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A parabolic system involving a quadratic gradient term related to the Boussinesq approximation

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Abstract. We propose a modification of the classical Boussinesq approximation for buoyancy-driven flows of viscous, incompressible fluids in situations where viscous heating cannot be neglected. This modification is motivated by unresolved issues regarding the global solvability of the original system. A very simple model problem leads to a coupled system of two parabolic equations with a source term involving the square of the gradient of one of the unknowns. Based on adequate notions of weak and strong solutions, we establish the global-in-time existence of weak solutions and the uniqueness of strong solutions.

Un sistema parabólico relacionado con la aproximación de Boussinesq conteniendo un término cuadrático sobre el gradiente de una de las componentes

Resumen. Se analiza la existencia global de soluciones para un sistema parabólico que responde a una formulación inspirada en el sistema de ecuaciones correspondiente a un fluido viscoso incompresible no isotérmico en el que los efectos de la fricción viscosa (conteniendo una expresión cuadrática del gradiente de la velocidad) en la ecuación del balance energético no son despreciados. Introducimos las nociones de soluciones débiles y fuertes adaptadas a ese sistema simplificado mostrando su existencia global en el tiempo (lo que es una de las mayores dificultades en el análisis matemático de este tipo de sistemas) y justificando, bajo hipótesis suplementarias, su unicidad.

1. Introduction

The flow of a viscous, heat-conducting fluid under the force of gravity is governed by a system of balance equations for momentum, mass, and internal energy [1, Ch. 4.1–4.3]. In the so-called Boussinesq approximation, the system is reduced to the Navier-Stokes equations for a homogeneous, incompressible fluid, coupled to a semilinear heat equation [13], [17]. The main coupling term is the buoyancy force (generation of momentum due to temperature gradients); viscous heating (heat production due to internal friction) is neglected. The resulting initial-boundary value problems are well posed in the same sense as for the classical Navier-Stokes equations; in particular, they have global-in-time weak solutions [8], [9], [14].

In many situations, viscous heating has a significant effect on the flow and cannot be neglected. The corresponding term in the balance of internal energy is quadratic in the velocity gradient, which greatly increases the mathematical difficulty of the problem, even if buoyancy effects are neglected [7], [12, Ch. 3.4], [15]. Various models have been proposed that incorporate both viscous heating and buoyancy, while still

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maintaining the relative simplicity of the Boussinesq approximation (see [11] and the references therein). However, there are unresolved issues regarding the global-in-time existence of (weak) solutions for large initial data. In the case of a Newtonian fluid, the only result in this direction appears to be [10, Theorem 2.1], where a two-dimensional Bénard problem is treated. Higher-dimensional analogues have been obtained only for a non-Newtonian model [16].

That global existence should be an issue, in this context, is not very surprising: while the primitive equations, supplemented with suitable boundary conditions, satisfy the principle of conservation of energy, the simplified equations do not (except in special cases, see [16, Remark 2]). This may well cause solutions to blow up in finite time (see [3] for a related problem with permanent blow-up at the boundary).

In this note we consider a rather simplistic model problem that may not be physically relevant, yet captures the characteristic mathematical difficulty of the full problem. We propose a modification of the classical Boussinesq approximation that allows us to establish the global-in-time existence of weak solutions of the resulting initial-boundary value problems without restrictions on the size of the initial data.

Consider a unidirectional flow of a viscous, incompressible fluid, independent of distance in the flow direction, in a channel parallel to the constant force of gravity. The flow can be described in terms of two scalar variables, a velocity v and a temperature θ; both are functions of time $t \in \mathbb{R}_+$ and position $x \in \Omega$, where Ω denotes the cross-section of the flow channel (a bounded domain in $\mathbb{R}^2$). The functions v and θ satisfy a pair of parabolic PDEs of the form

$$\rho v_t - \mu \Delta v = \rho g + f(t), \quad \rho c \theta_t - \kappa \Delta \theta = \mu |\nabla v|^2 \quad \text{in } \Omega, \tag{1}$$

