ASYMPTOTICS FOR SOME NONLINEAR DAMPED WAVE EQUATION: FINITE TIME CONVERGENCE VERSUS EXPONENTIAL DECAY RESULTS

EQUATION DES ONDES AVEC TERME D'AMORTISSEMENT NON-LINÉAIRE: CONVERGENCE EN TEMPS FINI VS DÉCROISSANCE EXPONENTIELLE

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Abstract. Given a bounded open set $\Omega \subset \mathbb{R}^n$ and a continuous convex function $\Phi : L^2(\Omega) \to \mathbb{R}$, let us consider the following damped wave equation

$$(S) \qquad u_{tt} - \Delta u + \partial\Phi\,(u_t) \ni 0, \qquad (t,x) \in (0,+\infty) \times \Omega,$$

under Dirichlet boundary conditions. The notation $\partial\Phi$ refers to the subdifferential of Φ in the sense of convex analysis. The nonlinear term $\partial\Phi$ allows to modelize a large variety of friction problems. Among them, the case $\Phi = |\,.\,|_{L^1}$ corresponds to a Coulomb friction, equal to the opposite of the velocity sign. After we have proved the existence and uniqueness of a solution to (S), our main purpose is to study the asymptotic properties of the dynamical system (S). In two significant situations, we bring to light an interesting phenomenon of dichotomy: either the solution converges in a finite time or the speed of convergence is exponential as $t \to +\infty$. We also give conditions which ensure the finite time stabilization of (S) toward some stationary solution.

Résumé. Etant donné un ouvert borné $\Omega \subset \mathbb{R}^n$ et une fonction convexe continue $\Phi : L^2(\Omega) \to \mathbb{R}$, considérons l'équation des ondes amorties suivante:

$$(S) \qquad u_{tt} - \Delta u + \partial\Phi\,(u_t) \ni 0, \qquad (t,x) \in (0,+\infty) \times \Omega,$$

avec conditions de Dirichlet au bord. La notation $\partial\Phi$ désigne le sous-différentiel de Φ au sens de l'analyse convexe. Le terme non-linéaire $\partial\Phi$ permet de modéliser une grande variété de problèmes avec frottement. Le cas $\Phi = |\,.\,|_{L^1}$ correspond au frottement de Coulomb, égal à l'opposé du signe de la vitesse. Après avoir établi l'existence et l'unicité d'une solution de (S), notre principal objectif est d'étudier les propriétés asymptotiques du système dynamique (S). Dans deux situations significatives, on met en évidence un phénomène intéressant de dichotomie: la solution converge en temps fini, ou bien la vitesse de convergence est exponentielle lorsque $t \to +\infty$. On donne également des conditions qui garantissent la stabilisation en temps fini de (S) vers une solution stationnaire.

1. Introduction

Throughout the paper, we denote by Ω a bounded open set in $\mathbb{R}^n$ with smooth boundary Γ. Given a continuous convex function $\Phi : L^2(\Omega) \to \mathbb{R}$, let us consider

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the following damped wave equation

$$(S) \qquad u_{tt} - \Delta u + \partial\Phi\,(u_t) \ni 0, \qquad (t,x) \in (0,+\infty) \times \Omega,$$

under Dirichlet boundary conditions

$$u(t,x) = 0, \qquad t \geq 0,\ x \in \Gamma, \tag{1.1}$$

and satisfying the following initial conditions

$$u(0,x) = u_0(x), \quad u_t(0,x) = v_0(x), \qquad x \in \Omega. \tag{1.2}$$

The operator $\partial\Phi : L^2(\Omega) \to \mathcal{P}(L^2(\Omega))$ is the subdifferential of Φ in the sense of convex analysis: for every $u \in L^2(\Omega)$,

$$\xi \in \partial\Phi(u) \subset \mathcal{P}(L^2(\Omega)) \qquad \Longleftrightarrow \qquad \forall v \in L^2(\Omega), \quad \Phi(v) \geq \Phi(u) + \langle \xi, v-u \rangle_{L^2}.$$

The nonlinear term $\partial\Phi$ allows to modelize a large variety of friction problems. The question of existence and uniqueness of a solution u satisfying (S) and (1.1)-(1.2) was settled in the thesis of Brézis [5], over an arbitrary finite time horizon. The problem of the asymptotic convergence when $t \to +\infty$ is delicate and has interested many authors. The linear case, corresponding to $\Phi = |\,.\,|^2_{L^2}$ (up to a constant) has given rise to a very abundant literature and the reader is referred to the classical textbooks [11, 15, 18, 22] for further details. The nonlinear problem is more subtle and one can distinguish at least two classes of interesting situations.
For the first one, let us introduce the convex function $j : \mathbb{R} \to \mathbb{R}$ and let us assume that $j(v) \in L^1(\Omega)$ for every $v \in L^2(\Omega)$. We define the convex function $\Phi : L^2(\Omega) \to \mathbb{R}$ by $\Phi(v) = \int_\Omega j(v(x))\,dx$. Following a classical result, we have $f \in \partial\Phi(v)$ if and only if $f(x) \in \partial j(v(x))$ for almost every $x \in \Omega$ (see for example [6, Proposition 2.16] or also [4, Proposition 2.7]). Setting $\beta := \partial j$, equation (S) can then be rewritten as

$$u_{tt} - \Delta u + \beta\,(u_t) \ni 0. \tag{1.3}$$

Given μ_c, $\mu_v \geq 0$, let us consider the particular case where the function j is defined by $j(r) = \mu_c\,|r| + \frac{\mu_v}{2}\,r^2$ for every $r \in \mathbb{R}$. The differential inclusion (1.3) then becomes

$$u_{tt} - \Delta u + \mu_c \operatorname{sgn}\,(u_t) + \mu_v\, u_t \ni 0, \tag{1.4}$$

where sgn $: \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is the set-valued sign function, defined by $\operatorname{sgn}(x) = 1$ if $x > 0$, $\operatorname{sgn}(x) = -1$ if $x < 0$ and $\operatorname{sgn}(0) = [-1,1]$. In this equation, the term $\mu_c \operatorname{sgn}\,(u_t)$ corresponds to the Coulomb friction while the term $\mu_v\, u_t$ represents a possible viscous component of the friction.
Coming back to the dynamical system (1.3), results of convergence were obtained by Haraux [16, 17], who used an argument of Dafermos-Slemrod [12]. Provided that $0 \notin \operatorname{int}(\beta^{-1}(0))$, he proved the convergence in $H^1_0(\Omega)$ of the solution u toward some stationary solution u_∞, along with the convergence in $L^2(\Omega)$ of the velocity u_t toward 0.

Another class of interest is given by the functions Φ which are positively homogeneous and convex. Such functions are not differentiable at the origin, and then induce a "nonsmooth" friction. Without extra difficulty, we can add a differentiable component in this model. For example, consider the function $\Phi : L^2(\Omega) \to \mathbb{R}$

defined by $\Phi = \mu_r \, |\,.\,|_{L^2} + \frac{\mu_v}{2} \, |\,.\,|^2_{L^2}$, for some μ_r, $\mu_v \geq 0$. In this case, the dynamical system (S) can be rewritten as the following global equation

$$\left\{ \begin{array}{llllll} u_{tt} - \Delta u & +\mu_r \, u_t/|u_t|_{L^2} & +\mu_v \, u_t & = 0 & \text{if} & u_t \neq 0, \\ u_{tt} - \Delta u & + \quad \mu_r \, \mathbb{B}_{L^2} & & \ni 0 & \text{if} & u_t = 0, \end{array} \right. \tag{1.5}$$

where $\mathbb{B}_{L^2}$ is the closed unit ball of $L^2(\Omega)$ centered at 0. In this model the radial friction $u_t/|u_t|_{L^2}$ has a non-local nature, due to the term $|u_t|_{L^2}$ which is computed on the whole space Ω. For that reason, system (1.5) will be referred to as the globally damped wave equation. As we shall mention later (see Remark 4.4) system (1.5) arises in the study of some control problems.

In Classical Mechanics there are many examples of finite-dimensional systems for which dry friction implies the stabilization in finite time of the underlying dynamics. At the beginning of the seventies, Haïm Brézis proposed the conjecture that the equilibrium position of a system like (1.4) is reached after a finite time (at least if $\mu_v = 0$). When the set Ω is one-dimensional (*e.g.* $\Omega =]0, 1[$), equation (1.4) modelizes the motion of a vibrating string subject to a friction. In this case, Cabannes [8, 9] obtained some partial results on finite time stabilization corresponding to particular initial data. The case of arbitrary initial data seems to be still an open problem. Motivated by this, and also suggested by the numerical approach of solutions, some easier formulations were considered in the literature, as for instance, the spatially discretized vibrating string via a finite difference scheme (see for example [3, 14]).

In this paper, we prove first that every solution u to (S) converges in $H^1_0(\Omega)$ toward some map $u_\infty \in H^2(\Omega)$ satisfying $\Delta u_\infty \in \partial\Phi(0)$. If, in addition, Δu_∞ belongs to the interior of the set $\partial\Phi(0)$, the dynamics is shown to stop definitively after a finite time. Counterexamples to finite time convergence exist when the Laplacian Δu_∞ belongs to the boundary of $\partial\Phi(0)$. We then focus our attention on this delicate case. For that purpose, we exhibit two types of asymptotic behaviors, for which we are able to evaluate the speed of convergence when finite time stabilization fails.

The first one, that we denote by (AE) (from "Asymptotic Expansion"), consists in assuming that the solution u to (S) can be asymptotically decomposed as the product of a time-dependent function by a space-dependent one, up to a negligible term. This hypothesis is satisfied in the overdamped linear case, for example. The second behavior (NV) ("Normal Velocity") is observed when the velocity vector $u_t(t)$ is normal to the set $\partial\Phi(0)$ at Δu_∞ for t large enough. A careful examination of (NV) shows that it is equivalent to a condition of uniform boundedness in time (see paragraph 6.1).

Due to the structural differences of (AE) and (NV), the estimates of the convergence rate rely on distinct arguments in each case. We prove in both situations a curious phenomenon of dichotomy: either the solution converges in a finite time or the speed of convergence is exponential. Our results are slightly more precise under (AE). We establish in this case that, if the excess of the set $\partial\Phi(v)$ over the set $\partial\Phi(0)$ tends to 0 sufficiently fast when $|v|_{L^2} \to 0$, then every solution to (S) stabilizes in a finite time. In concrete situations (*cf.* for example equation (1.4) or (1.5)), we obtain the existence of a critical coefficient for the viscous component, below which every solution stops definitively after a finite time. This critical coefficient is intimately connected with the first eigenvalue of the Laplacian Dirichlet operator $-\Delta$.

We point out that, as it can be easily shown, most of the results of this paper remain true in a more general framework (case of a general second order elliptic operator, different boundary conditions, etc.) but we shall not present it here for the sake of the exposition.

The paper is organized as follows. In section 2 we start with a general result of existence and uniqueness of solution for the inclusion (S) under the conditions (1.1)-(1.2). Section 3 is devoted to some spatially discretized version of (S). In this finite dimensional framework, we recall the main results of stabilization in finite time. We conclude the section by some numerical experiments illustrating the motion of a vibrating string (resp. membrane). In section 4 we prove that, if the function u_∞ fulfills some interior-like conditions, then the solution u stabilizes in a finite time. Sections 5 and 6 are devoted to the asymptotic analysis of (S), respectively in cases (AE) and (NV). These sections contain the major results of the paper, specially the phenomenon of dichotomy between finite time convergence and exponential decay rate.

2. General framework

Throughout the paper, we use the standard notations of convex analysis and the reader is referred to [21] for the general features relative to these notions.

