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On some Bernoulli free boundary type problems for general elliptic operators

J. I. Díaz
Departamento de Matemática Aplicada,
Universidad Complutense de Madrid, Plaza de Ciencias 3,
Ciudad Universitaria, 28040 Madrid, Spain
(ildefonso.diaz@mat.ucm.es)

J. F. Padial
Departamento de Matemática Aplicada, ETSA,
Universidad Politécnica de Madrid, Avenida Juan de Herrera 4,
28040 Madrid, Spain

J. M. Rakotoson
Laboratoire de Mathématiques, SP2MI, Université de Poitiers,
Boulevard Marie et Pierre Curie, Téléport 2, BP30179,
86962 Futuroscope Chasseneuil Cedex, France

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We consider some Bernoulli free boundary type problems for a general class of quasilinear elliptic (pseudomonotone) operators involving measures depending on unknown solutions. We treat those problems by applying the Ambrosetti–Rabinowitz minimax theorem to a sequence of approximate nonsingular problems and passing to the limit by some *a priori* estimates. We show, by means of some capacity results, that sometimes the measures are regular. Finally, we give some qualitative properties of the solutions and, for a special case, we construct a continuum of solutions.

1. Introduction

Although there are several connections between the problem considered and some formulations arising in different physical applications, the main motivation of this paper comes, initially, from a mathematical question. We have observed that most semi-linear problems, of the form $-\Delta u(x) = F(x, u(x))$, $x \in \Omega$ (where Ω is a given open bounded set in $\mathbb{R}^N$), with some boundary conditions on $\partial\Omega$, have been studied intensively in the literature when F is a given function from $\Omega \times \mathbb{R}$ into $\mathbb{R}$. Nevertheless, some relevant models in physics can be expressed as $-\Delta u(x) = \mu(x, u)$ in $\mathcal{D}'(\Omega)$, where $\mu(x, u)$ is a Radon measure depending on x but also on its own solution u.

One example of the above-mentioned problems, involving u-dependent measures, corresponds to the so-called *interior Bernoulli problem* on Ω. We recall that a 'classical' formulation of this problem is usually given as finding a set $A \subset \Omega \subset \mathbb{R}^N$

and a function $u : \bar{\Omega} \to \mathbb{R}$ such that

$$\left.\begin{aligned} -\Delta_p u &= 0 \quad \text{in } \Omega \setminus \bar{A}, \\ u &= 0 \quad \text{on } \partial\Omega, \\ u &= 1 \quad \text{on } \partial A, \\ -|\nabla u|^{p-2}\nabla u \cdot \boldsymbol{n} &= q \quad \text{on } \partial A, \end{aligned}\right\} \tag{IBP}$$

where $\Delta_p u := \operatorname{div}(|\nabla u|^{p-2}\nabla u)$, q is a given continuous function on $\bar{\Omega}$, $N \geqslant 2$ and $1 < p < \infty$. This problem arises, for instance (for $p = 2$), in the study of an inviscid incompressible irrotational horizontal flow in stationary regime. The vector velocity $\boldsymbol{v}$ is given through its stream function u by $\boldsymbol{v} = (\partial u/\partial y, -\partial u/\partial x)$. The incompressibility of the flow implies that u is a harmonic function. If we assume that the fluid circulates in Ω around a bubble of air A (of unknown location in Ω), since $\partial\Omega$ and ∂A are streamlines, after a normalization we can assume that $u \equiv 1$ on A and $u = 0$ on $\partial\Omega$. Moreover, the (Daniel) *Bernoulli principle* holds on ∂A, leading to

$$\tfrac{1}{2}|\boldsymbol{v}|^2 + \frac{p}{\rho} + gz = \text{const.}$$

and so $|\nabla u|$ must be constant on ∂A. For a mathematical treatment of the problem, see, for example, [1, 2, 9, 20] and [18], in which a long list of references and some applications to electrolytic drilling and galvanization can be found. Some other references, dealing with the case $p \neq 2$, are given in [20]. A different context leading to the formulation (IBP) is plasma physics, particularly the so-called *sharp problem*, in which a magnetically confined ideal fusion plasma is modelled by the Grad–Safranov equation under the constitutive law that the pressure is piecewise constant [11, 19].

In order to state the problem considered here, we note that we can reformulate problem (IBP) in terms of a measure with support on the subset $\partial(u^{-1}(1))$, the boundary of the set $\{x \in \Omega : u(x) = 1\}$, in the following way. Find $u \in C(\bar{\Omega}) \cap W_0^{1,p}(\Omega)$ such that

$$\left.\begin{aligned} \int_\Omega |\nabla u|^{p-2}\nabla u \cdot \nabla\varphi \, dx &= \langle \mu(\cdot, u), \varphi \rangle \quad \text{for all } \varphi \in C(\bar{\Omega}) \cap W_0^{1,p}(\Omega), \\ \langle \mu(\cdot, u), \varphi \rangle &= \int_{\partial(u^{-1}(1))} q(y)\varphi(y) \, dH_{N-1}(y). \end{aligned}\right\} \tag{IPB $-\ \mu$}$$

As usual, if Ω is an open bounded smooth set of $\mathbb{R}^N$, $N \geqslant 1$ (as we shall assume in the rest of the paper), $\partial(A)$ denotes the boundary of a subset A in Ω (that is $\partial(A) = \bar{A} - \mathring{A}$, where $\bar{A}$ is the closure of A in Ω and $\mathring{A}$ is its interior). We also use the notation $u^{-1}(1) = \{x \in \Omega : u(x) = 1\} := \{u = 1\}$. Here, H_m denotes the m-dimensional Hausdorff measure and, in general, we denote by $\langle \cdot, \cdot \rangle$ the duality product between some functional space V and its dual space V'.

The main result of this paper is to show that it is possible to carry out a mathematical treatment of this type of problem for suitable second- and higher-order quasilinear partial differential equations. For instance, more general second-order operators arise when the physical problem is formulated in terms of some special curvilinear coordinates. The price one has to pay by considering a larger generality on the partial differential operator comes from the question of the regularity of the solution u (and thus from the nature of the measure $\mu(x,u)$). Indeed, some spatial coefficients or nonlinear terms depending on the solution u and its derivatives may arise at the operator, presenting some singularities different from the term $\mu(x,u)$. Because of this, we must relax the notion of solution, allowing the presence of measures $\mu(x,u)$ more general than that given by

$$\langle \mu(\cdot,u),\varphi\rangle = \int_{\partial(u^{-1}(1))} q(y)\varphi(y)\,\mathrm{d}H_{N-1}(y).$$

More precisely, in this paper, we shall consider the following general formulation.