where ρ, μ, c, and κ, respectively, denote the density, viscosity, heat capacity, and thermal conductivity of the fluid; g is the gravitational acceleration (a positive constant). The function f represents the component of the pressure gradient opposite to the flow direction, which in this situation is independent of the spatial variable and plays the role of a given, externally applied force. The equations must be supplemented by suitable initial conditions at time $t = 0$ and boundary conditions on $\partial\Omega$, for example, a homogeneous Dirichlet condition for v and a homogeneous Neumann condition for θ in the case of impermeable, thermally insulated channel walls (n denotes the unit outward normal vector field on $\partial\Omega$):

$$v = 0, \quad \frac{\partial \theta}{\partial n} = 0 \quad \text{on } \partial\Omega, \tag{2}$$

$$v = v_0, \quad \theta = \theta_0 \quad \text{at } t = 0. \tag{3}$$

Since we are interested in buoyancy effects, we must assume that the density ρ is a (nonincreasing) function of temperature. In general, also the remaining coefficients, μ, c, and κ, may depend on temperature; but here, these are assumed to be positive constants. Now suppose that the temperature scale is chosen such that θ can be expected to fluctuate in a fairly narrow range about the reference temperature $\theta = 0$. Then, in a first-order approximation, ρ should decrease linearly with θ, and we can write

$$\rho = \rho_0(1 - \alpha\theta), \tag{4}$$

where $\rho_0 = \rho(0) > 0$ is the density at the reference temperature and $\alpha = -\rho'(0)/\rho(0) > 0$ is the thermal expansion coefficient at the reference temperature. The force of gravity is then given by

$$\rho g = \rho_0 g - \rho_0 \alpha \theta g. \tag{5}$$

The constant $\rho_0 g$ represents the hydrostatic pressure gradient and may be absorbed into the applied force f; the term $\rho_0 \alpha \theta g$ represents the force of buoyancy. Of course, (5) makes sense only as long as θ does not deviate too much from 0, and in particular, ρ must remain positive.

The ansatz (4) is one of the basic assumptions of the Boussinesq approximation; but it is used only in computing the force of gravity in accordance with (5) — everywhere else in the governing equations, ρ is set equal to ρ_0. In other words, the fluid is considered “thermally compressible, yet mechanically incompressible" (see [17] for a rigorous justification). In the case of a unidirectional flow parallel to gravity, as described by the system (1), this means that we have $\rho = \rho_0(1-\alpha\theta)$ on the right-hand side of the first equation, but $\rho = \rho_0$ in the terms involving the time derivatives of v and θ. This causes the characteristic difficulty alluded to earlier, and we are unable to prove the global-in-time existence of solutions, at least without restricting the size of the initial data.

It is natural to ask whether this problem can be remedied by using the ansatz (4), or a generalization thereof, not only in the force of gravity, but also in the rate of change of internal energy (the term involving θ_t). In the rate of change of momentum (the term involving v_t), which is of lesser importance in this context, we may either use (4) or set $\rho = \rho_0$. Assuming, for simplicity, that the constants ρ_0, μ, g, c, κ, and α are all equal to 1 and neglecting the (nonessential) applied force f, we are led to the systems

$$v_t - \Delta v = \rho(\theta), \quad \rho(\theta)\theta_t - \Delta\theta = |\nabla v|^2 \quad \text{in } \Omega \tag{1'}$$

or

$$\rho(\theta)v_t - \Delta v = \rho(\theta), \quad \rho(\theta)\theta_t - \Delta\theta = |\nabla v|^2 \quad \text{in } \Omega, \tag{1''}$$

respectively, where $\rho(\theta) = 1-\theta$ or, more generally, $\rho\colon \mathbb{R} \to \mathbb{R}$ is a nonincreasing function, strictly positive, say, on the interval $(-1,1)$ with $\rho(0) = 1$. Of course, we should assume that $|\theta_0| < 1$ and verify that the solutions we construct satisfy $|\theta| < 1$ for all time. Similar ideas were successfully employed in [2] and [6], albeit in situations without the quadratic gradient term.