2.1. **Existence and uniqueness.** Let Ω be a bounded open set in $\mathbb{R}^n$ with smooth boundary Γ. Given a continuous convex function $\Phi : L^2(\Omega) \to \mathbb{R}$, let us consider the following damped wave equation

$$(S) \qquad u_{tt} - \Delta u + \partial\Phi\,(u_t) \ni 0, \qquad (t,x) \in (0,+\infty) \times \Omega,$$

under Dirichlet boundary conditions (1.1) and initial conditions (1.2). Any solution u to (S) can be considered, either as a function $u : [0,+\infty) \times \Omega \to \mathbb{R}$ or as a function of time taking its values in a suitable functional space (such as $H^2(\Omega)$ or $H_0^1(\Omega)$ for example). Throughout the paper, we will essentially adopt the second point of view, so that the dependance with respect to the space variable x will be often omitted. We start with a general result of existence and uniqueness for the inclusion (S) under the conditions (1.1)-(1.2). Recall that, if C is a closed convex set of $L^2(\Omega)$, then C^0 denotes the element of minimal norm of C.

Theorem 2.1. *Let $\Phi : L^2(\Omega) \to \mathbb{R}$ be a continuous convex function. Assume that the initial data satisfy respectively $u_0 \in H^2(\Omega) \cap H_0^1(\Omega)$ and $v_0 \in H_0^1(\Omega)$. Then, the following assertions hold true:*
(i) There exists a unique map $u \in \mathcal{C}([0,+\infty) : H_0^1(\Omega))$, with $u_t \in \mathcal{C}([0,+\infty) : L^2(\Omega))$, such that:

(a) *$u_t \in L^\infty(0,+\infty : H_0^1(\Omega))$ and $u_{tt} \in L^\infty(0,+\infty : L^2(\Omega))$. More precisely, the following estimate holds for almost every $t \in (0,+\infty)$*

$$|\nabla u_t(t)|_{L^2}^2 + |u_{tt}(t)|_{L^2}^2 \leq |\nabla v_0|_{L^2}^2 + |(-\Delta u_0 + \partial\Phi(v_0))^0|_{L^2}^2.$$

(b) *(S) is satisfied for almost every $t \in (0,+\infty)$.*

(c) *$u(0) = u_0$ and $u_t(0) = v_0$.*

(ii) The map u satisfies $u \in L^\infty(0,+\infty : H^2(\Omega))$.
(iii) The map u_t is right differentiable on $(0,+\infty)$ and we have, for almost every $t \in (0,+\infty)$

$$\frac{d^+u_t}{dt}(t) + (-\Delta u(t) + \partial\Phi(u_t(t)))^0 = 0.$$

Proof. (i) is an immediate consequence of [5, Theorem III.1].
(ii) Since $u_t \in L^\infty(0,+\infty : H_0^1(\Omega))$ and since the imbedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact, the set $\{u_t(t), \quad t \in [0,+\infty)\}$ is relatively compact in $L^2(\Omega)$. On the other hand, we recall that the maximal monotone operator $\partial\Phi$ is bounded on every compact set of $L^2(\Omega)$ (see for example Brézis [6, §II. 5]). Then, we derive that the set $\partial\Phi(u_t(t))$ is uniformly bounded in $L^2(\Omega)$ when $t \in [0,+\infty)$. Since $u_{tt} \in L^\infty(0,+\infty : L^2(\Omega))$, we conclude in view of (S) that $\Delta u \in L^\infty(0,+\infty : L^2(\Omega))$.
(iii) is an immediate consequence of [5, Remark III.2]. □

We denote by I the subset of $[0,+\infty)$ on which the map u_t is derivable and the inclusion (S) is satisfied. Since the function u_t is absolutely continuous, it is clear that the set $[0,+\infty) \setminus I$ is negligible. A key tool in the asymptotic analysis of (S) is the existence of a Lyapounov function emanating from the mechanical interpretation of (S). Indeed, we define the energy-like function E by

$$E(t) = \frac{1}{2}|u_t(t)|_{L^2}^2 + \frac{1}{2}|\nabla u(t)|_{L^2}^2. \tag{2.1}$$

The decay rate of the function E is given by the following proposition.

Proposition 2.2. *Let $\Phi : L^2(\Omega) \to \mathbb{R}$ be a continuous convex function such that $0 \in \operatorname{argmin}\Phi$. Let u be the unique solution to (S) defined at Theorem 2.1. Then for every $t \in I$, we have*

$$\dot{E}(t) \leq -(\Phi(u_t(t)) - \Phi(0)) \leq 0. \tag{2.2}$$

Proof. By differentiating the expression of E, we find

$$\forall t \in I, \qquad \begin{aligned} \dot{E}(t) &= \langle u_t(t), u_{tt}(t)\rangle_{L^2} + \langle u_t(t), -\Delta u(t)\rangle_{L^2} \\ &= \langle u_t(t), u_{tt}(t) - \Delta u(t)\rangle_{L^2}. \end{aligned}$$

Since $-u_{tt}(t) + \Delta u(t) \in \partial\Phi(u_t(t))$, it suffices now to write the adequate subdifferential inequality. □

2.2. **Convergence toward a stationary solution.** We are going to prove that the solution u to (S) converges in $H^1(\Omega)$ and that its limit u_∞ is a stationary solution to (S), *i.e.* $\Delta u_\infty \in \partial\Phi(0)$. In a finite dimensional setting, a similar result has been established in [1, Theorem 3.1].

Theorem 2.3. *Let $\Phi : L^2(\Omega) \to \mathbb{R}$ be a continuous convex function such that $\operatorname{argmin}\Phi = \{0\}$. Let u be the unique solution to (S) defined at Theorem 2.1. Then, the following assertions hold true*
(i) There exists $u_\infty \in H_0^1(\Omega)$ such that

$$\lim_{t\to+\infty} |u(t) - u_\infty|_{H^1} = 0 \quad \textit{and} \quad \lim_{t\to+\infty} |u_t(t)|_{L^2} = 0.$$

(ii) We have $\lim_{t\to+\infty} u(t) = u_\infty$ weakly in $H^2(\Omega)$.
(iii) The limit u_∞ is a stationary solution to (S), i.e. $\Delta u_\infty \in \partial\Phi(0)$.

Proof. (i) Let us set $H = H_0^1(\Omega) \times L^2(\Omega)$ and let us define the operator $A : H \to \mathcal{P}(H)$ by

$$D(A) = \left(H^2(\Omega) \cap H_0^1(\Omega)\right) \times H_0^1(\Omega), \quad A(u,v) = (-v, -\Delta u + \partial\Phi(v)).$$

Setting $\mathbf{U}(t) = (u(t), u_t(t))$, it is immediate that the inclusion (S) can be rewritten as the following first-order in time system

$$\mathbf{U}_t(t) + A\,\mathbf{U}(t) \ni 0, \qquad t \geq 0. \tag{2.3}$$

Let us recall that, from Theorem 2.1 we have

$$u \in L^\infty(0, +\infty : H^2(\Omega)) \quad \text{and} \quad u_t \in L^\infty(0, +\infty : H_0^1(\Omega)).$$

This implies that the set $\{\mathbf{U}(t), \quad t \geq 0\}$ is precompact in the space H. By using an argument of Dafermos-Slemrod (see for example [12, Theorem 1] or [16, Theorem 1]), we derive the existence of some almost periodic solution ξ to (2.3) such that

$$\lim_{t \to +\infty} |\mathbf{U}(t) - \xi(t)|_H = 0. \tag{2.4}$$

By arguing as in [16, proof of Theorem 5], it is easy to check that

$$\xi_t(t) \in \operatorname{argmin} \Phi = \{0\} \qquad \text{a.e. on } (0, +\infty).$$

It ensures that $\xi_t \equiv \mathbf{0}$ and hence the vector function ξ is constant on $[0, +\infty)$. The conclusion is then an immediate consequence of (2.4).
(ii) Since $u \in L^\infty(0, +\infty : H^2(\Omega))$, there exists $\overline{u} \in H^2(\Omega)$ along with a sequence $(s_n) \subset (0, +\infty)$ tending to $+\infty$ such that $\lim_{n \to +\infty} u(s_n) = \overline{u}$ weakly in $H^2(\Omega)$, hence weakly in $H^1(\Omega)$. From (i) and the uniqueness of the limit, we derive that $u_\infty = \overline{u} \in H^2(\Omega)$. Since u_∞ is the unique limit point of the map $t \mapsto u(t)$ for the weak topology of $H^2(\Omega)$, we conclude that $\lim_{t \to +\infty} u(t) = u_\infty$ weakly in $H^2(\Omega)$.
(iii) Let us argue by contradiction and assume that the set $\partial\Phi(0) - \Delta u_\infty$ does not contain 0. It is then possible to strictly separate the convex compact set $\{0\}$ from the closed convex set $\partial\Phi(0) - \Delta u_\infty$. More precisely, there exist $p \in L^2(\Omega)$ and $m > 0$ such that

$$\forall \xi \in \partial\Phi(0) - \Delta u_\infty, \qquad \langle \xi, p \rangle > m. \tag{2.5}$$

Recall that the set $\{u_{tt}(t), \quad t \in I\}$ is bounded for the norm topology of $L^2(\Omega)$. Let $h \in L^2(\Omega)$ and let $(t_n) \subset I$ be a sequence tending to $+\infty$ such that $\lim_{n \to +\infty} u_{tt}(t_n) = h$ weakly in $L^2(\Omega)$. Since u is solution to (S), we have

$$-u_{tt}(t_n) + \Delta u(t_n) \in \partial\Phi(u_t(t_n)).$$

In view of (ii), the left-hand side of the above inclusion weakly converges to $-h + \Delta u_\infty$ in $L^2(\Omega)$. On the other hand, we have $\lim_{n \to +\infty} u_t(t_n) = 0$ strongly in $L^2(\Omega)$ and using the graph-closedness property of the operator $\partial\Phi$ in $s{-}L^2(\Omega) \times w{-}L^2(\Omega)$, we conclude that $-h + \Delta u_\infty \in \partial\Phi(0)$. In view of (2.5) we derive that $\langle h, p \rangle_{L^2} < -m$. This shows that the limit points of the map $t \mapsto \langle u_{tt}(t), p \rangle_{L^2}$ when $t \to +\infty$ are contained in the interval $]-\infty, -m[$. We deduce the existence of $t_* \geq 0$ such that, for almost every $t \geq t_*$, $\langle u_{tt}(t), p \rangle_{L^2} \leq -m$. By integrating this inequality, we immediately infer that $\lim_{t \to +\infty} \langle u_t(t), p \rangle_{L^2} = -\infty$, a contradiction with the fact that $u_t \in L^\infty(0, +\infty : L^2(\Omega))$. □

When $\partial\Phi(0) = \{0\}$ the stationary condition of Theorem 2.3 (iii) gives $\Delta u_\infty = 0$ and since $u_\infty \in H_0^1(\Omega)$, we conclude that $u_\infty = 0$. Suppose now that the function Φ is defined by $\Phi(v) = \int_\Omega j(v(x))\, dx$ for every $v \in L^2(\Omega)$. In this case, the set $\partial\Phi(0)$ equals $\{f \in L^2(\Omega), \quad f(x) \in \partial j(0) \text{ for a.e. } x \in \Omega\}$, so that Theorem 2.3 (iii) implies that $\Delta u_\infty(x) \in \partial j(0)$ for almost every $x \in \Omega$. Finally, in the case of the globally damped wave equation (1.5), we have $\partial\Phi(0) = \mu_r \mathbb{B}_{L^2}$ and the stationary condition becomes $|\Delta u_\infty|_{L^2} \leq \mu_r$.