PROBLEM 1.1. Given a Banach space V (of functions defined on Ω), a 'pseudo-monotone operator' $\mathcal{A}$ mapping V into its dual V' (see § 2 for details and some examples) and a function $q \in C(\bar{\Omega})$ with $q > 0$, find a function $u \in V$ with a non-void set $\partial(u^{-1}(1))$ and find a bounded Radon measure μ whose support is included in the set $\partial(u^{-1}(1))$ such that

$$\langle \mathcal{A}u,\varphi\rangle = \int_{\partial(u^{-1}(1))} q(x)\varphi(x)\,\mathrm{d}\mu(x) \quad \text{for all } \varphi \in V \cap C(\bar{\Omega}).$$

Our main result (theorem 2.8) shows that, under suitable assumptions on V, $\mathcal{A}$ and q, the above problem possesses at least one solution (u,μ) (we also obtain the additional information that $\mu \geqslant 0$ and that $\langle \mathcal{A}u,u\rangle > 0$). This is done by introducing a sequence of approximate quasilinear (non-singular) non-monotone equations

$$\mathcal{A}u_n = qF_n(u_n)$$

for some suitable functions F_n. We prove the applicability of the Ambrosetti–Rabinowitz minimax theorem, for each natural n and we pass to the limit by means of suitable *a priori* estimates.

In a separate step, by obtaining some capacity results we show (see proposition 3.2) that when u is more regular (for example, Lipschitz continuous) the measure $\mu(\cdot,u)$ is also more regular (with respect to the H_{N-1} measure) in the sense that there exists a function g, H_{N-1}-integrable on $\partial(u^{-1}(1))$, such that $\mathrm{d}\mu = g\,\mathrm{d}H_{N-1}$. Note that in that case the problem satisfied by (u,μ) can be formulated in a similar way to the interior Bernoulli-type problem (IBP) (but replacing the operator $\Delta_p u$ by $\mathcal{A}u$).

We point out that the dependence on u of the measure $\mu(\cdot,u)$ leads to very important differences with respect to the case of quasilinear problems involving prescribed measure data (independent of u) as source terms on the right-hand side of (IPB $-\ \mu$). As shown in the large literature on this case (see, for example, [3, 5–8, 22, 23]), it is possible to deal with more singular measures.

Note that the requirement that any solution must have a non-void set $\partial(u^{-1}(1))$ allows us to classify the problem studied in the class of *free-boundary problems* since the location of the set $\partial(u^{-1}(1)))$ is also unknown (see also remark 2.1 for the interpretation of problem 1.1 as a limit of some singular dead core problems). Notice also that, in some sense, the Radon measure μ can be viewed as a Lagrange multiplier associate to the constraint $\partial(u^{-1}(1)) \neq \emptyset$.

As it is natural, it is possible to get a sharper description of the solution (u, μ) when we know some additional information about the problem. So, in particular, we consider in § 5 the one-dimensional case, obtaining a complete description of (u, μ) when we also assume some symmetry conditions. This special formulation allows us to show that, in general, we cannot expect to have uniqueness of solutions for the formulation of problem 1.1, since when q is a positive constant we construct a continuum of solutions (u_λ, μ_λ) depending on a parameter λ.

2. Statement of the main results

Let Ω be an open bounded smooth set of $\mathbb{R}^N$ and for $1 \leqslant p < \infty$ let the usual Lebesgue space $L^p(\Omega)$ be endowed with its norm (denoted by $|\cdot|_p$). Let the Sobolev space $W^{m,p}(\Omega) = \{v \in L^p(\Omega),\ D^\alpha v \in L^p(\Omega),\ |\alpha| \leqslant m\}$ for $m \in \mathbb{N}$. We also recall the classical spaces $C^0(\bar{\Omega}) = C(\bar{\Omega}) = \{v : \bar{\Omega} \to \mathbb{R}, \text{a continuous function}\}$, $|v|_\infty = \max_{x \in \bar{\Omega}} |v(x)|$, $C^k(\bar{\Omega}) = \{v \in C(\bar{\Omega}) : D^\alpha v \in C(\bar{\Omega}),\ |\alpha| \leqslant k\}$ for $k \geqslant 1$ and $C_c^\infty(\Omega) = \{v$ is indefinitely differentiable with compact support in $\Omega\}$.

We denote by $(V, \|\cdot\|)$ a reflexive Banach space of dual V'. We assume that

$$V \hookrightarrow C(\bar{\Omega}) \text{ with } \textit{compact} \text{ embedding.}$$

REMARK 2.1. This assumption is not useful in the fundamental lemma 3.1. In the next paper in this series, [17], we will treat the case without this compact embedding.

We define

$$K = \inf_{|v|_\infty = 1} \|v\| > 0. \tag{2.1}$$

Concerning the operator $\mathcal{A}$ and function q arising in problem 1.1, we shall assume the following conditions.

ASSUMPTION 2.2. The mapping $\mathcal{A} : V \to V'$ is

(i) bounded (it maps bounded sets of V into bounded sets of V'),

(ii) strongly–weakly continuous (with the respective topology of V and V'),

(iii) *pseudomonotone* (i.e. if $v_j \rightharpoonup v$ weakly in V and $\limsup_j \langle \mathcal{A}v_j, v_j - v\rangle \leqslant 0$, then $v_j \to v$ strongly in V).

ASSUMPTION 2.3. For any $v \in V$ such that there is an open relatively compact set $\mathcal{O}$ in Ω with $v = 1$ on $\mathcal{O}$, we have $\langle \mathcal{A}v, \varphi\rangle = 0$ for all $\varphi \in V$ with support $(\varphi) \subset \mathcal{O}$. (Formally, we can state analogously that the restriction of $\mathcal{A}$ on any relatively compact set $\mathcal{O}$ in Ω to $v = 1$ is zero, i.e. $\mathcal{A}(1)|_{\mathcal{O}} = 0$.)

ASSUMPTION 2.4. If $\mathcal{A}v = 0$ for some $v \in V$, then $v = 0$.

ASSUMPTION 2.5. There exists a Gâteaux differentiable function $J : V \to \mathbb{R}$ such that

(i) $\langle \mathcal{A}u, \varphi\rangle = \langle J'(u), \varphi\rangle \left(:= \lim_{t \to 0} \dfrac{J(u + t\varphi) - J(u)}{t} \right)$ for all $\varphi \in V$ and all $u \in V$,

(ii) J is coercive in the sense that there exist non-negative constants α_i, $i = 0, 1, 2, 3$, such that, for all $v \in V$,

$$\alpha_0\|v\| - \alpha_1 \leqslant J(v) \leqslant \alpha_2\|v\|^p + \alpha_3 \quad \text{for some } p \geqslant 1,$$

and $\alpha_0 K > 2\alpha_1 + \alpha_3$ (K given by (2.1)),

(iii) $J(0) \leqslant 0$.

