Y}(h), \theta(h))$ in (2.2), classical arguments in passing to the limit as $h \rightarrow 0$ prove that

$$-\mathcal{L}v + \alpha v - U((1 - \theta x, \theta y)) \geq 0, \tag{2.3}$$

with equality if the optimal decision in the instant $t = 0$ is to protect the environment.

If the decision is to develop the environment into the private project, and $\Delta\theta$ represents the instantaneous increasing of the fraction of development, then $\theta(0^+) = \theta + \Delta\theta$. Applying the Fleming and Soner theory [5] to our problem we obtain: if τ is stopping time then the principle of dynamic programming is

$$v(x, y, \theta) = \sup_{\mathcal{A}_\theta} \mathbb{E} \left[\int_0^\tau e^{-\alpha s} U((1 - \theta(s))\mathcal{X}(s), \theta(s)\mathcal{Y}(s)) ds + e^{-\alpha \tau} v(\mathcal{X}(\tau), \mathcal{Y}(\tau), \theta(\tau)) \right].$$

Therefore,

$$v(x, y, \theta) \geq v(x, y, \theta(0^+)),$$

consequently

$$v_\theta \leq 0, \tag{2.4}$$

with equality if $\Delta\theta > 0$, in other words, with equality if the optimal decision at the instant $t = 0$ is developing the environment into the industrial project.

Writing (2.3) and (2.4) under the two possible options we obtain the Hamilton–Jacobi–Bellman equation:

$$\min\{-\mathcal{L}v + \alpha v - U((1 - \theta)x, \theta y), -v_\theta\} = 0 \quad \text{in } \Omega. \tag{2.5}$$

The correct formulation of (2.5) must be formulated by means of the viscosity solution theory (see [3]), but here we omit it.

In order to get some qualitative properties, we reformulate the problem in a different way. For $\Omega = \mathbb{R}_+^2$ we define the function

$$\widehat{U}(x, y, \theta) \doteq U((1 - \theta)x, \theta y).$$

Then, the problem under consideration can be stated as follows: find $v : \Omega \times [0, 1] \rightarrow \mathbb{R}$ such that

$$\begin{cases} \min\{-v_\theta, -\mathcal{L}v + \alpha v - \widehat{U}(x, y, \theta)\} = 0 & \text{in } \Omega \times (0, 1), \\ \mathcal{A}\nabla v \cdot \mathbf{v} = 0 & \text{on } \partial\Omega \times (0, 1), \\ v(x, y, 1) = \mathbb{E} \left[\int_0^\infty e^{-\alpha s} U(0, \mathcal{Y}(s)) ds \right] & \text{in } \Omega, \end{cases} \tag{2.6}$$

where \mathbf{v} is the unitary out-normal vector and \mathcal{A} is the matrix

$$\mathcal{A} = \begin{pmatrix} \sigma_1^2(x) & \sigma_1(x)\sigma_2(y)\varrho \\ \sigma_1(x)\sigma_2(y)\varrho & \sigma_2^2(y) \end{pmatrix}.$$

We introduce the change of variable $t = 1 - \theta$ and define the function

$$\overline{U}(x, y, t) \doteq \widehat{U}(tx, y, 1 - t) = U(tx, (1 - t)y)$$

for which we introduce the problem

$$\begin{cases} -\mathcal{L}f + \alpha f = \overline{U}(x, y, t) & \text{in } \Omega \times (0, 1), \\ \mathcal{A}\nabla f \cdot \mathbf{v} = 0 & \text{on } \partial\Omega \times (0, 1). \end{cases} \tag{2.7}$$