3. Stabilization in finite time via some discretized problem. Numerical illustrations

Motivated by the numerical approach of solutions, we consider in this section some discretized version of (S). To fix the ideas, suppose that we deal with the following one-dimensional equation, modelizing the motion of a vibrating string under friction:

$$u_{tt} - u_{xx} + \mu \operatorname{sgn}(u_t) + g(u_t) \ni 0, \qquad (t,x) \in (0,+\infty) \times (0,1), \tag{3.1}$$

where $\mu > 0$, $\operatorname{sgn} : \mathbb{R} \to \mathcal{P}(\mathbb{R})$ is the set-valued sign function and $g : \mathbb{R} \to \mathbb{R}$ is a Lipschitz continuous function such that $r\,g(r) \geq 0$ for every $r \in \mathbb{R}$. The term $\mu \operatorname{sgn}(u_t)$ represents the Coulomb friction while $g(u_t)$ represents another type of friction such as the one due to the viscosity of a possible surrounding fluid. The reader is referred to [7, 20] for general features about the Coulomb model. By using a finite differencing scheme, the spatial discretization of (3.1) leads to

$$\ddot{u}_i - \frac{u_{i+1} - 2\,u_i + u_{i-1}}{h^2} + \mu \operatorname{sgn}(\dot{u}_i) + g(\dot{u}_i) \ni 0, \qquad t \in (0,+\infty), \quad i = 1,2,\ldots,n, \tag{3.2}$$

where $h = 1/(n+1)$ denotes the space step. The previous inclusion can be rewritten as a vectorial problem by setting $\mathbf{U}(t) := (u_1(t),\ldots,u_n(t))^T$. For that purpose, let us define the function $\operatorname{Sgn} : \mathbb{R}^n \to \mathcal{P}(\mathbb{R}^n)$ by $\operatorname{Sgn}(u_1,\ldots,u_n) := (\operatorname{sgn}(u_1),\ldots,\operatorname{sgn}(u_n))^T$ and the function $G : \mathbb{R}^n \to \mathbb{R}^n$ by $G(u_1,\ldots,u_n) := (g(u_1),\ldots,g(u_n))^T$. We also define the symmetric positive definite matrix $A \in \mathcal{M}_n(\mathbb{R})$ by

$$A := \frac{1}{h^2}\begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & 0 & -1 & 2 & -1 \\ 0 & \cdots & 0 & -1 & 2 \end{pmatrix}.$$

With these notations, inclusion (3.2) is equivalent to

$$\ddot{\mathbf{U}}(t) + A\,\mathbf{U}(t) + \mu \operatorname{Sgn}(\dot{\mathbf{U}}(t)) + G(\dot{\mathbf{U}}(t)) \ni \mathbf{0}, \qquad t \in (0,+\infty). \tag{3.3}$$

This system also arises in the study of the vibration of n particles of equal mass. In fact, it was by passing to the limit in the number of particles (in absence of any friction) how the wave equation was obtained in 1746 by Jean Le Rond d'Alembert. The stabilization in a finite time, in absence of viscous friction ($G = 0$) was proved by Bamberger and Cabannes [3]. It was shown by Díaz and Millot [14] that the presence of a viscous friction (with a suitable behaviour of G near 0) may originate a qualitative distinction among the orbits in the sense that the state of the system may reach an equilibrium state in a finite time or merely in an asymptotic way (as $t \to +\infty$), according to the initial data $\mathbf{U}(0) = \mathbf{U}_0$ and $\dot{\mathbf{U}}(0) = \dot{\mathbf{U}}_0$. In the recent work [10], the author studies the general case of a friction equal to $-\partial\Psi(\dot{\mathbf{U}}(t))$, for some convex function $\Psi : \mathbb{R}^n \to \mathbb{R}$. The same phenomenon of dichotomy as above is observed and it is shown that either the solution converges in a finite time or the speed of convergence is exponential. Let us finally mention that a fully discretized version of (S) has been studied by Baji and Cabot [2], thus giving rise to an inertial proximal algorithm.

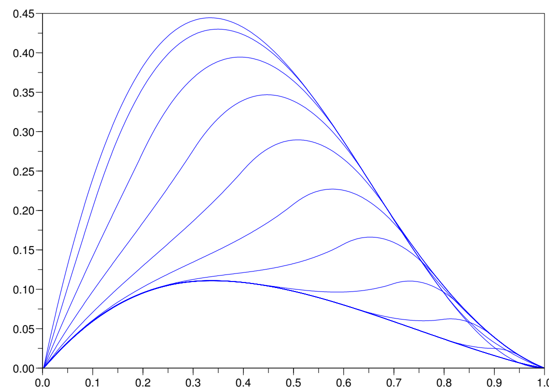

FIGURE 1. Vibrating string under Coulomb friction. Plotting of the solution $x \mapsto u(t,x)$ at successive instants: $t = 0,\, 0.1,\, 0.2,\, \ldots$

We end this paragraph with a few numerical experiments in the case of a pure dry friction (see equation (3.1) with $g = 0$). We use a finite differencing scheme, both in time and space. The plotting on Figure 1 corresponds to the initial conditions $u_0(x) = 3\,x\,(1-x)^2$ and $v_0(x) = 0$; the friction coefficient is taken equal to $\mu = 3$. We observe that the map $t \mapsto u(t)$ stabilizes after $t = 1$ toward a stationary solution satisfying $|u''_\infty|_{L^\infty} \le 3$.
Let us now turn to a two-dimensional example with $\Omega = (0,1) \times (0,1)$. We choose the initial conditions $u_0(x,y) = 9\,x\,y\,(1-x)^2\,(1-y)^2$ and $v_0(x,y) = 0$, and the friction coefficient equals $\mu = 2$. Figure 2 shows the evolution of the map $t \mapsto u(t)$ and it suggests the finite time convergence of $u(t)$ toward some stationary solution u_∞ satisfying $|\Delta u_\infty|_{L^\infty} \le 2$.

Remark 3.1. As pointed out in [14], the finite time stabilization can be also observed by using the finite element method.

4. Stabilization in a finite time under some interior-like conditions

In this section, we will assume that, for large values of t, the Laplacian $\Delta u(t)$ satisfies some interior condition with respect to the set $\partial\Phi(0)$. In a finite dimensional setting [1], this kind of condition implies the finite time stabilization of the dynamics. The extension of such a result to the damped wave equation leads us to the following theorem.

Theorem 4.1. *Let $\Phi : L^2(\Omega) \to \mathbb{R}$ be a continuous convex function and let u be the unique solution to (S) defined at Theorem 2.1. Assume that there exists $\varepsilon > 0$ and $t_0 \ge 0$ such that*

$$\Delta u(t) + \varepsilon\, \mathbb{B}_{L^2} \subset \partial\Phi(0), \quad \textit{for a.e. } t \ge t_0. \tag{4.1}$$

Then $u(t) = u_\infty$ for every $t \ge t_0 + |u_t(t_0)|_{L^2}/\varepsilon$.

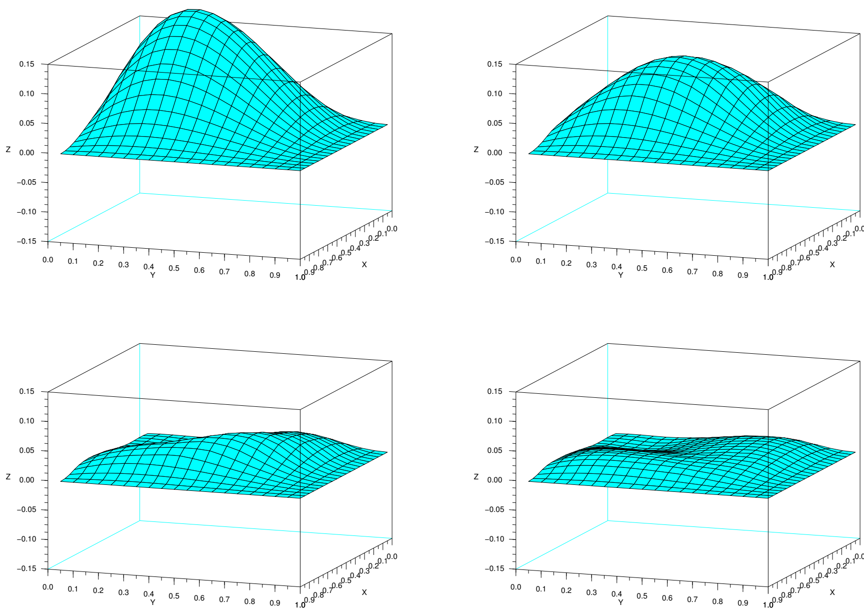

FIGURE 2. Vibrating membrane under Coulomb friction. Plotting of the map $(x, y) \mapsto u(t, x, y)$ at different instants: $t = 0$, $t = 0.25$, $t = 0.5$ and $t \to +\infty$.

Proof. For almost every $t \geq t_0$ and for every $v \in \mathbb{B}_{L^2}$, we have $\Delta u(t) + \varepsilon\, v \in \partial\Phi(0)$. Thus, for almost every $t \geq t_0$, we deduce

$$\Phi(u_t(t)) - \Phi(0) \geq \langle \Delta u(t) + \varepsilon\, v, u_t(t)\rangle_{L^2}, \quad \forall v \in \mathbb{B}_{L^2}.$$

Taking the supremum over $v \in \mathbb{B}_{L^2}$, we obtain for almost every $t \geq t_0$,

$$\Phi(u_t(t)) - \Phi(0) \geq \langle \Delta u(t), u_t(t)\rangle_{L^2} + \varepsilon\, |u_t(t)|_{L^2}. \tag{4.2}$$

On the other hand, the inequality (2.2) of energy decay can be rewritten as:

$$\frac{1}{2}\frac{d}{dt}|u_t(t)|^2_{L^2} - \langle \Delta u(t), u_t(t)\rangle_{L^2} + \Phi(u_t(t)) - \Phi(0) \leq 0 \qquad \text{a.e. on } (0, +\infty[. \tag{4.3}$$

By combining (4.2) and (4.3), we get

$$\frac{1}{2}\frac{d}{dt}|u_t(t)|^2_{L^2} + \varepsilon\, |u_t(t)|_{L^2} \leq 0. \tag{4.4}$$

By setting $h(t) := |u_t(t)|^2_{L^2}$, it is clear that relation (4.4) can be rewritten as the following differential inequality:

$$\dot{h}(t) + 2\,\varepsilon\, \sqrt{h(t)} \leq 0 \qquad \text{a.e. on } (t_0, +\infty[. \tag{4.5}$$

The solution of the differential equation $\dot{y} + 2\,\varepsilon\, \sqrt{y} = 0$ on $(t_0, +\infty[$ takes the zero value for $t = t_0 + \sqrt{y(t_0)}/\varepsilon$. In view of (4.5), a simple comparison argument then shows that there exists $t_1 \in [t_0, t_0 + \sqrt{h(t_0)}/\varepsilon]$ such that $h(t_1) = 0$. From (4.5), we deduce that $\dot{h}(t) \leq 0$ almost everywhere and hence $h(t) \leq h(t_1) = 0$, for every

$t \geq t_1$. We conclude that $|u_t(t)|_{L^2} = 0$ for every $t \in [t_1, +\infty[$, *i.e.* $u(t) = u_\infty$ for every $t \in [t_1, +\infty[$. □

We now derive two corollaries from the previous theorem. In the first one, we impose some interior-like condition on the limit u_∞. In the second one we will find suitable initial conditions ensuring that (4.1) is satisfied.

Corollary 4.2. *Let $\Phi : L^2(\Omega) \to \mathbb{R}$ be a continuous convex function and let u be the unique solution to (S) defined at Theorem 2.1. Assume that $\lim_{t\to+\infty} |u(t) - u_\infty|_{H^2} = 0$ for some $u_\infty \in H^2(\Omega)$. If $\Delta u_\infty \in \operatorname{int}_{L^2}(\partial\Phi(0))$, then there exists $t_1 \geq 0$ such that $u(t) = u_\infty$ for every $t \geq t_1$.*

Proof. The assumption $\Delta u_\infty \in \operatorname{int}_{L^2}(\partial\Phi(0))$ implies the existence of $\varepsilon > 0$ such that

$$\Delta u_\infty + 2\,\varepsilon\,\mathbb{B}_{L^2} \subset \partial\Phi(0).$$

On the other hand, since $\lim_{t\to+\infty} |u(t) - u_\infty|_{H^2} = 0$, there exists $t_0 \geq 0$ such that for every $t \geq t_0$, we have

$$\Delta u(t) \in \Delta u_\infty + \varepsilon\,\mathbb{B}_{L^2}.$$

Hence,

$$\Delta u(t) + \varepsilon\,\mathbb{B}_{L^2} \subset \Delta u_\infty + 2\,\varepsilon\,\mathbb{B}_{L^2} \subset \partial\Phi(0).$$

It suffices then to use Theorem 4.1. □

Let us now apply the previous corollary to the situation corresponding to $\Phi = \mu_r\,|\,.\,|_{L^2} + \frac{\mu_v}{2}\,|\,.\,|^2_{L^2}$ (see equation (1.5)). Recall that in this case we have $\partial\Phi(0) = \mu_r\,\mathbb{B}_{L^2}$. Under the hypotheses of Corollary 4.2, we deduce that

$$|\Delta u_\infty|_{L^2} < \mu_r \quad \Longrightarrow \quad u \text{ stabilizes in a finite time.}$$

Suppose now that the function Φ is defined by $\Phi(v) = \int_\Omega j(v(x))\,dx$ for every $v \in L^2(\Omega)$. In this case, the interior of the set $\partial\Phi(0) = \{f \in L^2(\Omega), \quad f(x) \in \partial j(0)$ for a.e. $x \in \Omega\}$ is empty, so that Corollary 4.2 cannot be applied.