We recall that an important subclass of pseudomonotone operators $\mathcal{A}$ is constituted by monotone operators satisfying some extra conditions. For instance, it is the case when A is monotone (i.e. $\langle \mathcal{A}u - \mathcal{A}v, u - v\rangle \geqslant 0$ for all u, v), hemi-continuous ($\lim_{t\to 0}\langle \mathcal{A}(u + t\varphi), v\rangle = \langle \mathcal{A}u, v\rangle$ for all u, φ, v) and bounded. In that case $\mathcal{A}$ is also strongly–weakly continuous [21]. On the other hand, if $V = V'$ is a Hilbert space and $\mathcal{A}$ is a maximal monotone operator, then $\mathcal{A}$ is strongly–weakly continuous. This is a consequence of a well-known property [10] for maximal monotone operators: if $x_n \rightharpoonup x$, $y_n \rightharpoonup y$ and $\mathcal{A}x_n = y_n$, $\liminf_n (y_n, x_n) \leqslant (y, x)$, then $\mathcal{A}x = y$.

We now give some examples of operators $\mathcal{A}$ and spaces V to which we can apply our result.

EXAMPLE 2.6. $V = W_0^{1,p}(\Omega)$, $p > N$, $\mathcal{A}v = -\operatorname{div}(\alpha(x)|\nabla v|^{p-2}\nabla v)$, $\alpha \in L^\infty(\Omega)$, $\alpha(x) \geqslant \alpha_0 > 0$, or

$$\mathcal{A}v = -\sum_{i=1}^{m} \frac{\partial}{\partial x_i}\left(\alpha_i\left|\frac{\partial v}{\partial x_i}\right|^{p_i-2}\frac{\partial v}{\partial x_i}\right),$$

for $\alpha_i \in L^\infty(\Omega)$, $\alpha_i(x) \geqslant \alpha_0 > 0$, $p_i > N$.

Note that the operators satisfy assumptions 2.2–2.5 with

$$J(v) = \frac{1}{p}\int_\Omega \alpha(x)|\nabla v|^p \,\mathrm{d}x$$

or

$$J(v) = \sum_{i=1}^{m} \frac{1}{p_i}\int_\Omega \alpha_i(x)\left|\frac{\partial v}{\partial x_i}\right|^{p_i} \mathrm{d}x \quad \text{for } v \in V = W_0^{1,p}(\Omega)$$

respectively, and that, from Sobolev embedding, $W^{1,p}(\Omega) \hookrightarrow C(\bar{\Omega})$ if $p > N$ with compact injection.

EXAMPLE 2.7. $V = W_0^{2,p}(\Omega)$, $p > \frac{1}{2}N$, $\mathcal{A}v = \Delta(|\Delta v|^{p-2}\Delta v) + b(x)\phi(v(x))$ for $v \in V$, where $b \in L^\infty(\Omega)$ and $\phi \in C(\mathbb{R}) \cap L^\infty(\mathbb{R})$ satisfy some additional conditions (see (2.2)).

It is easy to see that the operators given in the example satisfy assumptions 2.2–2.5 with

$$J(v) = \frac{1}{p}\int_\Omega |\Delta v|^p \,\mathrm{d}x + \int_\Omega b(x)\left(\int_0^{v(x)} \phi(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}x \quad \text{for } v \in V = W_0^{2,p}(\Omega),$$

where $\|v\| = |\Delta v|_p$, and to see that, from Sobolev embedding, $W^{2,p}(\Omega) \hookrightarrow C(\bar{\Omega})$ if $p > \frac{1}{2}N$ (this injection is compact).

A detailed proof of the pseudomonotonicity of the operators given in examples 2.6 and 2.7 can be found, for example, in [21]. The smallness condition

$$p|b|_\infty|\phi|_\infty < \sup_{v\neq 0, v\in V} \frac{|\Delta v|^p_{L^p(\Omega)}}{|v|_{L^1(\Omega)}} \tag{2.2}$$

on b and ϕ and the growth of J imply condition 2.4. Assumption 2.3 holds, since $\phi(1) = 0$.

Our main existence result is the following.

THEOREM 2.8. *Suppose that assumptions 2.2–2.5 hold. Then there exists a* (u, μ) *solution of problem 1.1 with* $u \in V$ *and* μ *a non-negative bounded Radon measure whose support is non-void and contained in* $\partial(u^{-1}(1))$ *such that*

$$\langle Au, \varphi\rangle = \int_{\partial(u^{-1}(1))} q(x)\varphi(x)\,\mathrm{d}\mu(x) \quad \textit{for all } \varphi \in V. \tag{2.3}$$

Moreover,

$$\langle Au, u\rangle > 0. \tag{2.4}$$

REMARK 2.9. Problem 1.1 is (sometimes) regarded as some kind of limit case of singular dead core problems. Indeed, some relevant problems in chemical engineering lead to the formulation

$$\begin{aligned} -\Delta_p v + qv^{-\alpha} = 0 &\quad \text{in } \Omega, \\ v = 1 &\quad \text{on } \partial\Omega, \end{aligned}$$

for some $\alpha \in (0, 1)$ (see, for example, [13] and references therein). Note that function $u = 1 - v$ satisfies

$$\begin{aligned} -\Delta_p u = \frac{q}{(1-u)^\alpha} &\quad \text{in } \Omega, \\ u = 0 &\quad \text{on } \partial\Omega. \end{aligned}$$

The variational formulation leads to the minimization of the functional

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x + \frac{q}{1-\alpha}\int_\Omega (1-u)^{1-\alpha}\,\mathrm{d}x.$$

Making $\alpha \to 1$, we (formally) obtain the problem of minimizing

$$J(u) = \frac{1}{p}\int_\Omega |\nabla u|^p\,\mathrm{d}x + q\int_\Omega H(1-u)\,\mathrm{d}x,$$

with $H(u)$ the Heaviside function, which is another formulation of the Bernoulli problem [1].

3. Proof of the main theorem and its corollaries

In order to show the existence result we shall use an *Ambrosetti–Rabinowitz minimax*-type argument: we shall introduce an approximate sequence of functionals, we shall prove the existence of a critical point for each one of then and, finally, using some *a priori* estimates on these critical points, we shall pass to the limit.

To construct an approximate sequence for equation (2.3), let $\omega \subset\subset \Omega$ be an open relatively compact subset of Ω. Let $z_1 \in V$ be such that $0 \leqslant z_1 \leqslant 1$, $z_1 = 1$ on ω and let $\delta = \delta(\omega, z_1) > 0$ be such that $J(z_1) \leqslant \delta \int_\omega q \, \mathrm{d}x < \infty$.