Due to the Lipschitz condition assumed on the functions μ_i and σ_i one proves that the solution of (2.7) verifies $f \in H_{loc}^2(\mathcal{A}, \Omega)$ (see [4]). Finally, for the function

$$u(x, y, t) \doteq v(x, y, 1 - t) - f(x, y, t)$$

problem (2.6) becomes

$$\begin{cases} \min\{u_t + f_t, -\mathcal{L}u + \alpha u\} = 0 & \text{in } \Omega \times (0, 1), \\ \mathcal{A}\nabla u \cdot \mathbf{v} = 0 & \text{on } \partial\Omega \times (0, 1), \\ u(x, y, 0) = 0 & \text{in } \Omega. \end{cases} \tag{2.8}$$

Other general formulation on the evolution problem (2.8) can be obtained by the multivalued operator theory (see [2]). It is known that we may write the evolution problem as

$$\begin{cases} \frac{\partial u}{\partial t} + \gamma(-\mathcal{L}u + \alpha u) \ni g(x, y, t) & \text{in } \Omega \times (0, 1), \\ \mathcal{A}\nabla u \cdot \mathbf{v} = 0 & \text{on } \partial\Omega \times (0, 1), \\ u(x, y, 0) = 0 & \text{in } \Omega, \end{cases} \tag{2.9}$$

where $g(x, y, t) = -\partial f/\partial t(x, y, t)$ and γ is the maximal monotone graph given by

$$\gamma(u) = \begin{cases} \emptyset & \text{if } u < 0, \\ (-\infty, 0] & \text{if } u = 0, \\ 0 & \text{if } u > 0. \end{cases}$$

3. Existence of solution of problem (2.9)

Here we introduce the Banach space

$$L_w^\infty(\Omega) = \{v : wv \in L^\infty(\Omega)\},$$

equipped with the norm

$$\|u\|_{L_w^\infty(\Omega)} = \|wu\|_{L^\infty(\Omega)}.$$

Analogously one introduces the space $L_w^2(\Omega)$, $H_w^1(\mathcal{A}, \Omega)$ and $H_{loc}^2(\Omega)$ (see [4]).

Next we define the multivalued operator $\mathcal{C} : D(\mathcal{C}) \rightarrow \mathcal{P}(L^\infty(\Omega))$ by

$$\begin{cases} D(\mathcal{C}) = \{v \in L^\infty(\Omega) \cap H^1(\mathcal{A}, \Omega) : -\mathcal{L}v + \alpha v \in L^\infty(\Omega), \gamma(-\mathcal{L}v + \alpha v) \in L^\infty(\Omega)\}, \\ \mathcal{C}u = \gamma(-\mathcal{L}v + \alpha v) \quad \text{if } u \in D(\mathcal{C}). \end{cases}$$

Lemma 3.1. *The multivalued operator \mathcal{C} is m -T-accretive in $L^\infty(\Omega)$.*

Proof. First of all, we note that the operator \mathcal{C} is T-accretive in $L^\infty(\Omega)$ if and only if for all $(u_1, v_1), (u_2, v_2) \in \mathcal{C}$ the inequality

$$\tau^+(u_1 - u_2, v_1 - v_2) \geq 0$$

holds, where τ^+ is given by

$$\tau^+(f, g) = \max \left\{ \lim_{\lambda \downarrow 0} \text{ess sup} [\alpha(x)g(x) : x \in \Omega(f, \lambda)], \alpha \in L^\infty(\Omega), \alpha(x) \in \text{sign}^+ f(x) \text{ a.e.} \right\}$$

with $\Omega(f, \lambda) = \{x \in \Omega : |f(x)| \leq |f|_{L^\infty} - \lambda\}$ and

$$\text{sign}^+(v) = \begin{cases} 1 & \text{if } v > 0, \\ [0, 1] & \text{if } v = 0, \\ 0 & \text{if } v < 0. \end{cases}$$