Let us now state another consequence of Theorem 4.1, which is more specifically devoted to the globally damped wave equation (*cf.* inclusion (1.5)).

Corollary 4.3. *Given $\mu_r > 0$, $\mu_v \geq 0$, we define the function $\Phi : L^2(\Omega) \to \mathbb{R}$ by $\Phi = \mu_r\,|\,.\,|_{L^2} + \frac{\mu_v}{2}\,|\,.\,|^2_{L^2}$. Let u be the unique solution to (S) defined at Theorem 2.1. If the initial conditions (u_0, v_0) satisfy $|\Delta u_0|_{L^2} + |\nabla v_0|_{L^2} < \mu_r/C$, for some $C \geq 1$, then we have $u(t) = u_\infty$ for every*

$$t \geq \frac{|v_0|_{L^2}}{\mu_r - C\,(|\Delta u_0|_{L^2} + |\nabla v_0|_{L^2})}.$$

Proof. From [5, Theorem III.2], the following estimate holds true for almost every $t \geq 0$

$$|\Delta u(t)|_{L^2} \leq C\,(|\Delta u_0|_{L^2} + |\nabla v_0|_{L^2}),$$

for some $C \geq 1$. Recalling that $\partial\Phi(0) = \mu_r\,\mathbb{B}_{L^2}$, we deduce that condition (4.1) is satisfied with

$$\varepsilon = \mu_r - C\,(|\Delta u_0|_{L^2} + |\nabla v_0|_{L^2}) > 0 \quad \text{ and } t_0 = 0.$$

It suffices then to apply Theorem 4.1. □

Remark 4.4. The above mentioned results of finite time stabilization may have interesting applications in control theory. Indeed, let $T > 0$, $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $v_0 \in H_0^1(\Omega)$ be given. Consider the $H^2(\Omega)$-approximate controllability question stated in the following terms: given $\varepsilon > 0$ (arbitrarily small) find a *feedback type control* $f(t,x)$ such that the solution $u(t,x\,;f)$ of the linear wave equation

$$\begin{cases} u_{tt} - \Delta u = f & \text{in } (0,T)\times\Omega, \\ u = 0 & \text{on } (0,T)\times\partial\Omega, \\ u(0,x) = u_0(x) & \text{on } \Omega, \\ u_t(0,x) = v_0(x) & \text{on } \Omega, \end{cases}$$

satisfies

$$u_t(T,.\,;f) = 0 \text{ on } \Omega \quad \text{and} \quad |u(T,.\,;f)|_{H^2} \leq \varepsilon.$$

Then, if the finite time stabilization result for the globally damped wave equation (1.5) holds, the desired control function can be chosen as follows:

$$f(t,x) = -\mu_r \, \frac{u_t(t,x)}{|u_t(t,.)|_{L^2}},$$

with $\mu_r = \frac{\varepsilon}{C}$ and $C = C(\Omega)$ given as the constant such that $|h|_{H^2} \leq C\,|\Delta h|_{L^2}$, for any $h \in H_0^1(\Omega) \cap H^2(\Omega)$. In contrast with other problems dealing with the exact controllability for the wave equation with globally distributed controls (see, e.g. [19] and [13]), this result does not hold for any arbitrary $T > 0$ but only for some $T > 0$ large enough depending on the initial data $u_0 \in H_0^1(\Omega) \cap H^2(\Omega)$ and $v_0 \in H_0^1(\Omega)$.

5. On the dichotomy phenomenon under some expansion condition

5.1. **Illustration of the dichotomy phenomenon.** Given μ_c, $\mu_v > 0$, let us consider the following damped wave equation

$$u_{tt} - \Delta u + \mu_c \operatorname{sgn}(u_t) + \mu_v\, u_t \ni 0, \tag{5.1}$$

where the friction term is decomposed as the sum of a dry component and a viscous one. Let us assume that $\mu_v \geq 2\sqrt{\lambda_1}$, with $\lambda_1 > 0$ the first eigenvalue of the Dirichlet-Laplacian operator $-\Delta$. Then we can find some solutions to (5.1) which exponentially converge toward their limit and also some solutions which stabilize in a finite time. We construct the first type of solutions in the form

$$u(t,x) = \xi(x) + a(t)\, e_1(x),$$

where $e_1 \in H_0^1(\Omega)$ is an eigenfunction of $-\Delta$ associated to λ_1 such that $e_1 > 0$ in Ω, the function $\xi \in H_0^1(\Omega)$ is the solution to $\Delta\xi = \mu_c$ in Ω and $a(t)$ is a solution of the ODE

$$\ddot{a} + \mu_v \dot{a} + \lambda_1\, a = 0, \tag{5.2}$$

such that $\dot{a}(t) > 0$ for every $t \geq 0$ (which is possible since $\mu_v \geq 2\sqrt{\lambda_1}$). Then, we get a solution u which tends toward $u_\infty = \xi$ and the convergence rate is exponential. By the contrary, if we choose $b(t)$ as a solution of (5.2) such that $\dot{b}(t) > 0$ for all $t \in [0,1)$, $\dot{b}(1) = 0$ and $b(1) = K > 0$ with $K \leq \frac{\mu_c}{\lambda_1 |e_1|_{L^\infty}}$ and take $a(t) = b(t)$ if $t \leq 1$ and $a(t) = K$ for $t \geq 1$ we get a solution which attains the stationary state $u_\infty(x) = \xi(x) + K e_1(x)$ after $t = 1$.

5.2. **Assumption (AE) and preliminary results.** Inspired by the previous paragraph, we assume from now on that the solution u to (S) admits the following asymptotic expansion when $t \to +\infty$

$$u(t,x) = u_\infty(x) + a(t)\, w(x) + R(t,x), \tag{AE}$$

where the functions a, u_∞, w and R satisfy the following set of hypotheses:

$$(\mathcal{H}) \quad \begin{cases} \bullet \text{ The map } a : [0,+\infty) \to [0,+\infty) \text{ is differentiable, nonincreasing and} \\ \qquad\qquad \lim_{t\to+\infty} a(t) = \lim_{t\to+\infty} \dot a(t) = 0. \\ \bullet \text{ The map } u_\infty \text{ satisfies } u_\infty \in H_0^1(\Omega) \cap H^2(\Omega). \\ \bullet \text{ The map } w \text{ satisfies } w \in H_0^1(\Omega) \setminus \{0\}. \\ \bullet \text{ The map } R \text{ is such that } R \in W^{1,\infty}(0,+\infty : H_0^1(\Omega)) \text{ and} \\ \qquad |R(t)|_{H^1} = o(a(t)) \quad \text{and} \quad |R_t(t)|_{L^2} = o(\dot a(t)) \text{ when } t \to +\infty. \end{cases}$$

The terminology (AE) stands for "Asymptotic Expansion". Let us justify the assumption (AE) in the case of the following linear damped wave equation with the forcing term $h \in L^2(\Omega)$

$$u_{tt} - \Delta u + \mu\, u_t = h, \qquad \mu > 0.$$

We assume that $\mu > 2\sqrt{\lambda_1}$ (overdamped case), where $\lambda_1 > 0$ is the first eigenvalue of the Dirichlet-Laplacian operator $-\Delta$. Let $e_1 \in H_0^1(\Omega)$ be an eigenfunction of $-\Delta$ associated to λ_1 and define the function $\xi \in H_0^1(\Omega)$ as the solution to $-\Delta\xi = h$ in Ω. By using the Fourier decomposition of solutions on the basis of the eigenfunctions associated to the Laplacian operator, one can check that

$$u(t,x) = \xi(x) + A\, e^{(-\mu+\sqrt{\mu^2-4\lambda_1})\frac{t}{2}}\, e_1(x) + R(t,x),$$

where $A \in \mathbb{R}$ and the function R satisfies

$$|R(t)|_{H^1} = o\left(e^{(-\mu+\sqrt{\mu^2-4\lambda_1})\frac{t}{2}}\right) \text{ and } \quad |R_t(t)|_{L^2} = o\left(e^{(-\mu+\sqrt{\mu^2-4\lambda_1})\frac{t}{2}}\right).$$

Therefore assertion (AE) holds true with $u_\infty(x) = \xi(x)$, $a(t) = e^{(-\mu+\sqrt{\mu^2-4\lambda_1})\frac{t}{2}}$ and $w(x) = A\, e_1(x)$ (provided that $A \neq 0$).
Coming back to the general case, let us now study the topological structure of the set $\mathcal{D} = \{t \in (0,+\infty), \quad |u_t(t)|_{L^2} = 0\}$.

Proposition 5.1. *Let $\Phi : L^2(\Omega) \to \mathbb{R}$ be a continuous convex function and let u be the unique solution to (S) defined at Theorem 2.1. Then, either the set $\mathcal{D}$ equals the interval $[t_0,+\infty[$ for some $t_0 \geq 0$ or the set $\mathcal{D}$ is discrete and countable (hence of zero measure).*

Proof. Assume that $\mathcal{D}$ is not equal to any interval $[t_0,+\infty[$ with $t_0 \geq 0$. Consider any $t_* > 0$ satisfying $|u_t(t_*)|_{L^2} = 0$ (if such an element does not exist, the conclusion is trivial) and let us prove that it is an isolated point of $\mathcal{D}$. Let us first remark that we necessarily have $\Delta u(t_*) \notin \partial\Phi(0)$. Indeed, if $\Delta u(t_*) \in \partial\Phi(0)$, then the constant function equal to $u(t_*)$ on $[t_*,+\infty[$ is solution to (S), and from the uniqueness property we derive that $u(t) = u(t_*)$ for every $t \geq t_*$, a contradiction.
Since $\Delta u(t_*) \notin \partial\Phi(0)$, it is possible to strictly separate the convex compact set $\{0\}$ from the closed convex set $\partial\Phi(0) - \Delta u(t_*)$. More precisely, there exist $p \in L^2(\Omega)$ and $m > 0$ such that:

$$\forall \xi \in \partial\Phi(0) - \Delta u(t_*), \qquad \langle \xi, p\rangle > m. \tag{5.3}$$

From Theorem 2.1 (i), the set $\{u_{tt}(t), \quad t \in I\}$ is bounded for the norm topology of $L^2(\Omega)$. Let $h \in L^2(\Omega)$ and let $(t_n) \subset I$ be a sequence tending to t_* such that $\lim_{n\to+\infty} u_{tt}(t_n) = h$ weakly in $L^2(\Omega)$. Since u is solution to (S), we have

$$-u_{tt}(t_n) + \Delta u(t_n) \in \partial\Phi(u_t(t_n)).$$

It is immediate to check that $\lim_{t\to t_*} \Delta u(t) = \Delta u(t_*)$ weakly in $L^2(\Omega)$. Hence the left-hand side of the above inclusion weakly converges to $-h+\Delta u(t_*)$ in $L^2(\Omega)$. On the other hand, we have $\lim_{n\to+\infty} u_t(t_n) = u_t(t_*) = 0$ strongly in $L^2(\Omega)$ and using the graph-closedness property of the operator $\partial\Phi$ in $s - L^2(\Omega) \times w - L^2(\Omega)$, we conclude that $-h + \Delta u(t_*) \in \partial\Phi(0)$. In view of (5.3) we derive that $\langle h, p\rangle_{L^2} < -m$. This shows that the limit points of the map $t \mapsto \langle u_{tt}(t), p\rangle_{L^2}$ when $t \to t_*$ are contained in the interval $]-\infty, -m[$. We deduce the existence of $\varepsilon > 0$ such that, for almost every $t \in]t_* - \varepsilon, t_* + \varepsilon[$, $\langle u_{tt}(t), p\rangle_{L^2} \leq -m$. Let us integrate this inequality on $[t_*, t]$ to obtain:

$$\forall t \in]t_* - \varepsilon, t_* + \varepsilon[, \quad |\langle u_t(t), p\rangle_{L^2}| \geq m\,|t - t_*|.$$