For $n \geqslant \delta$, we shall consider the following non-negative continuous function F_n on $\mathbb{R}$, for any $\sigma \in \mathbb{R}$:

$$F_n(\sigma) = \begin{cases} n & \text{if } \sigma = 1, \\ 0 & \text{if } \sigma \leqslant 1 - \dfrac{\delta}{n} \text{ or } \sigma \geqslant 1 + \dfrac{\delta}{n}, \end{cases}$$

and such that

$$\int_{-\infty}^{1} F_n(\sigma) \, \mathrm{d}\sigma = \int_{1}^{+\infty} F_n(\sigma) \, \mathrm{d}\sigma = \delta.$$

Note that the exact value of F_n on $[1 - (\delta/n), 1 + (\delta/n)]$ is not relevant here.

We consider the sequence of functionals mapping $J_n : V \to \mathbb{R}$ by setting, for $v \in V$,

$$J_n(v) = J(v) - \int_\Omega q(x) \left(\int_0^{v(x)} F_n(\sigma) \, \mathrm{d}\sigma \right) \mathrm{d}x.$$

We have the following lemma.

LEMMA 3.1 (fundamental sequence). *Suppose that assumptions 2.2–2.5 hold and that V is only compactly embedded in $L^1(\Omega)$. Then there is a sequence $u_n \in V$ such that*

$$Au_n = qF_n(u_n) \quad \text{in } V'. \tag{3.1}$$

Moreover,

$$0 < \tfrac{1}{2}\alpha_0 K - \alpha_1 \leqslant J_n(u_n) \leqslant \alpha_2 \|z_1\|^p + \alpha_3 + 2\delta \int_\Omega q \, \mathrm{d}x. \tag{3.2}$$

The proof will again rely on the Ambrosetti–Rabinowitz minimax theorem [4]. We first have the following proposition.

PROPOSITION 3.2. $J_n(z_1) \leqslant 0$.

Proof of proposition 3.2. Since $z_1 \geqslant 0$ and $F_n \geqslant 0$ we have

$$\int_\Omega q(x) \left(\int_0^{z_1(x)} F_n(\sigma) \, \mathrm{d}\sigma \right) \mathrm{d}x \geqslant \int_\omega q(x) \left(\int_0^{1} F_n(\sigma) \, \mathrm{d}\sigma \right) \mathrm{d}x = \delta \int_\omega q(x) \, \mathrm{d}x.$$

Then, we obtain

$$J_n(z_1) = J(z_1) - \int_\Omega q(x) \left(\int_0^{z_1(x)} F_n(\sigma) \, \mathrm{d}\sigma \right) \mathrm{d}x \leqslant J(z_1) - \delta \int_\omega q(x) \, \mathrm{d}x \leqslant 0.$$

□

Proof of lemma 3.1. We consider the set of functions

$$\Gamma = \{ f \in C([0,1], V) : f(0) = 0, \ f(1) = z_1 \}$$

and define

$$\gamma_n := \min_{f\in\Gamma}\max_{s\in[0,1]} J_n(f(s)).$$

Since $J(0)\leqslant 0$ (see assumption 2.5), we then have $\max(J_n(z_1), J_n(0))\leqslant 0$. Let us show that $\gamma_n > 0$ for $n \geqslant 2\delta$. Let $0<\mu\leqslant K/(2\|z_1\|)$ and $f\in\Gamma$. Then there exists $s_f \in]0,1[$ such that $\|f(s_f)\| = \mu\|z_1\|$ (note that $\mu\leqslant\frac{1}{2}$). By the definition of K, we deduce that

$$|f(s_f)|_\infty \leqslant \frac{\mu}{K}\|z_1\| \leqslant \frac{1}{2} < 1-\frac{\delta}{n} \quad \text{if } n\geqslant 2\delta.$$

Moreover, from the definition of F_n, we have $J_n(f(s_f)) = J(f(s_f))$. Thus, we have

$$\max_{s\in[0,1]} J_n(f(s)) \geqslant \max_{\mu\in[0,K/(2\|z_1\|)]}\Big(\min_{\|v\|=\mu\|z_1\|} J(v)\Big).$$

By the growth assumption on J, we deduce that

$$\max_{s\in[0,1]} J_n(f(s)) \geqslant \max_{\mu\in[0,K/(2\|z_1\|)]}(\alpha_0\mu\|z_1\| - \alpha_1) = \tfrac{1}{2}a_0K - \alpha_1 > 0.$$

Thus,

$$\gamma_n \geqslant \tfrac{1}{2}\alpha_0 K - \alpha_1 > 0 = \max(J_n(z_1), J_n(0)). \tag{3.3}$$

Next, we want to show that for a fixed $n\geqslant 2\delta$ the functional J_n satisfies the Palais–Smale condition (here denoted by $(\mathrm{PS})_\gamma$) for any value $\gamma\in\mathbb{R}$. Let $(v_j)_j$ be a sequence in V such that

$$J_n v_j \xrightarrow[j\to+\infty]{} \gamma,$$

and

$$J_n' v_j \to 0 \quad \text{strongly in } V'.$$

Then there is a constant $\beta_n>0$ such that $|J_n(v_j)|\leqslant\beta_n$ for all $j\in\mathbb{N}$. By the growth condition on J and the definition of F_n, there exists a constant $\beta_n'>0$ (also depending on δ) such that $\|v_j\|\leqslant\beta_n'$ for all j. Since V is reflexive and the embedding $V\hookrightarrow L^1(\Omega)$ is compact, we deduce that there is a function $v\in V$ and a subsequence (still labelled as v_j) such that

$$v_j \rightharpoonup v \text{ weakly in } V \quad\text{and}\quad v_j\to v \text{ in } L^1(\Omega).$$

Since $J_n'(v_j) = \mathcal{A}v_j - qF_n(v_j)$, we then have

$$\langle \mathcal{A}v_j, v_j - v\rangle = \int_\Omega qF_n(v_j)(v_j-v)\,\mathrm{d}x + \langle J_n'(v_j), v_j - v\rangle,$$

from which we deduce that

$$|\langle \mathcal{A}v_j, v_j - v\rangle| \leqslant c_n|v_j - v|_1 + |J_n'(v_j)|_{V'}|v_j - v|_V \tag{3.4}$$

with $c_n = |q|_\infty \max_\sigma |F_n(\sigma)| < \infty$. Since $(v_j)_j$ remains in a bounded set of V, with the condition that $\lim_j |J_n' v_j|_{V'} = 0$, we then have $\limsup_{j\to+\infty}\langle\mathcal{A}v_j, v_j - v\rangle\leqslant 0$. The fact that $\mathcal{A}$ is pseudomonotone implies that $v_j\to v$ strongly in V. This shows that we have the $(\mathrm{PS})_\gamma$ condition for J_n. So, by the Ambrosetti–Rabinowitz mountain-pass lemma [4] we deduce that γ_n is a critical value, i.e. there exists $u_n\in V$ such that $J_n(u_n)=\gamma_n$ and $J_n'(u_n)=0$. This last statement implies (3.1).