It will be shown that this approach, while admittedly ad hoc rather than physics-based, is potentially useful in that both of the above systems (1') and (1''), along with the boundary and initial conditions (2) and (3), are globally solvable. Based on adequate notions of weak and strong solutions, and under minimal assumptions on the function ρ, we establish the global-in-time existence of weak solutions without restrictions on the size of the initial data. We also obtain some additional regularity properties of weak solutions and the uniqueness of strong solutions. Detailed proofs will appear in [4]. In [5], the same approach will be used to treat the full Navier-Stokes-Boussinesq system of equations.

2. Notation, assumptions, and main results

Let $V = H_0^1(\Omega)$, $H = H^1(\Omega)$, $\Omega \subset \mathbb{R}^N$, a smooth bounded set, with $N = 2$ or 3. We shall use the following eigenfunctions which are in $C^\infty(\Omega) \cap H^2(\Omega)$:

$$-\Delta\varphi_j = \lambda_j^D \varphi_j \ in\ \Omega, \quad \varphi_j = 0 \ on\ \partial\Omega,\ j = 1, 2, \dots.$$

$$-\Delta\psi_j + \psi_j = \lambda_j^N \psi_j \ in\ \Omega, \quad \frac{\partial\psi_j}{\partial n} = 0 \ on\ \partial\Omega \quad j = 1, 2, \dots.$$

(we note that $\psi_1 = 1$). For $T > 0$, we set $Q_T =]0,T[\times\Omega$.
We set $V_m = \operatorname{span}\{\varphi_j, j \leqslant m\},\ H_m = \operatorname{span}\{\psi_j,\ j \leqslant m\}$ for $m \geqslant 1$.
We recall that $\bigcup_{m\geqslant 1} V_m$ (resp. $\bigcup_{m\geqslant 1} H_m$) is dense in V (resp. in H). We will use the following orthogonal projections: $P_m : L^2(\Omega) \to V_m, Q_m : L^2(\Omega) \to H_m$.

We consider a function ρ and a number $a > 0$ such that $\rho : [0,a) \to]0,+\infty[$ continuous, nonincreasing. We denote by Φ a primitive of ρ on $[0,a]$. We want to prove that:

Theorem 1 *Let $(\theta_0, v_0) \in H^1(\Omega) \times H_0^1(\Omega),\ 0 \leqslant \theta_0 \leqslant a$. For any $T > 0$, there exists at least (θ, v) such that $\theta \in L^2(0,T;H^1(\Omega))$, $0 \leqslant \theta \leqslant a$, $\theta \in C([0,T], L^2(\Omega))$, $v \in L^2(0,T;H^2(\Omega)) \cap C([0,T], H_0^1(\Omega))$ satisfying:*

$$\frac{d}{dt}\int_\Omega v\varphi dx + \int_\Omega \nabla v \cdot \nabla\varphi dx = \int_\Omega \rho(\theta)\varphi dx,\ in\ \mathcal{D}'(0,T),\ \forall\varphi \in H_0^1(\Omega),$$

$$\frac{d}{dt}\int_\Omega \Phi(\theta)\psi dx + \int_\Omega \nabla\theta\cdot\nabla\psi dx = \int_\Omega g_v\psi dx,\ \textit{in } \mathcal{D}'(0,T),\ \forall\psi\in H^1(\Omega),$$

with $v(0)=v_0,\ \theta(0)=\theta_0, g_v\in[|\nabla v|^2\chi_{\theta<a}, |\nabla v|^2]$ *a.e in* Q_T.