We will use the auxiliary function

$$\gamma_\varepsilon(x) = \varepsilon \inf\{0, x\} = \varepsilon x^-$$

verifying $\gamma_\varepsilon \rightarrow \gamma$, as $\varepsilon \rightarrow 0$. So that, for $(u_1, v_1), (u_2, v_2) \in \mathcal{C}$ we claim

$$\tau^+(u_1 - u_2, \gamma_\varepsilon(-\mathcal{L}u_1 + \alpha u_1) - \gamma_\varepsilon(-\mathcal{L}u_2 + \alpha u_2)) \geq 0 \quad \text{for all } \varepsilon > 0. \tag{3.1}$$

Indeed, if there exists $\lambda > 0$ with

$$\gamma_\varepsilon(-\mathcal{L}u_1 + \alpha u_1) - \gamma_\varepsilon(-\mathcal{L}u_2 + \alpha u_2) < 0 \quad \text{c.p.t. } \Omega((u_1 - u_2)^+, \lambda)$$

the monotonicity of the operator γ_ε leads to

$$-\mathcal{L}(u_1 - u_2) + \alpha(u_1 - u_2) \leq 0 \quad \text{in } \Omega((u_1 - u_2)^+, \lambda).$$

As on the boundary of $\Omega((u_1 - u_2)^+, \lambda)$ we have $u_1 - u_2 = |(u_1 - u_2)^+|_{L^\infty} - \lambda$ the maximum principle derives a contradiction. So, taking limit in (3.1) as $\varepsilon \downarrow 0$ we obtain

$$\tau^+(u_1 - u_2, v_1 - v_2) \geq 0.$$

On the other hand, in proving the m-accretivity, we note that the problem

$$\begin{cases} \gamma(-\mathcal{L}u + \alpha u) + \lambda u \ni F & \text{in } \Omega, \\ \mathcal{A}\nabla u \cdot \nu = 0 & \text{on } \partial\Omega \end{cases}$$

has one and only one solution if $F \in L^\infty$ and $\lambda > 0$. Indeed, this problem is equivalent to

$$\begin{cases} -\mathcal{L}u + \alpha u + \gamma(\lambda u - F) \ni 0 & \text{in } \Omega, \\ \mathcal{A}\nabla u \cdot \nu = 0 & \text{on } \partial\Omega, \end{cases}$$

studied in [4]. \square

4. The environment preservation domain

In this section we shall obtain an estimate of the vanishing domain of the operator $-\mathcal{L}u + \alpha u$ representing the domain where the optimal control problem is to protect the environment.

Theorem 4.1. *Let $f(x, y, t)$ be the unique solution of problem (2.7) and assume that*

$$\begin{cases} \exists t_0 \in [0, 1] \text{ that } \rho(t) \geq 0 \quad \forall t \in [t_0, 1] \text{ for} \\ \rho(t) \doteq \min_{(x,y) \in \bar{\Omega}} \left(\frac{\partial f}{\partial t}(x, y, t) \right) = - \max_{(x,y) \in \bar{\Omega}} (g(x, y, t)) \end{cases} \quad (4.1)$$

and

$$\exists t_1 \in [t_0, 1] \text{ that } \int_0^{t_0} \|g(s)\|_{L^\infty(\Omega)} ds \leq \int_{t_0}^{t_1} -\max[g(x, y, \tau)] d\tau. \quad (4.2)$$

Then

$$v(x, y, \theta) = f(x, y, 1 - \theta) \quad \forall \theta \in [0, 1 - t_1] \text{ and a.e. } (x, y) \in \Omega. \quad (4.3)$$

Remark 4.2. The result says that v can be determined in terms of U in the interval $[0, 1 - t_1]$.

Remark 4.3. In the case of $\sigma_i(z) = \sigma_i z$ and $\mu_i(z) = \mu_i z$ the hypotheses (4.1) and (4.2) are verified, for instance, for $U(x, y, \theta) = \log((1 - \theta)x) - \log(\theta y) + \sigma_1^2 - \sigma_2^2 - \mu_1 + \mu_2$.