From relation (3.3), we have $J_n(u_n) = \gamma_n \geqslant \frac{1}{2}\alpha_0 K - \alpha_1$. Since the map $g : [0, 1] \to V$ defined by $g(s) = sz_1$ belongs to Γ, we then have

$$\gamma_n \leqslant \max_s J_n(g(s)) \leqslant \alpha_2 \|z_1\|^p + \alpha_3 + \int_\Omega q(x)\left(\int_{-\infty}^{+\infty} F_n(\sigma)\,\mathrm{d}\sigma\right)\mathrm{d}x,$$

i.e.

$$J_n(u_n) \leqslant \alpha_2 \|z_1\|^p + \alpha_3 + 2\delta \int_\Omega q(x)\,\mathrm{d}x.$$

□

COROLLARY 3.3.

(i) *The sequence u_n is uniformly bounded in V for any $n \in \mathbb{N}$.*

(ii) *$F_n(u_n)$ remains in a bounded set of $L^1_+(\Omega)$ for any $n \in \mathbb{N}$.*

Proof. The first statement is a consequence of the growth assumption (assumption 2.5), the construction of F_n and (3.2). On the other hand, since $\inf_{\bar{\Omega}} q = m > 0$, for $n \geqslant 2\delta$ we have

$$0 \leqslant \frac{m}{2}\int_\Omega F_n(u_n) \leqslant \int_\Omega q u_n F_n(u_n)\,\mathrm{d}x = \langle \mathcal{A}u_n, u_n\rangle.$$

Thus,

$$0 \leqslant \int_\Omega F_n(u_n)\,\mathrm{d}x \leqslant \frac{2}{m}|\mathcal{A}u_n|_{V'}\|u_n\|. \tag{3.5}$$

Since $\mathcal{A}$ is a bounded operator, there exists a constant $c > 0$ (independent of n) such that

$$|\mathcal{A}u_n|_{V'}\|u_n\| \leqslant c, \quad 0 \leqslant \int_\Omega F_n(u_n) \leqslant \frac{2}{m}c. \tag{3.6}$$

□

From now, we assume that $V \hookrightarrow C(\bar{\Omega})$ with compact embedding.

As consequence of the above result, we may assume that there exist a subsequence, still denoted u_n, and $F_n(u_n)$, a function $u \in V$ and a non-negative measure μ such that $F_n(u_n) \rightharpoonup \mu$, vaguely in the set of bounded Radon measures and weakly in V' and $u_n \rightharpoonup u$ weakly in V (and with $|u_n - u|_\infty \to 0$ by the compact embedding).

We then have the following corollary.

COROLLARY 3.4.

(i) $\operatorname{supp}(\mu) \subset \{u = 1\}$.

(ii) *$u_n \to u$ strongly in V.*

(iii) *$q\mu \in V'$, $\mathcal{A}u = q\mu$ in V' and $\langle \mathcal{A}u, u\rangle \geqslant 0$.*

Proof. To prove (i) we first will show that $\operatorname{supp}(\mu) \subset \{u = 1\}$. Let $\varphi \in C_c(\Omega)$ such that $\operatorname{supp}\varphi \subset \{u < 1\}$ and $x_M \in \operatorname{supp}\varphi$ such that

$$u(x_M) = \max\{u(x),\ x \in \operatorname{supp}\varphi\}.$$

Choosing $0 < \eta < \frac{1}{2}(1 - u(x_M))$, for sufficiently large n, $n \geqslant n_{\eta,\delta}$, we have

$$\operatorname{supp}\varphi \subset \{u_n < 1 - \eta\} \subset \left\{u_n < 1 - \frac{\delta}{n}\right\}.$$

Thus,

$$\int_\Omega \varphi q F_n(u_n)\,\mathrm{d}x = 0.$$

This implies that $\langle \mu, \varphi q\rangle = 0$, since $q > 0$ on $\bar{\Omega}$ and $q \in C(\bar{\Omega})$; thus, $\operatorname{supp}(\mu) \subset \{u \geqslant 1\}$. Considering $\psi \in C_c(\Omega)$ such that $\operatorname{supp}(\psi) \subset \{u > 1\}$, we have $\min\{u(x) : x \in \operatorname{supp}(\psi)\} > 1$. Then, for large n, $\operatorname{supp}(\psi) \subset \{u_n \geqslant 1 + (\delta/n)\}$. Thus,

$$\langle \mu, q\psi\rangle = 0$$

implies that

$$\operatorname{supp}(\mu) \subset \{u = 1\}.$$

To prove part (ii), we point out that

$$\langle \mathcal{A}u_n, u_n - u\rangle = \int_\Omega qF_n(u_n)(u_n - u)\,\mathrm{d}x \leqslant c|u_n - u|_\infty$$

and thus

$$\limsup_n \langle \mathcal{A}u_n, u_n - u\rangle \leqslant 0,$$

since $\mathcal{A}$ is pseudomonotone. Then, $u_n \to u$ strongly in V.

Finally, since $\mathcal{A}$ maps V-strong into V'-weak, we then have $\mathcal{A}u_n \rightharpoonup \mathcal{A}u$ in V'-weak. Thus, $\langle \mathcal{A}u_n, u_n\rangle = \int_\Omega qF_n(u_n) \geqslant 0$ implies that $\langle \mathcal{A}u, u\rangle \geqslant 0$ and $\mathcal{A}u = q\mu$, which proves (iii). □

The proof of theorem 2.8 is a consequence of the above results together with the following properties.

LEMMA 3.5.

(i) $\operatorname{supp}\mu \subset \{u = 1\} \setminus \overset{\circ}{\overline{\{u = 1\}}} = \partial(u^{-1}(1))$.

(ii) $\operatorname{supp}(\mu)$ *is non-void and* $\langle \mathcal{A}u, u\rangle = \int_{\partial(u^{-1}(1))} q\,\mathrm{d}\mu > 0$.