Therefore, we have $|u_t(t)|_{L^2} \neq 0$ for every $t \in]t_* - \varepsilon, t_* + \varepsilon[$ and hence $\mathcal{D} \cap]t_* - \varepsilon, t_* + \varepsilon[= \{t_*\}$, *i.e.* t_* is isolated in $\mathcal{D}$. Since this is true for every $t_* \in \mathcal{D}$, the set $\mathcal{D}$ is discrete. On the other hand, the set $\mathcal{D}$ is clearly closed in view of the continuity of the map $t \mapsto |u_t(t)|_{L^2}$. We infer that every bounded subset of $\mathcal{D}$ is finite. We conclude that the set $\mathcal{D}$ is countable as a countable union of finite sets. □

By differentiating expression (AE) with respect to time, we obtain $u_t(t) = \dot{a}(t)\,w + R_t(t)$. Since $|R_t(t)|_{L^2} = o(\dot{a}(t))$, it is immediate that for t large enough, $|u_t(t)|_{L^2} = 0$ if and only if $\dot{a}(t) = 0$. If the solution u does not converge in a finite time, we infer from Proposition 5.1 that the set $\mathcal{D} = \{t \in (0, +\infty), \quad \dot{a}(t) = 0\}$ is discrete and countable. In this case, we have

$$\lim_{t\to+\infty,\, t\notin\mathcal{D}} \frac{u_t(t)}{\dot{a}(t)} = w \qquad \text{strongly in } L^2(\Omega). \tag{5.4}$$

We now establish that the function w must be normal to the set $\partial\Phi(0)$ at Δu_∞. Let us recall that, for a convex subset $C \subset L^2(\Omega)$ and $u \in C$, the normal cone of C at u is defined by $N_C(u) = \{\xi \in L^2(\Omega), \quad \langle \xi, v - u\rangle_{L^2} \leq 0 \text{ for all } v \in C\}$.

Proposition 5.2. *Let $\Phi : L^2(\Omega) \to \mathbb{R}$ be a continuous convex function and let u be the unique solution to (S) defined at Theorem 2.1. Assume that assertion (AE) holds with the functions a, u_∞, w and R satisfying hypotheses $(\mathcal{H})$. If the solution u does not converge in a finite time, then we have $-w \in N_{\partial\Phi(0)}(\Delta u_\infty)$.*

Proof. First recall that, since the solution u does not converge in a finite time, the set $\mathcal{D} = \{t \in (0, +\infty), \quad |u_t(t)|_{L^2} = 0\}$ is discrete and countable (see Proposition 5.1). Let us argue by contradiction and assume that $-w \notin N_{\partial\Phi(0)}(\Delta u_\infty)$. This implies the existence of $\xi \in \partial\Phi(0)$ such that the quantity $m := \langle \xi - \Delta u_\infty, -w\rangle_{L^2}$ is positive. From Theorem 2.1 (i), the set $\{u_{tt}(t), \quad t \in I\}$ is bounded for the norm topology of $L^2(\Omega)$. Let $h \in L^2(\Omega)$ and let $(t_n) \subset I \setminus \mathcal{D}$ be a subsequence tending to $+\infty$ such that $\lim_{n\to+\infty} u_{tt}(t_n) = h$ weakly in $L^2(\Omega)$. Since $-u_{tt}(t_n) + \Delta u(t_n) \in \partial\Phi(u_t(t_n))$ and $\xi \in \partial\Phi(0)$, we deduce from the monotonicity of $\partial\Phi$ that $\langle -u_{tt}(t_n) + \Delta u(t_n) - \xi\,,\, u_t(t_n)\rangle_{L^2} \geq 0$. Recalling that $\dot{a}(t) < 0$ for every $t \in (0, +\infty) \setminus \mathcal{D}$, we derive that

$$\left\langle -u_{tt}(t_n) + \Delta u(t_n) - \xi\,,\, \frac{u_t(t_n)}{\dot{a}(t_n)} \right\rangle_{L^2} \leq 0.$$

Since $\lim_{t\to+\infty} \Delta u(t) = \Delta u_\infty$ weakly in $L^2(\Omega)$, the first term of the above bracket weakly converges in $L^2(\Omega)$ toward $-h+\Delta u_\infty-\xi$. In view of (5.4), the right member of the same bracket strongly converges in $L^2(\Omega)$ toward w. Hence, we obtain at the limit when $n \to +\infty$

$$\langle -h + \Delta u_\infty - \xi, w\rangle_{L^2} \le 0,$$

or equivalently $\langle h, w\rangle_{L^2} \ge m > 0$. This shows that the limit points of

$$\{\langle u_{tt}(t), w\rangle_{L^2}, \quad t \in I \setminus \mathcal{D}\} \qquad \text{when } t \to +\infty$$

are contained in the interval $[m, +\infty[$. Since the set $\mathcal{D}$ is negligible, we deduce the existence of $t_* \ge 0$ such that, for almost every $t \ge t_*$, $\langle u_{tt}(t), w\rangle_{L^2} \ge m/2$. By integrating this inequality, we immediately infer that $\lim_{t\to+\infty}\langle u_t(t), w\rangle_{L^2} = +\infty$, a contradiction with the fact that $u_t \in L^\infty(0, +\infty : L^2(\Omega))$. □

5.3. **Convergence rate estimates.** The next result shows that under assumption (AE), either the solutions to (S) converge in a finite time or the convergence rate is exponential. This result is an extension of [10, Theorem 5.2], which has been established in a finite dimensional setting. Given a subset $A \subset L^2(\Omega)$, we denote by $d(., A)$ the distance function to the set A: $d(x, A) = \inf_{y\in A} |x - y|_{L^2}$ for every $x \in L^2(\Omega)$. Given another subset $B \subset L^2(\Omega)$, we define the excess $e(A, B)$ of A over B by: $e(A, B) = \sup_{x\in A} d(x, B)$.

Theorem 5.3. *Let $\Phi : L^2(\Omega) \to \mathbb{R}$ be a continuous convex function and suppose that there exist $\eta > 0$ and $\alpha \ge 0$ such that*

$$|v|_{L^2} \le \eta \quad \Longrightarrow \quad e\Big(\partial\Phi(v), \partial\Phi(0)\Big) \le \alpha\, |v|_{L^2}. \tag{5.5}$$

Let u be the unique solution to (S) defined at Theorem 2.1. Assume that assertion (AE) holds with the functions a, u_∞, w and R satisfying hypotheses $(\mathcal{H})$. Then, one of the following cases holds:
(i) There exists $t_0 \ge 0$ such that $u(t) = u_\infty$ for every $t \ge t_0$.
(ii) There exist $t_1 \ge 0$, A, $B > 0$, and γ, $\delta > 0$ such that for every $t \ge t_1$,

$$|u(t) - u_\infty|_{L^2} \ge A\, e^{-\gamma t} \quad \text{and} \quad \int_t^{+\infty} |u(s) - u_\infty|_{L^2}\, ds \le B\, e^{-\delta t}. \tag{5.6}$$

Denoting by λ_1 the first eigenvalue of the operator $-\Delta$, any positive exponent γ (resp. δ) such that $\gamma > \alpha$ (resp. $\delta < \lambda_1/\alpha$) satisfies the previous estimate. If moreover $\alpha < \sqrt{\lambda_1}$, then case (i) necessarily occurs.

Proof. Let us assume that case (i) does not hold, *i.e.* the solution u does not converge toward u_∞ in a finite time. For every $t \in I$, we have: $-u_{tt}(t) + \Delta u(t) \in \partial\Phi(u_t(t))$. Let us define $\xi(t)$ as the unique element of $\partial\Phi(0)$ such that

$$d\Big(-u_{tt}(t) + \Delta u(t), \partial\Phi(0)\Big) = |\xi(t) + u_{tt}(t) - \Delta u(t)|_{L^2}.$$

Let us write that

$$\begin{aligned}\langle u_{tt}(t), w\rangle_{L^2} &= \langle \xi(t) + u_{tt}(t) - \Delta u(t), w\rangle_{L^2} + \langle \xi(t) - \Delta u_\infty, -w\rangle_{L^2}\\ &\quad + \langle \Delta u(t) - \Delta u_\infty, w\rangle_{L^2},\end{aligned} \tag{5.7}$$

and let us evaluate each term of the right member. From the definition of $\xi(t)$ we have for every $t \in I$:

$$|\xi(t) + u_{tt}(t) - \Delta u(t)|_{L^2} \le \sup_{v\in\partial\Phi(u_t(t))} d(v, \partial\Phi(0)) = e\Big(\partial\Phi(u_t(t)), \partial\Phi(0)\Big).$$

Since $\lim_{t\to+\infty} |u_t(t)|_{L^2} = 0$, there exists $t_0 \geq 0$ such that $|u_t(t)|_{L^2} \leq \eta$ for every $t \geq t_0$. Hence we deduce from assumption (5.5) and the previous inequality that

$$\forall t \in [t_0, +\infty[\cap I, \quad |\xi(t) + u_{tt}(t) - \Delta u(t)|_{L^2} \leq \alpha\, |u_t(t)|_{L^2}. \tag{5.8}$$

In view of Proposition 5.2, we have $-w \in N_{\partial\Phi(0)}(\Delta u_\infty)$ and since $\xi(t) \in \partial\Phi(0)$, we infer

$$\langle \xi(t) - \Delta u_\infty, -w\rangle_{L^2} \leq 0. \tag{5.9}$$

Let us evaluate the term $\langle \Delta u(t) - \Delta u_\infty, w\rangle_{L^2}$ by using the assumption (AE)

$$\begin{aligned}\langle \Delta u(t) - \Delta u_\infty, w\rangle_{L^2} &= -\langle \nabla u(t) - \nabla u_\infty, \nabla w\rangle_{L^2}\\ &= -|\nabla w|_{L^2}^2\, a(t) - \langle \nabla R(t), \nabla w\rangle_{L^2}\\ &\leq -\lambda_1\, |w|_{L^2}^2\, a(t) - \langle \nabla R(t), \nabla w\rangle_{L^2}\, . \end{aligned}\tag{5.10}$$

The last inequality is a consequence of the Poincaré inequality $|\nabla v|_{L^2}^2 \geq \lambda_1\, |v|_{L^2}^2$ for every $v \in H_0^1(\Omega)$. By assumption, we have $|\nabla R(t)|_{L^2} = o(a(t))$ when $t \to +\infty$ and therefore inequality (5.10) can be rewritten as:

$$\langle \Delta u(t) - \Delta u_\infty, w\rangle_{L^2} \leq -\lambda_1\, |w|_{L^2}^2\, a(t) + o(a(t)). \tag{5.11}$$

In view of (5.7), we deduce from (5.8), (5.9) and (5.11) that

$$\langle u_{tt}(t), w\rangle_{L^2} \leq \alpha\, |w|_{L^2}\, |u_t(t)|_{L^2} - \lambda_1\, |w|_{L^2}^2\, a(t) + o(a(t)).$$

