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A variant of the Smale's model for flocks formation

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Abstract. Motivated by a recent lecture of S. Smale, we propose here a variant of the model considered by F. Cucker and S. Smale for the formation of uniform flocks allowing to prove that the formation of an uniform structure take place in a finite time (and not merely when t converges to infinity).

Una variación sobre un modelo de Smale para la fomación de bandadas

Resumen. Motivado por una reciente conferencia de S. Smale, proponemos aqui una variación del modelo considerado por F. Cucker y S. Smale que permite mostrar la formación de bandadas con un movimiento sincronizado en un tiempo finito (y no meramente cuando t converge a infinito).

1 Introduction

This note is motivated by a part of a lecture by S. Smale ([7]) in which he reported some results on his previous work [2], in collaboration with F. Cucker, on the formation of folks by birds (or fish), according a model which proposed by Vicsek et al. [8]. The motivation comes from the observation that under some conditions (for example on the structure of its dynamics) the state of the flock converges to one in which all birds fly with the same velocity. The goal of the above mentioned papers was to provide some justification of this observation.

We propose here a slight variation of the simplified model for the case of two birds

$$P(\alpha,\beta)\begin{cases}\dot{x}=v,\\ \dot{v}=-\beta\frac{v^{(2-\frac{1}{\beta})}}{(1+x)^{\alpha}},\\ x(0)=x_0,\\ v(0)=v_0,\end{cases} \tag{1}$$

where $\alpha,\beta\in(0,1]$.

The case $\beta=1$ and $\alpha>1$ was considered in [2, Section 4] (see also the general results, including the case $\alpha\in(0,1)$ in [2, Section 3]). They prove that the uniform folk is formed asymptotically when $t\to+\infty$.

Our main goal is to prove the finite (continuous) time convergence to a flock (we recall that the convergence in discrete time was obtained in [2, Section 5] for a discrete version of a general model, including (1) with $\beta=1$ as a special case).

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Theorem 1 *Assume $\alpha \in (0,1)$, $v_0 > 0$ and $x_0 > 0$. Then there exists $T_f = T_f(x_0, v_0) \in (0, +\infty]$ such that, when $t \to T_f$, $v(t) \to 0$ and*

$$x(t) \to x_\infty := \Big((1-\alpha)v_0^{1/\beta} + (1+x_0)^{1-\alpha}\Big)^{\frac{1}{1-\alpha}} - 1. \tag{2}$$

Moreover, $x(t)$ is strictly increasing on $[0, T_f)$ and we have that $T_f = +\infty$ if $\beta = 1$ and $T_f < +\infty$ if $\beta \in (0,1)$. In the case,$\beta \in (0,1)$, the solution satisfies that

$$x(t) = x_\infty \quad \textit{and} \quad v(t) = 0 \quad \textit{for any } t \in [T_f, +\infty).$$

IDEA OF THE PROOF. We start by recalling that the existence of a local solution come from the continuity of the right hand side of both equations. The uniqueness of solution can be obtained (even for the case $\alpha \in (0,1)$ and $\beta \in (0,1)$) thanks to the monotonicity of the application $v \to v^{(2-\frac{1}{\beta})}$ and by using that the application $x \to \frac{1}{(1+x)^\alpha}$ is locally Lipschitz continuous (recall that $v_0 > 0$ and $x_0 > 0$). We point out that, if we define

$$V = v^{1/\beta}$$

then the system can be reformulated as

$$\widetilde{P}(\alpha,\beta) \begin{cases} \dot{x} = V^\beta, \\ \dot{V} = -\dfrac{V^\beta}{(1+x)^\alpha}, \\ x(0) = x_0, \\ V(0) = V_0 := v_0^{1/\beta}. \end{cases} \tag{3}$$

As a matter of fact, we shall prove our conclusion by using this equivalent formulation. Since the system $\widetilde{P}(\alpha,\beta)$ is autonomous we can eliminate the time on each orbit and write, equivalently the orbit as solution of the scalar equation

$$\frac{dx}{dV}(x) = -(1+x)^\alpha. \tag{4}$$

Thus, we have that

$$V(t) - V_0 = \frac{1}{(1-\alpha)}[(1+x_0)^{1-\alpha} - (1+x(t))^{1-\alpha}]. \tag{5}$$