Proof. Let $\varphi \in C_c^\infty(\overset{\circ}{\overline{\{u = 1\}}})$. Since $u = 1$ on the open set $\mathcal{O} = \overset{\circ}{\overline{\{u = 1\}}}$. Thus, $\langle \mathcal{A}u, \varphi\rangle = 0$ (by assumption 2.4), i.e. $\langle \mu, \varphi q\rangle = 0$, which implies that $\mu = 0$ on $\mathcal{O}$. This shows (i). Moreover, if $\operatorname{supp}\mu = \emptyset$, then $\mathcal{A}u = 0$, $u \in V$ and, according to assumption 2.4, we will have $u = 0$. However, from lemma 3.1 and the strong convergence of corollary 3.4, from the growth condition in assumption 2.5 we have $0 < \frac{1}{2}\alpha_0 K - \alpha_1 - \alpha_3 \leqslant \alpha_2\|u_n\|^p$. This implies that $0 < \frac{1}{2}a_0 K - \alpha_1 - \alpha_3 \leqslant \alpha_2\|u\|^p$, which contradicts the fact that $u \equiv 0$. □

4. Some qualitative properties of u and μ

We can now summarize the estimates obtained in the above section.

COROLLARY 4.1. *Let u be the solution of the main problem given by theorem 2.8. Then*

$$0 < M_0 \leqslant \|u\| \leqslant M_1,$$

where

$$M_0 = \left(\frac{1}{\alpha_2}\left[\frac{\alpha_0 K}{2} - \alpha_1 - \alpha_3\right]\right)^{1/p}$$

with α_i, $i = 0, 1, 2, 3, 4$, and K given in assumption 2.5; and

$$M_1 = \frac{1}{\alpha_0}\left[2\delta \int_\Omega q\,\mathrm{d}x + \alpha_2\|z_1\|^p + \alpha_3 + \alpha_1\right]$$

for some $\delta > 0$.

To go further in our study, we shall assume some additional properties on the operator $\mathcal{A}$.

ASSUMPTION 4.2. Assume that, for all v, φ in V,

(i) $\langle \mathcal{A}\varphi, \varphi\rangle \geqslant 0$;

(ii) $\langle \mathcal{A}v, \varphi\rangle \leqslant \langle \mathcal{A}v, v\rangle^{1/p'}\langle \mathcal{A}\varphi, \varphi\rangle^{1/p}, \quad \dfrac{1}{p} + \dfrac{1}{p'} = 1.$

DEFINITION 4.3. We define the $\mathcal{A}$-capacity of a set $E \subset \Omega$ by

$$\operatorname{cap}_{\mathcal{A}}(E) = \inf\{\langle \mathcal{A}\varphi, \varphi\rangle,\ \varphi \in V,\ \varphi \geqslant 1 \text{ on } E\}.$$

PROPOSITION 4.4. *Suppose that assumptions 2.2–2.5 and 4.2 hold and let (u, μ) be a solution of problem 1.1 with $\mu \geqslant 0$. Then, for any set $E \subset \Omega$,*

$$\mu(E) \leqslant \frac{\langle \mathcal{A}u, u\rangle^{1/p'}}{m} \operatorname{cap}_{\mathcal{A}}(E)^{1/p},$$

with $m = \inf_{\bar\Omega} q > 0$.

Proof of proposition 4.4. Let $\varphi \in V$ such that $\varphi \geqslant 1$ on $E \subset \Omega$. Then

$$m\mu(E) \leqslant \int_\Omega q\varphi\,\mathrm{d}\mu = \langle \mathcal{A}u, \varphi\rangle \leqslant \langle \mathcal{A}u, u\rangle^{1/p'}\langle \mathcal{A}\varphi, \varphi\rangle^{1/p}.$$

Thus,

$$\mu(E) \leqslant \frac{\langle \mathcal{A}u, u\rangle^{1/p'}}{m} \inf\{\langle \mathcal{A}\varphi, \varphi\rangle, \varphi \in V,\ \varphi \geqslant 1\}^{1/p}.$$

□

COROLLARY 4.5. *Under the same assumptions as proposition 4.4, we have*

$$\operatorname{cap}_{\mathcal{A}}(\partial(u^{-1}(1))) > 0.$$

DEFINITION 4.6 (maximum principle property). We will say that the couple $(\mathcal{A}, V)$ satisfies the maximum principle property if, from the inequality $\langle \mathcal{A}v, \phi(v)\rangle \leqslant 0$ (for any $\phi : \mathbb{R} \to \mathbb{R}$ non-decreasing globally Lipschitz continuous and $v \in V$ such that $\phi(v) \in V$), we obtain $v(x) \notin \operatorname{supp}(\phi')$ almost everywhere (a.e.) $x \in \Omega$.

PROPOSITION 4.7. *Let*

$$\operatorname{Lip}^0(\mathbb{R}) = \{\phi : \mathbb{R} \to \mathbb{R}, \textit{globally Lipschitz continuous with } \phi(0) = 0\}.$$

Suppose that assumptions 2.2–2.5 hold and let V be such that, for all $v \in V$, $\phi(v) \in V$ whenever $\phi \in \operatorname{Lip}^0(\mathbb{R})$. Assume that $(\mathcal{A}, V)$ satisfies the maximum principle property. Then, any solution of problem 1.1 satisfies

$$0 \leqslant u \leqslant 1.$$

Proof. Let $\phi_1(\sigma) = (\sigma - 1)_+$. It is clear that $\phi_1 \in \operatorname{Lip}^0(\mathbb{R})$. Since $\phi_1(u) \in V$ for any u solution of problem 1.1 and since $\operatorname{supp}(\mu) \subset \{u = 1\}$, we have $\langle \mathcal{A}u, \phi(u)\rangle = \int \phi(u) q \, \mathrm{d}\mu = 0$, which implies that $u(x) \notin \operatorname{supp}(\phi_1')$ a.e. $x \in \Omega$ and equivalently $u(x) \leqslant 1$ a.e. $x \in \Omega$. On the other hand, by using $\phi_2(\sigma) = -\sigma_-$, we deduce that $\langle \mathcal{A}u, \phi_2(u)\rangle = 0$ and so $u(x) \notin \operatorname{supp}(\phi_2')$ a.e. $x \in \Omega$ and equivalently $u(x) \geqslant 0$ a.e. $x \in \Omega$. □

REMARK 4.8. It is easy to see that if, for $\Gamma_1 \subset \partial\Omega$ with $\mathrm{d}H_{N-1}(\Gamma_1) > 0$, we consider the space $V = \{v \in W^{1,p}(\Omega), v = 0 \text{ on } \Gamma_1\}$, $p > N$ and the operator

$$Av = -\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(\alpha_i \left|\frac{\partial u}{\partial x_i}\right|^{p-2} \frac{\partial u}{\partial x_i}\right),$$

with $\alpha_i \in L^\infty(\Omega)$, $i = 1, \dots, N$, $\alpha_i(x) \geqslant \alpha_0 > 0$ a.e., then (A, V) satisfies the maximum principle property and V is stable under the action of any element of $\operatorname{Lip}^0(\mathbb{R})$ (i.e. $\phi(v) \in V$ for any $v \in V$ and $\phi \in \operatorname{Lip}^0(\mathbb{R})$). Many variants associated with second-order elliptic operators also satisfy the above-mentioned properties.