Since the differentiation of expression (AE) gives

$$u_t(t) = \dot a(t)\, w + R_t(t) \quad \text{ with } \quad |R_t(t)|_{L^2} = o(\dot a(t)), \tag{5.12}$$

the above inequality yields

$$\langle u_{tt}(t), w\rangle_{L^2} \leq -\alpha\, |w|_{L^2}^2\, \dot a(t) + o(\dot a(t)) - \lambda_1\, |w|_{L^2}^2\, a(t) + o(a(t)). \tag{5.13}$$

Observing that $\int_0^{+\infty} a(s)\, ds < +\infty$, let us integrate inequality (5.13) on $[t, +\infty[$ to find:

$$-\langle u_t(t), w\rangle_{L^2} \leq \alpha\, |w|_{L^2}^2\, a(t) + o(a(t)) - \lambda_1\, |w|_{L^2}^2 \left(\int_t^{+\infty} a(s)\, ds\right) + o\left(\int_t^{+\infty} a(s)\, ds\right).$$

From equality (5.12), we infer that

$$-|w|_{L^2}^2\, \dot a(t) + o(\dot a(t)) + \lambda_1\, |w|_{L^2}^2 \left(\int_t^{+\infty} a(s)\, ds\right) + o\left(\int_t^{+\infty} a(s)\, ds\right) \leq \alpha\, |w|_{L^2}^2\, a(t) + o(a(t)).$$

Since $a(t) \geq 0$ and $\dot a(t) \leq 0$ for every $t \geq 0$, the previous inequality entails

$$-\dot a(t) + o(\dot a(t)) \leq \alpha\, a(t) + o(a(t)) \tag{5.14}$$

$$\lambda_1 \left(\int_t^{+\infty} a(s)\, ds\right) + o\left(\int_t^{+\infty} a(s)\, ds\right) \leq \alpha\, a(t) + o(a(t)). \tag{5.15}$$

Consider some positive exponents γ and δ such that $\gamma > \alpha$ and $\delta < \lambda_1/\alpha$. In view of (5.14)-(5.15), there exists $t_1 \geq t_0$ such that for every $t \geq t_1$

$$-\dot a(t) \leq \gamma\, a(t) \quad \text{ and } \quad \delta \left(\int_t^{+\infty} a(s)\, ds\right) \leq a(t).$$

An elementary integration of the previous inequalities on $[t_1, t]$ yields respectively:

$$a(t) \geq a(t_1)\, e^{-\gamma(t-t_1)} \quad \text{ and } \quad \left(\int_t^{+\infty} a(s)\, ds\right) \leq \left(\int_{t_1}^{+\infty} a(s)\, ds\right) e^{-\delta(t-t_1)}.$$

Inequalities (5.6) immediately follow from the equivalence $|u(t)-u_\infty|_{L^2} \sim |w|_{L^2}\, a(t)$ when $t \to +\infty$.
Let us now prove the last assertion of the theorem. Let us argue by contradiction and assume that case (ii) holds. An immediate integration of the first inequality of (5.6) on $[t, +\infty[$ shows that

$$\forall t \geq t_1, \qquad \int_t^{+\infty} |u(s) - u_\infty|\, ds \geq \frac{A}{\gamma}\; e^{-\gamma\, t}.$$

In view of the second inequality of (5.6), the exponents must satisfy the following relation: $\delta \leq \gamma$. Since this is true for every $\gamma > \alpha$ and $\delta < \lambda_1/\alpha$, we conclude that $\lambda_1 \leq \alpha^2$, which contradicts the assumption. □

In this theorem, condition (5.5) plays a central role. We are now going to show that this condition is satisfied in at least two interesting situations.

Corollary 5.4. *Let $j : \mathbb{R} \to \mathbb{R}$ be a convex function and assume that there exists $\alpha \geq 0$ such that*

$$\forall r \in \mathbb{R}, \quad e\Big(\partial j(r), \partial j(0)\Big) \leq \alpha\, |r|. \tag{5.16}$$

Suppose that $j(v) \in L^1(\Omega)$ for every $v \in L^2(\Omega)$, and define the function $\Phi : L^2(\Omega) \to \mathbb{R}$ by $\Phi(v) = \int_\Omega j(v(x))\, dx$. Let u be the unique solution to (S) defined at Theorem 2.1. If assertion (AE) holds, then we have the same conclusions as in Theorem 5.3.

Proof. Given $v \in L^2(\Omega)$, let us compute the excess $e\,(\partial\Phi(v), \partial\Phi(0))$. For every $g \in \partial\Phi(v)$ and for almost every $x \in \Omega$, let us define $\tilde{g}(x)$ as the unique element of the set $\partial j(0)$ such that $|g(x) - \tilde{g}(x)| = d(g(x), \partial j(0))$. Since $\tilde{g}(x) \in \partial j(0)$ for almost every $x \in \Omega$, we have $\tilde{g} \in \partial\Phi(0)$. Hence we deduce

$$d(g, \partial\Phi(0)) \leq |g - \tilde{g}|_{L^2} = \left(\int_\Omega d(g(x), \partial j(0))^2\, dx\right)^{1/2}. \tag{5.17}$$

Since $g \in \partial\Phi(v)$, we have $g(x) \in \partial j(v(x))$ for almost every $x \in \Omega$. It ensues that $d(g(x), \partial j(0)) \leq e\Big(\partial j(v(x)), \partial j(0)\Big)$ and by taking into account assumption (5.16), we infer that $d(g(x), \partial j(0)) \leq \alpha\, |v(x)|$ for almost every $x \in \Omega$. In view of (5.17), we deduce that $d(g, \partial\Phi(0)) \leq \alpha\, |v|_{L^2}$. Since this is true for every $g \in \partial\Phi(v)$, we conclude that $e\,(\partial\Phi(v), \partial\Phi(0)) \leq \alpha\, |v|_{L^2}$. Hence condition (5.5) is satisfied and we can now apply Theorem 5.3. □

Recall that the support function $\sigma_C : L^2(\Omega) \to \mathbb{R} \cup \{+\infty\}$ of a set $C \subset L^2(\Omega)$ is defined by $\sigma_C(u) = \sup_{v \in C} \langle u, v \rangle_{L^2}$ for every $u \in L^2(\Omega)$. Assume that the set $C \subset L^2(\Omega)$ is closed, convex and bounded for the norm topology of $L^2(\Omega)$. We let the reader check that the function σ_C is then positively homogeneous, convex and finite-valued (hence continuous). Observe that the support function $\sigma_{\mathbb{B}_{L^2}}$ coincides with the norm $|\,.\,|_{L^2}$. Given a convex function $\Psi : L^2(\Omega) \to \mathbb{R}$ of class $\mathcal{C}^1$, we define the function $\Phi : L^2(\Omega) \to \mathbb{R}$ by $\Phi = \sigma_C + \Psi$. In this particular framework, condition (5.5) takes a simplified form, as shown by the following corollary.

Corollary 5.5. *Let $C \subset L^2(\Omega)$ be a closed convex subset which is bounded for the strong topology of $L^2(\Omega)$. Consider a convex function $\Psi : L^2(\Omega) \to \mathbb{R}$ of class $\mathcal{C}^1$ such that there exist $\eta > 0$ and $\alpha \geq 0$ satisfying*

$$|v|_{L^2} \leq \eta \quad \Longrightarrow \quad |\nabla\Psi(v)|_{L^2} \leq \alpha\, |v|_{L^2}. \tag{5.18}$$

Defining the function $\Phi : L^2(\Omega) \to \mathbb{R}$ by $\Phi = \sigma_C + \Psi$, let u be the unique solution to (S) given by Theorem 2.1. If assertion (AE) holds, then we have the same conclusions as in Theorem 5.3.

Proof. Let us compute the excess $e\left(\partial\Phi(v), \partial\Phi(0)\right)$. It is immediate to check that $\partial\sigma_C(0) = C$ and $\partial\sigma_C(v) \subset C$ for every $v \in L^2(\Omega)$. Hence,

$$e\Big(\partial\Phi(v), \partial\Phi(0)\Big) \leq e\Big(\nabla\Psi(v) + C, C\Big) = \sup_{w\in C} d(\nabla\Psi(v) + w, C)$$
$$\leq \sup_{w\in C} |\nabla\Psi(v) + w - w|_{L^2} = |\nabla\Psi(v)|_{L^2} \leq \alpha\, |v|_{L^2}.$$

Hence condition (5.5) is satisfied and we can now apply Theorem 5.3. □

Assume that the term Ψ corresponds to a viscous friction, *i.e.* $\Psi = \frac{\mu_v}{2}\,|\,.\,|^2_{L^2}$ for some $\mu_v \geq 0$. Under assumption (AE), Corollary 5.5 shows that, if $\mu_v \in [0, \sqrt{\lambda_1}[$ then the solution u stabilizes in a finite time. This means that the dynamics stops after a finite time when the viscous component of the friction is small enough.

6. On the dichotomy phenomenon under some condition of normal velocity

In this section, we study the asymptotic properties of (S) under the following assumption:

$$\text{(NV)} \qquad u_t(t) \in N_{\partial\Phi(0)}(\Delta u_\infty) \quad \text{for } t \text{ large enough.}$$

Assertion (NV) says that the velocity $u_t(t)$ is normal to the set $\partial\Phi(0)$ when $t \to +\infty$. Let us first remark that, if $\partial\Phi(0) = \{0\}$, we have $\Delta u_\infty = 0$ and hence $N_{\partial\Phi(0)}(\Delta u_\infty) = N_{\{0\}}(0) = L^2(\Omega)$. It ensues that (NV) is automatically satisfied in this case.

6.1. **Interpretation of assumption (NV).** Let $j : \mathbb{R} \to \mathbb{R}$ be a convex function and let us assume that, for every $v \in L^2(\Omega)$, we have $j(v) \in L^1(\Omega)$. The function $\Phi : L^2(\Omega) \to \mathbb{R}$ is defined by $\Phi(v) = \int_\Omega j(v(x))\,dx$, for every $v \in L^2(\Omega)$. Let us set $\beta := \partial j$ and assume that $0 \in \operatorname{int}(\beta(0))$. Suppose that

$$\lim_{t\to+\infty} \Delta u(t,x) = \Delta u_\infty(x) \qquad \text{for almost every } x \in \Omega.$$

Let us fix $\overline{x} \in \Omega$ such that the previous relation is satisfied. Let us write the inclusion (1.3) with $x = \overline{x}$

$$u_{tt}\,(t,\overline{x}) - \Delta u\,(t,\overline{x}) + \beta\,(u_t(t,\overline{x})) \ni 0.$$

By arguing as in [14, Lemma 2], we deduce the existence of $t_{\overline{x}} \geq 0$ such that

$$\forall t \geq t_{\overline{x}}, \quad u_t(t,\overline{x}) \in N_{\beta(0)}(\Delta u_\infty(\overline{x})).$$

Without loss of generality, we can assume that $t_{\overline{x}}$ is the smallest time such that the previous inclusion holds true. Suppose moreover that $\overline{x} \mapsto t_{\overline{x}}$ is essentially bounded on Ω and let $T := \text{ess-}\sup_{\overline{x}\in\Omega} t_{\overline{x}} < +\infty$. We then have

$$\text{for every } t \geq T, \text{ for almost every } \overline{x} \in \Omega, \qquad u_t(t,\overline{x}) \in N_{\beta(0)}(\Delta u_\infty(\overline{x})).$$

Recalling that $\partial\Phi(0) = \{f \in L^2(\Omega), \quad f(x) \in \beta(0) \text{ for almost every } x \in \Omega\}$ and using a classical result relative to the subdifferential of convex integral functionals (see for example [6, Proposition 2.16]), we deduce that $u_t(t) \in N_{\partial\Phi(0)}(\Delta u_\infty)$ for every $t \geq T$, which is exactly (NV).