Idea of Proof: For fixed ε we construct first a sequence $(\theta_m, v_m)\in C^1([0,T];H_m)\times C^1([0,T],V_m)$ satisfying

$$\frac{\partial v_m}{\partial t} = \Delta v_m + P_m(\rho_+(\theta_m)_+) \tag{6}$$

$$\frac{\partial \theta_m}{\partial t} = Q_m\left(F_{\varepsilon,m}(\theta_m, v_m)\right) \tag{7}$$

where $F_{\varepsilon,m}(\theta_m,v_m) = \dfrac{S_\varepsilon(\theta_m)|\nabla v_m|^2}{(1+\varepsilon|\nabla v_m|^2)\rho_{\varepsilon,m}(\theta_m)} + \dfrac{\Delta\theta_m}{\rho_{\varepsilon,m}(\theta_m)}$ and $\rho_{\varepsilon,m},\ S_\varepsilon$ are suitable functions. Letting first m go to infinity and then ε to zero, we show using usual compactness that there is a couple (θ, v) satisfying the above equations.

Definition 1 *A couple (θ,v) satisfying the regularity and equations of theorem 1 is called a weak solution for system (1') with boundary and initial conditions (2)-(3).*
A weak solution (θ,v) is called a strong solution if $\theta\in L^2(0,T;H^2(\Omega))$ and it satisfies the following condition:

$$|\nabla v|^2 = |\nabla v|^2\chi_{\{\theta<a\}}.$$

3. Some extensions and corollaries

Corollary 1 *Let $-1\leqslant\theta_0\leqslant 1,\ (\theta_0,v_0)\in H^1(\Omega)\times H_0^1(\Omega)$. Then for all $T>0$, there exists a function $\theta\in L^2(0,T;H^1(\Omega)),\ -1\leqslant\theta\leqslant 1$ with $\theta\in C([0,T],L^2(\Omega))$, $v\in C([0,T];H_0^1(\Omega))\cap L^2(0,T;H^2(\Omega))$ satisfying in $\mathcal{D}'(0,T)$: $\forall\varphi\in H_0^1(\Omega),\forall\psi\in H^1(\Omega)$*

$$\frac{d}{dt}\int_\Omega v(t,x)\varphi(x)dx + \int_\Omega \nabla\varphi(x)\nabla v(t,x)dx = \int_\Omega \varphi(x)(1-\theta(t,x))dx$$

and

$$-\frac{1}{2}\frac{d}{dt}\int_\Omega (1-\theta)^2\psi(x)dx + \int_\Omega \nabla\psi(x)\nabla\theta(t,x)(t,x)dx = \int_\Omega \psi(x)g_v(t,x)dx,$$

$v(0)=v_0,\ \theta(0)=\theta_0$, *with* $g_v\in[\nabla v|^2\chi_{\{\theta<1\}}, \nabla v|^2]$.

Proof: Let $\widetilde{\theta}_0 = \theta_0+1$. From the main theorem, there exist $\widetilde{\theta}\in C([0,T];L^2(\Omega)),\ 0\leqslant\widetilde{\theta}\leqslant 2$, and $v\in L^2(0,T;H^2(\Omega))\cap C([0,T],H_0^1(\Omega))$ such that:

$$\begin{cases} -\dfrac{1}{2}\dfrac{d}{dt}\displaystyle\int_\Omega (2-\widetilde{\theta})^2\psi(x)dx + \int_\Omega \nabla\psi\cdot\nabla\widetilde{\theta} = \int_\Omega \psi g_v(t,x)dx, \\ \dfrac{d}{dt}\displaystyle\int_\Omega v(t,x)\varphi(x)dx + \int_\Omega \nabla\varphi\nabla v(t,x)dx = \int_\Omega \varphi(x)(2-\widetilde{\theta}(t,x))^2dx. \end{cases}$$

Setting $\widetilde{\theta} = 1+\theta$ thus, $-1\leqslant\theta\leqslant +1$ satisfying the equation of corollary 1.