To end this section, if we consider the special case of

$$\mathcal{A}v = -\Delta_p v, \qquad V = W_0^{1,p}(\Omega) \quad \text{with } p > N \geqslant 2, \tag{4.1}$$

the following result allows a better identification of the measure solution μ.

PROPOSITION 4.9. *Assume (4.1) and let (u, μ) be the solution of problem 1.1 such that u is Lipschitz continuous in Ω (i.e. $\nabla u \in L^\infty(\Omega)^N$). There is then a function $g \geqslant 0$, H_{N-1}-integrable on $\partial(u^{-1}(1))$, such that $\mathrm{d}\mu = g \, \mathrm{d}H_{N-1}$. In particular,*

$$\int_\Omega |\nabla u|^{p-2} \nabla u \cdot \nabla \varphi \, \mathrm{d}x = \int_{\partial(u^{-1}(1))} q\varphi g \, \mathrm{d}H_{N-1} \quad \textit{for all } \varphi \in W_0^{1,p}(\Omega).$$

Proof. Given a set $E \subset \Omega$, let us denote by

$$\gamma_1(E) = \inf\left\{\int_\Omega |\nabla\varphi| \, \mathrm{d}x, \ \varphi \geqslant 1 \text{ on } E, \ \varphi \in C_c^\infty(\Omega)\right\}$$

the 1-capacity of E. Then we have

$$\mu(E) \leqslant \frac{1}{m}\int_\Omega q\varphi \,\mathrm{d}\mu = \frac{1}{m}\int_\Omega |\nabla u|^{p-2}\nabla u\cdot\nabla\varphi\,\mathrm{d}x \leqslant \frac{|\nabla u|_\infty^{p-1}}{m}\int_\Omega |\nabla\varphi|\,\mathrm{d}x$$

whenever $E \subset \Omega$, $\varphi \geqslant 1$ on E, $\varphi \in C_c^\infty(\Omega)$. This shows that

$$\mu(E) \leqslant \frac{|\nabla u|_\infty^{p-1}}{m}\gamma_1(E).$$

According to Fleming's result [25], we deduce that if $H_{N-1}(E) = 0$, then $\gamma_1(E) = 0$ and thus $\mu(E) = 0$. So the application of the Radon–Nikodym theorem gives the result. □

5. The one-dimensional case: the shape of μ and a continuum of solutions

In this section we shall present a simple case (the case of $N = 1$ plus some additional conditions) for which it is possible to carry out the computation of the measure $\mu(u)$. We assume, for simplicity, that the operator $\mathcal{A}$ satisfies the following assumption.

ASSUMPTION 5.1. For any $v \in V$ such that $v(a) = v(b)$ and $\langle \mathcal{A}v, \varphi\rangle = 0$ for all $\varphi \in C_c^\infty[a,b]$, and some $[a,b] \subset]0,1[$, we have $v(x) = v(a)$ for all $x \in [a,b]$.

REMARK 5.2. An example of the operator $\mathcal{A}$ satisfying assumption 5.1 (and also conditions in assumptions 2.2–2.5 and 4.2) is given by

$$\mathcal{A}v = -\sum_{i=1}^{m}(\alpha_i|v'|^{p_i-2}v')',$$

where $1 < p_1 \leqslant p_2 \leqslant \cdots \leqslant p_m$, $\alpha_i \in L^\infty(0,1)$, $i = 1,\dots,m$, $\alpha_i(x) \geqslant \alpha_0 > 0$ a.e. Note that a natural choice is then $V = \{v \in W^{1,p_m}(0,1) : v(0) = 0\}$ or $V = W_0^{1,p_m}(0,1)$.

We have the following proposition.

PROPOSITION 5.3. *Suppose that assumptions 2.2–2.5 and 5.1 hold. Then, for any solution (u,μ) of problem 1.1, there exist two points $(a_0,a_1) \in [0,1]\times[0,1]$ such that*

$$\{u = 1\} = [a_0,a_1] \subsetneqq [0,1].$$

In particular, $\partial(u^{-1}(1)) = \{a_0,a_1\}$.

Proof. Let $a_0 = \min\{x \in]0,1[: u(x) = 1\} = \min\{u = 1\}$ and let $a_1 = \max\{u = 1\}$. Let us show that $]a_0,a_1[\cap \{x : u(x) < 1\} = \emptyset$. First of all, we note that if $a_0 = a_1$, there is nothing to be proved. Now, assume that $a_0 < a_1$ and suppose that $]a_0,a_1[\cap \{u < 1\} \neq \emptyset$. Let $x_0 \in]a_0,a_1[$ such that $u(x_0) < 1$ and let us denote by $I(x_0)$ the biggest interval containing x_0 and $\mathring{I}(x_0) \subset]a_0,a_1[\cap \{u < 1\}$. Then, on the boundary of $\mathring{I}(x_0)$, $\partial\mathring{I}(x_0)$, we have $u = 1$. Thus, since $\operatorname{supp}(\mu) \subset \{u = 1\}$, from the differential equation we find that u satisfies the boundary-value problem on $\mathring{I}(x_0)$ given by

$$\begin{aligned} \langle \mathcal{A}u, \varphi \rangle &= 0 \qquad \text{for all } \varphi \in C_c^\infty(\mathring{I}(x_0)), \\ u(x) &= 1, \qquad x \in \partial \mathring{I}(x_0). \end{aligned}$$

Thus, from assumption 5.1, we deduce that $u = 1$ on $I(x_0)$, which is a contradiction since $u(x_0) < 1$. This shows that $[a_0, a_1] \subset \{u \geqslant 1\}$. A similar argument shows that $]a_0, a_1[\cap \{u > 1\} = \emptyset$. Thus, we have $[a_0, a_1] \subset \{u = 1\}$ and, by the definition of a_0, a_1 we have $\{u = 1\} \subset [a_0, a_1]$, i.e. $\{u = 1\} = [a_0, a_1]$. This implies that $\partial\{u = 1\} = \{a_0, a_1\}$. □

COROLLARY 5.4. *Suppose that assumptions 2.2–2.5 and 5.1 hold. There then exist two non-negative constants* $\bar{\lambda}_0$, $\bar{\lambda}_1$ *with* $\bar{\lambda}_0 + \bar{\lambda}_1 > 0$, *such that* $\mu = \bar{\lambda}_0 \delta_{a_0} + \bar{\lambda}_1 \delta_{a_1}$.