6.2. **Minorization by an exponential decay rate.** Let us define the energy-like function F by

$$F(t) = \frac{1}{2}|u_t(t)|^2_{L^2} + \frac{1}{2}|\nabla u(t) - \nabla u_\infty|^2_{L^2}. \tag{6.1}$$

The function F is related to the energy function E by the following formula

$$F(t) = E(t) + \langle u(t), \Delta u_\infty\rangle_{L^2} + \frac{1}{2}|\nabla u_\infty|^2_{L^2}.$$

The map F is non increasing; indeed, from (2.2) we deduce that

$$\forall t \in I, \quad \dot{F}(t) \leq -(\Phi(u_t(t)) - \Phi(0)) + \langle u_t(t), \Delta u_\infty\rangle_{L^2}.$$

In view of Theorem 2.3 (iii), we have $\Delta u_\infty \in \partial\Phi(0)$. It ensues that $\langle u_t(t), \Delta u_\infty\rangle_{L^2} \leq \Phi(u_t(t)) - \Phi(0)$ and the announced result follows. The lyapounov function F will play an essential role throughout this section. The next result asserts that under assertion (NV), either the solutions to (S) converge in a finite time or the convergence rate is minorized by some negative exponential. Given two subsets A, $B \subset L^2(\Omega)$, we recall that the excess of A over B is defined by $e(A, B) = \sup_{v\in A} d(v, B)$.

Theorem 6.1. *Let* $\Phi : L^2(\Omega) \to \mathbb{R}$ *be a continuous convex function such that* $\operatorname{argmin}\Phi = \{0\}$. *Suppose that there exist* $\eta > 0$ *and* $\alpha \geq 0$ *such that*

$$|v|_{L^2} \leq \eta \quad \Longrightarrow \quad e\Big(\partial\Phi(v), \partial\Phi(0)\Big) \leq \alpha\,|v|_{L^2}. \tag{6.2}$$

Let u be the unique solution to (S) defined at Theorem 2.1 and let u_∞ denote its limit in $H^1(\Omega)$ as $t \to +\infty$. If assertion (NV) is satisfied, then one of the following cases holds:
(i) There exists $t_0 \geq 0$ such that $u(t) = u_\infty$ for every $t \geq t_0$.
(ii) There exist $t_1 \geq 0$ and $A > 0$ such that

$$\forall t \geq t_1, \qquad \int_t^{+\infty} |u_t(s)|^2_{L^2}\, ds \geq A\, e^{-2\,\alpha\, t}.$$

If moreover $\alpha = 0$ then case (i) necessarily holds, i.e. the solution u is stabilized in a finite time.

Proof. Let us first remark that, if $|u_t|_{L^2} \notin L^2(0, +\infty : \mathbb{R})$ then $\int_t^{+\infty} |u_t(s)|^2_{L^2}\, ds = +\infty$ for every $t \geq 0$, so that item (ii) is trivially satisfied. Hence we can assume without loss of generality that $|u_t|_{L^2} \in L^2(0, +\infty : \mathbb{R})$. Consider the function F defined by (6.1); we have for every $t \in I$

$$\dot{F}(t) = \langle u_t(t), u_{tt}(t) - \Delta u(t) + \Delta u_\infty\rangle_{L^2}. \tag{6.3}$$

For every $t \in I$, we have $-u_{tt}(t) + \Delta u(t) \in \partial\Phi(u_t(t))$. Let us define $\xi(t)$ as the unique element of $\partial\Phi(0)$ such that

$$d\Big(- u_{tt}(t) + \Delta u(t), \partial\Phi(0)\Big) = |\xi(t) + u_{tt}(t) - \Delta u(t)|_{L^2}.$$

It is then clear that, for every $t \in I$

$$|\xi(t) + u_{tt}(t) - \Delta u(t)|_{L^2} \leq \sup_{y\in\partial\Phi(u_t(t))} d(y, \partial\Phi(0)) = e\Big(\partial\Phi(u_t(t)), \partial\Phi(0)\Big).$$

Since $\lim_{t\to+\infty} |u_t(t)|_{L^2} = 0$, there exists $t_0 \geq 0$ such that $|u_t(t)|_{L^2} \leq \eta$ for every $t \geq t_0$. Hence we deduce from assumption (6.2) and the previous inequality that

$$\forall t \in [t_0, +\infty[\cap I, \quad |\xi(t) + u_{tt}(t) - \Delta u(t)|_{L^2} \leq \alpha\,|u_t(t)|_{L^2}. \tag{6.4}$$

From assertion (NV), there exists $t_1 \geq t_0$ such that for every $t \geq t_1$, we have $u_t(t) \in N_{\partial\Phi(0)}(\Delta u_\infty)$. Since $\xi(t) \in \partial\Phi(0)$, we infer that

$$\langle u_t(t), \xi(t) - \Delta u_\infty \rangle_{L^2} \leq 0. \tag{6.5}$$

In view of (6.3), (6.4) and (6.5), we conclude that

$$\forall t \in [t_1, +\infty[\, \cap I, \quad \dot{F}(t) \geq -\alpha\, |u_t(t)|^2_{L^2}.$$

Recalling that $|u_t|_{L^2} \in L^2(0, +\infty : \mathbb{R})$, we can integrate the previous inequality on $[t, +\infty[$. Since $\lim_{t\to+\infty} |u_t(t)|_{L^2} = 0$ and $\lim_{t\to+\infty} |\nabla u(t) - \nabla u_\infty|_{L^2} = 0$, we obtain:

$$\forall t \geq t_1, \quad \frac{1}{2}|u_t(t)|^2_{L^2} + \frac{1}{2}|\nabla u(t) - \nabla u_\infty|^2_{L^2} \leq \alpha \int_t^{+\infty} |u_t(s)|^2_{L^2}\, ds,$$

and hence $|u_t(t)|^2_{L^2} \leq 2\,\alpha \int_t^{+\infty} |u_t(s)|^2_{L^2}\, ds$, for every $t \geq t_1$. An immediate integration on $[t_1, t]$ shows that

$$\forall t \geq t_1, \quad \int_t^{+\infty} |u_t(s)|^2_{L^2}\, ds \geq \left(\int_{t_1}^{+\infty} |u_t(s)|^2_{L^2}\, ds\right)\, e^{-2\,\alpha(t-t_1)}.$$

If $\int_{t_1}^{+\infty} |u_t(s)|^2_{L^2}\, ds = 0$, then clearly $|u_t(t)|_{L^2} = 0$ for every $t \geq t_1$. If $\int_{t_1}^{+\infty} |u_t(s)|^2_{L^2}\, ds > 0$, the expected formula is obtained by setting $A := \left(\int_{t_1}^{+\infty} |u_t(s)|^2_{L^2}\, ds\right)\, e^{2\,\alpha\, t_1}$.
Now assume that $\alpha = 0$. Let us argue by contradiction and assume that case (ii) holds, *i.e.* there exists $A > 0$ such that $\int_t^{+\infty} |u_t(s)|^2_{L^2}\, ds \geq A$ for t large enough. This clearly contradicts the fact that $\lim_{t\to+\infty} \int_t^{+\infty} |u_t(s)|^2_{L^2}\, ds = 0$ and we conclude that $u(t) = u_\infty$ for t large enough. □

It is immediate to apply Theorem 6.1 to the situations corresponding respectively to equations (1.3) and (1.5).

6.3. **Majorization by an exponential decay rate.** We are going to prove that under suitable conditions the convergence rate of $|u(t) - u_\infty|_{H^1}$ toward 0 is majorized by some negative exponential. The key assumption of the next theorem is the existence of a symmetric positive operator L such that[1]

$$e\Big(\partial\Phi(v), \partial[\Phi'(0;.)](v) + L\, v\Big) = O(|v|^2_{L^2}) \qquad \text{when } |v|_{L^2} \to 0. \tag{6.6}$$

Suppose that the function Φ equals $\Phi := \sigma_C + \Psi$ for some convex set $C \subset L^2(\Omega)$ and some convex function $\Psi : L^2(\Omega) \to \mathbb{R}$ of class $\mathcal{C}^3$ such that $\nabla\Psi(0) = 0$. In this case, equality (6.6) is satisfied with $L := \nabla^2\Psi(0)$. This can be easily obtained from a second-order Taylor expansion of the function $\nabla\Psi$ in the neighborhood of 0.

Theorem 6.2. *Let $L : L^2(\Omega) \to L^2(\Omega)$ be a symmetric operator satisfying*

$$\forall v \in L^2(\Omega), \qquad m\,|v|^2_{L^2} \leq \langle Lv, v\rangle_{L^2} \leq M\,|v|^2_{L^2}, \tag{6.7}$$

for some m, $M > 0$. Assume that $\Phi : L^2(\Omega) \to \mathbb{R}$ is a continuous convex function satisfying (6.6) and such that $\operatorname{argmin}\Phi = \{0\}$. *Let u be the unique solution to (S) defined at Theorem 2.1 and let u_∞ denote its limit in $H^1(\Omega)$ as $t \to +\infty$. If assertion (NV) holds, then there exist $C, \gamma > 0$ and $t_0 \geq 0$ such that*

$$\forall t \geq t_0, \qquad |u_t(t)|^2_{L^2} + |\nabla u(t) - \nabla u_\infty|^2_{L^2} \leq C\, e^{-\gamma t}.$$

[1] Let us recall that the directional derivative $\Phi'(u;.)$ of Φ at $u \in L^2(\Omega)$ is defined by $\Phi'(u;h) = \lim_{t\to0^+} (\Phi(u + t\,h) - \Phi(u))\,/t$ for every $h \in L^2(\Omega)$.

Denoting by λ_1 the first eigenvalue of the Dirichlet-Laplacian operator $-\Delta$, any positive exponent γ such that $\gamma < \frac{m\,(\sqrt{M^2+4\,\lambda_1}-M)}{m+\sqrt{M^2+4\,\lambda_1}-M}$ satisfies the previous estimate.

Proof. For every $t \in I$, we have: $-u_{tt}(t) + \Delta u(t) \in \partial\Phi(u_t(t))$. Let us define $\xi(t)$ as the unique element of the set $\partial[\Phi'(0;.)](u_t(t)) + L\,u_t(t)$ such that

$$d\Big(-u_{tt}(t) + \Delta u(t), \partial[\Phi'(0;.)](u_t(t)) + L\,u_t(t)\Big) = |\xi(t) + u_{tt}(t) - \Delta u(t)|_{L^2}.$$

In view of assumption (6.6) we have, for every $t \in I$

(6.8)

$$|\xi(t) + u_{tt}(t) - \Delta u(t)|_{L^2} \le e\Big(\partial\Phi(u_t(t)), \partial[\Phi'(0;.)](u_t(t)) + L\,u_t(t)\Big) = O(|u_t(t)|^2_{L^2}).$$

Let us define the auxiliary function G by:

$$G(t) := \langle u_t(t), u(t) - u_\infty\rangle_{L^2} + \frac{1}{2}\langle L(u(t) - u_\infty), u(t) - u_\infty\rangle_{L^2}.$$

An elementary computation shows that for every $t \in I$

$$\begin{aligned}\dot G(t) &= |u_t(t)|^2_{L^2} + \langle u_{tt}(t) + L\,u_t(t), u(t) - u_\infty\rangle_{L^2}\\ &= |u_t(t)|^2_{L^2} + \langle u_{tt}(t) + \xi(t) - \Delta u(t), u(t) - u_\infty\rangle_{L^2}\\ &\qquad - \langle \xi(t) - L\,u_t(t) - \Delta u_\infty, u(t) - u_\infty\rangle_{L^2}\\ &\qquad + \langle \Delta u(t) - \Delta u_\infty, u(t) - u_\infty\rangle_{L^2}.\end{aligned}$$

Since $\langle \Delta u(t) - \Delta u_\infty, u(t) - u_\infty\rangle_{L^2} = -|\nabla u(t) - \nabla u_\infty|^2_{L^2}$, the previous inequality can be rewritten as

$$\begin{aligned}\dot G(t) + 2\,F(t) = 2\,|u_t(t)|^2_{L^2} &+ \langle u_{tt}(t) + \xi(t) - \Delta u(t), u(t) - u_\infty\rangle_{L^2}\\ &- \langle \xi(t) - L\,u_t(t) - \Delta u_\infty, u(t) - u_\infty\rangle_{L^2},\end{aligned} \tag{6.9}$$

where the function F is defined by (6.1). Let us fix some $\eta \in]0, m[$. Since $\lim_{t\to+\infty} u(t) = u_\infty$ strongly in $L^2(\Omega)$, we obtain in view of inequality (6.8) that there exists $t_1 \ge 0$ such that, for every $t \in [t_1, +\infty[\cap I$

$$|\langle u_{tt}(t) + \xi(t) - \Delta u(t), u(t) - u_\infty\rangle_{L^2}| \le \eta\,|u_t(t)|^2_{L^2}. \tag{6.10}$$