From here and the fact that $v_0 > 0$ and $x_0 > 0$ we deduce that the pair $(x(t), V(t))$ remains in a bounded set of $\{(x,V) \in \mathbb{R}^2, \quad x \geq 0, V \geq 0\}$ when $t \in [0, +\infty)$ and since $\dot{x} \geq 0$ and $\dot{V} \leq 0$ we get that, necessarily, $x(t) \to x_\infty$, for some $x_\infty \geq x_0$, and $v(t) \to 0$ when $t \to +\infty$. Moreover, from (5) we get that x_∞ is given by (2). Assume now that $\beta \in (0,1)$. Let us prove that the limits are attaint in a finite time. By introducing

$$\widehat{x}(t) = 1 + x(t)$$

and using the first equation of (3) we get that

$$\frac{d\widehat{x}}{dt}(t) = \frac{1}{(1-\alpha)}(\widehat{x}_\infty^{1-\alpha} - \widehat{x}(t)^{1-\alpha})^\beta,$$

where $\widehat{x}_\infty = 1 + x_\infty$. In particular, for any $t \geq 0$

$$\int_{\widehat{x}_0}^{\widehat{x}(t)} \frac{ds}{(\widehat{x}_\infty^{1-\alpha} - s^{1-\alpha})^\beta} = t,$$

where $\widehat{x}_0 = x_0 + 1$. Thus, $x(t) \to x_\infty$ when $t \to T_f$, for a finite time $T_f > 0$, if and only if the following improper integral is convergent

$$\int_{\widehat{x}_0}^{\widehat{x}_\infty} \frac{ds}{(\widehat{x}_\infty^{1-\alpha} - s^{1-\alpha})^\beta} < +\infty.$$

But this can be easily checked (make the change of variables $u = \widehat{x}_\infty^{1-\alpha} - s^{1-\alpha}$) thanks to the assumption $\beta \in (0, 1)$. $\square$

Remark 1 *A very rich situation arises when $\alpha \in (0, 1)$ and $x_0 = 0$ (even for $\beta = 1$), which seems to be unadvertised in [2]. As we deduce from the auxiliary equation*

$$\frac{d\widehat{x}}{dV}(x) = -\widehat{x}(V)^\alpha$$

now for $\widehat{x}(V_0) = 0$ we get a continuum of solutions (which may remain satisfying $\widehat{x}(V) = 0$ on the interval $[V_s, V_0]$ for an arbitrary $V_s \in (0, V_0]$ and $\widehat{x}(V) > 0$ and $\widehat{x}(V) \to \widehat{x}_\infty$ when $V \to 0$). The above convergence $\widehat{x}(V) \to \widehat{x}_\infty$ when $V \to 0$ takes place in a infinite (respectively a finite time) if $\beta = 1$ (respectively $\beta \in (0, 1)$). A curious fact, considered in [1], is that this continuum of solutions still exists for equations of the form

$$\frac{d\widehat{x}}{dV}(x) = -\widehat{x}(V)^\alpha + f(\widehat{x}(V))$$

when

$$|f(s)| \leq cs^{\frac{\alpha}{1-\alpha}}$$

for some $c > 0$ small enough.

Remark 2 *As mentioned in* [2]*, model $P(\alpha, \beta)$ (even for $\beta = 1$) is only a simplification of a vectorial model of the type*

$$\mathbf{P}(L) \begin{cases} \dot{\mathbf{x}} = \mathbf{v}, \\ \dot{\mathbf{v}} = -L(\mathbf{x})(\mathbf{v}), \\ \mathbf{x}(0) = \mathbf{x}_0, \\ \mathbf{v}(0) = \mathbf{v}_0, \end{cases} \tag{6}$$

where $\mathbf{x}, \mathbf{v} \in \mathbb{R}^{3k}$ and $L(\mathbf{x})$ denotes a matrix, called as the Laplacian of the "adjacency matrix" A

$$A(\mathbf{x}) = (a_{ij}) = (\eta(\|\mathbf{x}_i - \mathbf{x}j\|^2)$$

where $\mathbf{x}_i \in \mathbb{R}^3$, $\mathbf{x} =(\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_k) \in \mathbb{R}^{3k}$ and

$$\eta(y) = \frac{K}{(\sigma^2 + y)^\alpha}.$$

In fact, in the case of two birds following parallel lines (the scalar case) we must understand that $x = x_1 - x_2$ and $v = v_1 - v_2$. By using some energy arguments (as in [1]*), it seems possible to extend Theorem* 1 *to some suitable systems of the type $\mathbf{P}(\alpha, \beta)$ when the matrix $L(\mathbf{x})$ depends also on $\mathbf{v}$ is a suitable way.*

Remark 3 *Discrete models corresponding to a time discretization of $\mathbf{P}(L)$ are relevant in learning and language evolution (*[3, 6, 7]*). It is not difficult to extend the conclusions of Theorem* 1 *to the discrete model associated to problem $P(\alpha, \beta)$ (see, some related results in* [4] *and its references).*

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