Proof. Since $\mu \geqslant 0$ and $\operatorname{supp}(\mu) \subset \partial(u^{-1}(1)) = \{a_0, a_2\}$, we deduce that there exist $\bar{\lambda}_0 \geqslant 0$ and $\bar{\lambda}_1 \geqslant 0$ such that $\mu = \bar{\lambda}_0 \delta_{a_0} + \bar{\lambda}_1 \delta_{a_1}$. Moreover, the fact that $\mu \not\equiv 0$ implies that $\bar{\lambda}_0 + \bar{\lambda}_1 > 0$. □

As we have observed, the shape of μ depends on $(\mathcal{A}, V)$. Let us give an example for which $a_1 = 1 - a_0$, $\bar{\lambda}_0 = \bar{\lambda}_1$. We first define some set of symmetric functions and the notion of a symmetric operator.

DEFINITION 5.5.

$$\begin{aligned} L_s^p(0,1) &= \{v \in L^p(0,1),\ v(x) = v(1-x) \text{ a.e.}\}, \\ C_s[0,1] &= \{v \in C[0,1],\ v(x) = v(1-x) \text{ for all } x\}. \end{aligned}$$

We set $V_s = V \cap C_s[0,1]$. The elements of $L_s^p(0,1)$ are called 'symmetric' functions. An operator $\mathcal{A} : V \to V'$ is called 'symmetric' if, for all $\varphi \in V$ and all $v \in V_s$, $\langle \mathcal{A}v, \varphi \rangle = \langle \mathcal{A}v, \varphi_s \rangle$, where $\varphi_s(x) = \frac{1}{2}(\varphi(x) + \varphi(1-x))$ and $\varphi_s \in V_s$.

REMARK 5.6. It is easy to see that if $\alpha_i \in L_s^\infty(0,1)$, the operator $\mathcal{A}$ defined in remark 5.2 is a symmetric operator (and the conditions in assumptions 2.2–2.5 are satisfied, for example, on the space $V_s = W_0^{1,p_m}(0,1) \cap C_s[0,1]$).

According to the results of theorem 2.8 and proposition 5.3, we have the following proposition.

PROPOSITION 5.7. *Suppose that assumptions 2.2–2.5 and 5.1 hold, and that* $q \in C_s[0,1]$ *with* $q > 0$. *There then exist* $u \in V_s$, $\bar{\lambda}_0 \geqslant 0$, $\bar{\lambda}_1 \geqslant 0$ *and* $a_0 \in [0, \frac{1}{2}]$ *such that*

$$\mathcal{A}u = q(a_0)(\bar{\lambda}_0 \delta_{a_0} + \bar{\lambda}_1 \delta_{1-a_0}) \quad \text{in } V_s'.$$

Moreover, if $\mathcal{A}$ *is symmetric, then*

$$\mu = \frac{\langle \mathcal{A}u, u \rangle}{2q(a_0)} (\delta_{a_0} + \delta_{1-a_0}).$$

Proof. Since V_s is a reflexive Banach space (since it is a closed subspace of V), by applying theorem 2.8 and all the above results, we find the existence of $u \in V_s$,

$0 \leqslant a_0 \leqslant a_1 \leqslant 1$, $a_i \in \partial(u^{-1}(1))$ and $\bar{\lambda}_0 \geqslant 0$, $\bar{\lambda}_1 \geqslant 0$ such that

$$\mathcal{A}u = q(a_0)\bar{\lambda}_0\delta_{a_0} + \bar{\lambda}_1 q(a_1)\delta_{a_1}.$$

Since u is symmetric, $a_1 = 1 - a_0$ and then $q(a_0) = q(a_1)$, $0 \leqslant a_0 \leqslant \frac{1}{2}$. Finally, if $\mathcal{A}$ is symmetric, then

$$\langle \mathcal{A}u, \varphi\rangle = \langle \mathcal{A}u, \varphi_s\rangle = q(a_0)(\bar{\lambda}_0 + \bar{\lambda}_1)\left(\frac{\varphi(a_0) + \varphi(1 - a_0)}{2}\right)$$

for any $\varphi \in V$. In particular if $\varphi = u$, then

$$\langle \mathcal{A}u, u\rangle = q(a_0)(\bar{\lambda}_0 + \bar{\lambda}_1).$$

Thus,

$$\mathcal{A}u = q(a_0)\frac{\langle \mathcal{A}u, u\rangle}{2q(a_0)}(\delta_{a_0} + \delta_{1-a_0}).$$

□

REMARK 5.8. The result of proposition 5.7 shows that the problem can be interpreted as a nonlinear problem of eigenvalue type.

The above result allows us to show that, in general, we cannot expect to have uniqueness of solutions (u, μ) of problem 1.1. Indeed, when q is a positive constant the above arguments lead to the explicit construction a family of solutions, as follows.

PROPOSITION 5.9. *Assume that q is positive constant. Then, for any $\lambda > 2/q$, the pair (u_λ, μ_λ) is a solution of the special problem 1.1,*

$$\int_0^1 u'\varphi'\,\mathrm{d}x = \int_{\partial(u^{-1}(1))} q\varphi\,\mathrm{d}\mu \quad \textit{for all } \varphi \in H_0^1(0,1),\ u \in H_0^1(0,1),$$

where u_λ is given by

$$u_\lambda(x) = \begin{cases} \lambda q x & \textit{if } 0 \leqslant x \leqslant \dfrac{1}{\lambda q}, \\[2ex] 1 & \textit{if } \dfrac{1}{\lambda q} \leqslant x \leqslant 1 - \dfrac{1}{\lambda q}, \\[2ex] \lambda q(1-x) & \textit{if } 1 - \dfrac{1}{\lambda q} \leqslant x \leqslant 1, \end{cases}$$

and μ_λ is given by $\mu_\lambda = \lambda(\delta_{a_0} + \delta_{1-a_0})$ with $a_0 = 1/(\lambda q)$.

It is possible to find the uniqueness of solution (u, μ) of problem 1.1, at least under the simple formulation given in the above proposition, once we add some extra condition, for example, by prescribing the value of

$$\int_0^1 u'^2(x)\,\mathrm{d}x$$

in the correct way. In particular, the above arguments lead to the following result.

PROPOSITION 5.10. *Let $\mathcal{A}v = -v''$ and $\lambda > 2/q$. Then, the following problem possesses a unique solution.*

PROBLEM 5.11. Find $u \in H_0^1(0,1)$ and a measure $\mu \geqslant 0$ satisfying

$$-u'' = q\mu, \qquad \operatorname{supp}(\mu) \subset \partial(u^{-1}(1)),$$
$$\frac{1}{2q}\int_0^1 u'^2(x)\,dx = \lambda.$$

REMARK 5.12. See [12] for some results on the uniqueness of solutions for suitable Bernoulli-type problems. A continuum of solutions also arises in some (non-singular) problems with a free boundary (see [14]).

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