Proposition 1 *Let θ be the function given in theorem 1. If $\theta\in L^2(0,T;H^2(\Omega))$ then:*

$$g_v = |\nabla v|^2\chi_{\{\theta<a\}}.$$

Theorem 2 (The system (1”) for $N = 2$.) *Let $\rho,\ \theta_0,\ v_0$ be as in theorem 1. Assume that $\rho(a) > 0$. Then, there exists a couple (θ, v) having the same property as in theroem 1 and being a solution of*

$$\begin{cases} \rho(\theta)\dfrac{\partial v}{\partial t} - \Delta v = \rho(\theta), \\ \rho(\theta\,)\,\dfrac{\partial \theta}{\partial t} - \Delta\theta = |\nabla v|^2\chi_{\{\theta<a\}} \end{cases} \tag{8}$$

with

$$\begin{cases} (v,\theta) \in L^2(0,T;H^2(\Omega))^2 \times C([0,T],H^1(\Omega))^2, \\ \dfrac{\partial\theta}{\partial t},\ \dfrac{\partial v}{\partial t} \text{ are in } L^2(Q_T), \\ v(0) = v_0,\ \theta(0) = \theta_0, \\ \dfrac{\partial\theta}{\partial n} = v = 0 \text{ on } (0,T)\times\partial\Omega. \end{cases}$$

Corollary 2 *Let $\theta_0 \in C(\overline{\Omega}) \cap H^1(\Omega)$ with $\operatorname*{Max}_{\overline{\Omega}} |\theta_0| < 1$ and $v_0 \in H^1_0(\Omega)$. Then there is a couple (θ, v) being in $L^2(0,T;H^2(\Omega))^2 \times (C[0,T];H^1(\Omega))^2$, with $\dfrac{\partial\theta}{\partial t}$ and $\dfrac{\partial v}{\partial t}$ in $L^2(Q_T)$, satisfying:*

$$\begin{cases} (1-\theta)^m\dfrac{\partial v}{\partial t} - \Delta v = 1-\theta, \\ (1-\theta\,)\,\dfrac{\partial\theta}{\partial t} - \Delta\theta = |\nabla v|^2\chi_{\{\theta<|\theta_0|_\infty\}}, \\ \dfrac{\partial\theta}{\partial n} = v = 0 \text{ on } (0,T)\times\partial\Omega, \\ \theta(0) = \theta_0, : v(0) = v_0, \end{cases}$$

where $m = 0$ or $m = 1$. Moreover, $|\theta|_\infty \leqslant |\theta_0|_\infty < 1$.

Proof: It suffices to prove it for $m = 1$, the proof is the same in the other case. Let $a = |\theta_0|_\infty$, and let us set $\rho(\sigma) = 1 + a - \sigma$ then $\rho(2a) = 1 - a > 0$. From the above theorem we have a couple $(\widetilde{\theta}, v) \in L^2(0,T;H^2(\Omega)) \times C([0,T];H^1(\Omega))^2$ such that:

$$\begin{cases} \rho(\widetilde{\theta})\dfrac{\partial v}{\partial t} - \Delta v = \rho(\widetilde{\theta}), \\ \rho(\widetilde{\theta}\,)\,\dfrac{\partial\widetilde{\theta}}{\partial t} - \Delta\widetilde{\theta} = |\nabla v|^2\chi_{\{\widetilde{\theta}<2a\}}, \\ \dfrac{\partial\widetilde{\theta}}{\partial t} \text{ and } \dfrac{\partial v}{\partial t} \text{ are in } L^2(Q_T), \\ \dfrac{\partial\widetilde{\theta}}{\partial n} = v = 0 \text{ on } (0,T)\times\partial\Omega, \\ \widetilde{\theta}(0) = \theta_0 + a,\ v(0) = v_0,\ 0 \leqslant \widetilde{\theta} \leqslant 2a. \end{cases}$$

We set $\theta = \widetilde{\theta} - a$. Then $\rho(\widetilde{\theta}) = \rho(\theta + a) = 1 - \theta$ and $\theta(0) = \theta_0$, and the result follows. We can have a uniqueness result related to a strong solution as we state in the following proposition:

Proposition 2 *Let $N = 2,\ m = 0$ in corollary 2. Assume that the couple (θ, v) found in corollary 2 satisfies*

$$|\nabla v|^2 = \nabla v|^2\chi_{\{\theta<|\theta_0|_\infty\}},\ |\theta_0|_\infty < 1.$$

Then the couple (θ, v) is unique.

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