Proof. We will suppose that $u(\cdot, \cdot, t) \neq 0$ for all $t \in (0, 1)$, because if there exists t^* such that $u(\cdot, \cdot, t^*) = 0$ then the function

$$\widehat{u}(\cdot, \cdot, t) = \begin{cases} u(\cdot, \cdot, t) & \text{if } t \in (0, t^*), \\ 0 & \text{if } t \in (t^*, 1) \end{cases}$$

is also a solution of problem (2.9) and then the uniqueness of solutions yields the conclusion. We note that by assumption (4.1) one has

$$B_{\rho(t)}(g(\cdot, \cdot, t)) \subset \mathcal{C}(0) = \{w \in L^\infty(\Omega) : w \leq 0 \text{ in } \Omega\}. \quad (4.4)$$

By the accretivity of the operator \mathcal{C} in $L^\infty(\Omega)$ we obtain that

$$|u(t) - \tilde{v}(t)| \leq |u(r) - \tilde{v}(r)| + \int_r^t \tau(u(s) - v(s), g(s) - h(s)) ds \quad \forall r \in [0, 1], \quad (4.5)$$

for any \tilde{v} solution of $d\tilde{v}/dt(t) + \mathcal{C}\tilde{v}(t) \ni h(t)$ (see [1]). Choosing $\tilde{v} = 0$ and $h(s) = g(s) + \rho(s)\varepsilon(s)$ with $|\varepsilon(s)|_{L^\infty(\Omega)} \leq 1$ and $r = t_0$ we obtain

$$|u(t)| \leq |u(t_0)| + \int_{t_0}^t \tau(u(s), -\rho(s)\varepsilon(s)) ds.$$

Now from

$$\tau(x, \lambda y) = \lambda \tau(x, y) \quad \forall \lambda \in \mathbb{R} \text{ and } \forall x, y \in L^\infty(\Omega),$$

we obtain

$$|u(t)| \leq |u(t_0)| + \int_{t_0}^t -\rho(s) \tau(u(s), \varepsilon(s)) \, ds.$$

So, taking $\varepsilon(s) = u(s)/|u(s)|$ the property $\tau(x, x) = |x|$, $\forall x \in L^\infty(\Omega)$, implies

$$|u(t)| \leq |u(t_0)| - \int_{t_0}^t \rho(s) \, ds.$$

On the other hand, by choosing $\tilde{v} = 0$ and $h = 0$ in (4.5) and using that $\tau(x, y) \leq \|y\|_{L^\infty(\Omega)}$ we deduce

$$|u(t_0)| \leq \int_0^{t_0} \|g(s)\|_{L^\infty(\Omega)} \, ds.$$

Therefore

$$|u(t)| \leq \int_0^{t_0} \|g(s)\|_{L^\infty(\Omega)} \, ds - \int_{t_0}^t \rho(s) \, ds$$

whence assumption (4.2) leaves to the contradiction $|u(t_1)| = 0$. \square

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References

- [1] Ph. Benilan, Equations d' evolution dans un espace de Banach quelconque et applications, These Univ. Paris XI, Orsay, 1972.
- [2] H. Brezis, Problèmes Unilatéraux, J. Math. Pures Appl. 51 (1972) 1–168.
- [3] M.G. Crandall, H. Ishii, P.L. Lions, User's guide to viscosity solutions of second order partial differential equations, Bull. Amer. Math. Soc. 27 (1992) 1–42.
- [4] J.I. Díaz, Ch. Faghloumi, Analysis of a nonlinear elliptic problem arising in the study of policies on projects altering the environment, Appl. Math. Optim. 45 (2002) 251–267.
- [5] W.H. Fleming, H.M. Soner, Controlled Markov Processes and Viscosity Solutions, Springer, New York, 1993.
- [6] J.A. Scheinkman, Th. Zariphopoulou, Optimal environmental management in the presence of irreversibilities, J. Econom. Theory 96 (1–2) (2001) 180–207.