From the definition of $\xi(t)$, we have for every $t \in I$

$$\xi(t) - L\,u_t(t) \in \partial[\Phi'(0;.)](u_t(t)) \subset \partial\Phi(0). \tag{6.11}$$

From assertion (NV), there exists $t_2 \ge t_1$ such that $u_t(t) \in N_{\partial\Phi(0)}(\Delta u_\infty)$ for every $t \ge t_2$. An immediate integration on $[t, +\infty[$ shows that $u(t) - u_\infty \in -N_{\partial\Phi(0)}(\Delta u_\infty)$ for every $t \ge t_2$. In view of (6.11), this implies that, for every $t \in [t_2, +\infty[\cap I$,

$$\langle \xi(t) - L\,u_t(t) - \Delta u_\infty, u(t) - u_\infty\rangle_{L^2} \ge 0. \tag{6.12}$$

By combining (6.9), (6.10) and (6.12), we find

$$\forall t \in [t_2, +\infty[\cap I, \quad \dot G(t) + 2\,F(t) \le (2 + \eta)\,|u_t(t)|^2_{L^2}. \tag{6.13}$$

Let us now differentiate the function F; we find for every $t \in I$

$$\begin{aligned}\dot F(t) &= \langle u_{tt}(t), u_t(t)\rangle_{L^2} - \langle \Delta u(t) - \Delta u_\infty, u_t(t)\rangle_{L^2}\\ &= \langle u_{tt}(t) - \Delta u(t) + \xi(t), u_t(t)\rangle_{L^2} + \langle \Delta u_\infty - \xi(t) + L\,u_t(t), u_t(t)\rangle_{L^2}\\ &\quad - \langle L\,u_t(t), u_t(t)\rangle_{L^2}.\end{aligned} \tag{6.14}$$

In view of (6.8), we have

$$|\langle u_{tt}(t) - \Delta u(t) + \xi(t), u_t(t)\rangle_{L^2}| = O\left(|u_t(t)|^3_{L^2}\right) \qquad \text{when } t \to +\infty. \tag{6.15}$$

From the definition of $\xi(t)$, we have $u_t(t) \in N_{\partial\Phi(0)}(\xi(t) - L\,u_t(t))$ for every $t \in I$. Since $\Delta u_\infty \in \partial\Phi(0)$, we infer that

$$\langle \Delta u_\infty - \xi(t) + L\,u_t(t), u_t(t)\rangle_{L^2} \leq 0. \tag{6.16}$$

From (6.14), (6.15) and (6.16), we conclude that

$$\dot F(t) \leq -\langle L\,u_t(t), u_t(t)\rangle_{L^2} + O\left(|u_t(t)|^3_{L^2}\right).$$

Since $\langle L\,u_t(t), u_t(t)\rangle_{L^2} \geq m\,|u_t(t)|^2_{L^2}$ and $\lim_{t\to+\infty} |u_t(t)|_{L^2} = 0$, there exists $t_3 \geq t_2$ such that for every $t \in [t_3, +\infty[\cap I$,

$$\dot F(t) \leq -(m-\eta)\,|u_t(t)|^2_{L^2}. \tag{6.17}$$

Let us multiply (6.13) by $A_\eta := (m-\eta)/(2+\eta)$ and add to (6.17); we obtain

$$\dot F(t) + A_\eta\,\dot G(t) + 2\,A_\eta\,F(t) \leq 0. \tag{6.18}$$

Our purpose now is to deduce from (6.18) a differential equation involving a single function. This is made possible owing to the following relations between the functions G and F

$$\forall t \geq 0, \qquad G(t) \geq -F(t)/m \quad \text{and} \quad F(t) \geq B\,G(t), \tag{6.19}$$

where B is a positive real that we are going to determine. We classically have, for all $\theta > 0$,

$$|\langle u_t(t), u(t) - u_\infty\rangle_{L^2}| \leq \frac{|u_t(t)|^2_{L^2}}{2\theta} + \frac{\theta}{2}|u(t) - u_\infty|^2_{L^2}.$$

In view of assumption (6.7), we infer that

$$-\frac{|u_t(t)|^2_{L^2}}{2\theta} + \frac{-\theta+m}{2}|u(t)-u_\infty|^2_{L^2} \leq G(t) \leq \frac{|u_t(t)|^2_{L^2}}{2\theta} + \frac{\theta+M}{2}|u(t)-u_\infty|^2_{L^2}. \tag{6.20}$$

Taking $\theta = m$ in the first inequality of (6.20), we obtain $G(t) \geq -|u_t(t)|^2_{L^2}/(2\,m) \geq -F(t)/m$, which is the first inequality of (6.19). On the other hand, since $\lambda_1 > 0$ is the first eigenvalue of the operator $-\Delta$, we have $|\nabla v|^2_{L^2} \geq \lambda_1\,|v|^2_{L^2}$ for every $v \in H^1_0(\Omega)$ and hence

$$F(t) \geq \frac{1}{2}\,|u_t(t)|^2_{L^2} + \frac{\lambda_1}{2}\,|u(t) - u_\infty|^2_{L^2}. \tag{6.21}$$

Setting $\tau(\theta) := \min\{\theta, \lambda_1/(\theta+M)\}$, we deduce from the second inequality of (6.20) and (6.21) that

$$F(t) \geq \tau(\theta)\,G(t). \tag{6.22}$$

We let the reader check that the function $\tau : (0,+\infty) \to \mathbb{R}$ achieves its maximum at $B := (\sqrt{M^2 + 4\,\lambda_1} - M)/2$ and that $\tau(B) = B$. Taking $\theta = B$ in inequality (6.22), we obtain the second inequality of (6.19). We deduce from (6.18) and the second inequality of (6.19) that

$$\dot F(t) + A_\eta\,\dot G(t) + 2\,A_\eta\,B\,G(t) \leq 0. \tag{6.23}$$

Let us multiply (6.18) by B and (6.23) by A_η; adding the two inequalities and setting $H(t) := F(t) + A_\eta\,G(t)$, this yields:

$$\forall t \in [t_3, +\infty[\cap I, \qquad (A_\eta + B)\,\dot H(t) + 2\,A_\eta\,B\,H(t) \leq 0.$$

An elementary integration on $[t_3, t]$ gives:

$$\forall t \in [t_3, +\infty[, \qquad H(t) \leq H(t_3)\,e^{-\frac{2\,A_\eta\,B}{A_\eta+B}(t-t_3)}. \tag{6.24}$$

From the first inequality of (6.19), we have $H(t) \geq F(t) - A_\eta/m\, F(t)$. Since $A_\eta \leq m/2$, we finally obtain $H(t) \geq \frac{1}{2} F(t) = \frac{1}{4} |u_t(t)|^2_{L^2} + \frac{1}{4} |\nabla u(t) - \nabla u_\infty|^2_{L^2}$. Setting $C := 4\, H(t_3)\, e^{\frac{2 A_\eta B}{A_\eta + B} t_3}$, we deduce in view of (6.24) that

$$\forall t \in [t_3, +\infty[, \qquad |u_t(t)|^2_{L^2} + |\nabla u(t) - \nabla u_\infty|^2_{L^2} \leq C\, e^{-\frac{2 A_\eta B}{A_\eta + B} t}.$$

Since

$$\lim_{\eta \to 0} \frac{2\, A_\eta\, B}{A_\eta + B} = \frac{m\, B}{\frac{m}{2} + B} = \frac{m\, (\sqrt{M^2 + 4\, \lambda_1} - M)}{m + \sqrt{M^2 + 4\, \lambda_1} - M},$$

any positive exponent γ such that $\gamma < \frac{m\, (\sqrt{M^2 + 4\lambda_1} - M)}{m + \sqrt{M^2 + 4\lambda_1} - M}$ satisfies the estimate of the statement. □

Remark 6.3. Assume that $\partial\Phi(0) = \{0\}$. We have already noticed at the beginning of this section that assertion (NV) automatically holds in this case. On the other hand, we have $\Phi'(0; .) \equiv 0$ and hence condition (6.6) can be rewritten as

$$e\Big(\partial\Phi(v), L\, v\Big) = O(|v|^2_{L^2}) \qquad \text{when } |v|_{L^2} \to 0.$$

Finally, since $\Delta u_\infty \in \partial\Phi(0) = \{0\}$, we have $\Delta u_\infty = 0$ and hence the vector $u_\infty \in H^1_0(\Omega)$ satisfies $u_\infty = 0$. Therefore, the conclusion of Theorem 6.2 becomes: $|u_t(t)|^2_{L^2} + |\nabla u(t)|^2_{L^2} \leq C\, e^{-\gamma t}$ for t large enough. This remark applies in particular to the case where the map Φ is defined by $\Phi(v) = \frac{1}{2} \langle Lv, v\rangle_{L^2}$. In this case, the dynamical system (S) reduces to the linearly damped wave equation $u_{tt}(t) - \Delta u(t) + L u_t(t) = 0$.

Let us notice that the key condition (6.6) of Theorem 6.2 entails condition (6.2) of Theorem 6.1. This remark gives rise to the following corollary.

Corollary 6.4. *Under the assumptions of Theorem 6.2, one of the following cases holds:*
(i) There exists $t_0 \geq 0$ such that $u(t) = u_\infty$ for every $t \geq t_0$.
(ii) There exist $t_1 \geq 0$ and C, D, γ, $\delta > 0$ such that for every $t \geq t_1$,
(6.25)

$$|u_t(t)|^2_{L^2} + |\nabla u(t) - \nabla u_\infty|^2_{L^2} \leq C\, e^{-\gamma\, t} \quad \textit{and} \quad \int_t^{+\infty} |u_t(s)|^2_{L^2}\, ds \geq D\, e^{-\delta\, t}.$$

Any positive exponent γ (resp. δ) such that $\gamma < \frac{m\, (\sqrt{M^2 + 4\lambda_1} - M)}{m + \sqrt{M^2 + 4\lambda_1} - M}$ (resp. $\delta > 2\, M$) satisfies the previous estimate.

Proof. The first inequality of (6.25) results immediately from Theorem 6.2. Since $\partial[\Phi'(0; .)](v) \subset \partial\Phi(0)$ for every $v \in L^2(\Omega)$, we have

$$e(\partial\Phi(v), \partial\Phi(0)) \leq e\Big(\partial\Phi(v), \partial[\Phi'(0; .)](v)\Big). \tag{6.26}$$

For every $w \in \partial\Phi(v)$, we have $d\Big(w, \partial[\Phi'(0; .)](v)\Big) \leq |Lv|_{L^2} + d\Big(w, \partial[\Phi'(0; .)](v) + Lv\Big)$ and taking the supremum when $w \in \partial\Phi(v)$, we infer that

$$e\Big(\partial\Phi(v), \partial[\Phi'(0; .)](v)\Big) \leq |Lv|_{L^2} + e\Big(\partial\Phi(v), \partial[\Phi'(0; .)](v) + Lv\Big). \tag{6.27}$$

In view of condition (6.6), inequalities (6.26), (6.27) and the fact that $|Lv|_{L^2} \leq M\, |v|_{L^2}$ for every $v \in L^2(\Omega)$, we conclude that

$$e(\partial\Phi(v), \partial\Phi(0)) \leq M\, |v|_{L^2} + O(|v|^2_{L^2}).$$

Hence condition (6.2) of Theorem 6.1 is satisfied with $\alpha := M + \varepsilon/2$ for any $\varepsilon > 0$. We deduce that, either the solution u to (S) converges in a finite time or $\int_t^{+\infty} |u_t(s)|_{L^2}^2\, ds \geq D\, e^{-(2\,M+\varepsilon)\,t}$ for some positive D and t large enough. □

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