ON A CLIMATE MODEL WITH A DYNAMIC NONLINEAR DIFFUSIVE BOUNDARY CONDITION

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ABSTRACT. This work studies the sensitivity of a global climate model with deep ocean effect to the variations of a Solar parameter Q. The model incorporates a dynamic and diffusive boundary condition. We study the number of stationary solutions according to the positive parameter Q.

1. **Introduction.** We are concerned with a two dimensional climate model (latitude – depth) which models the coupling mean surface temperature with ocean temperature. Watts and Morantine [23] proposed a model consisting of an equation of parabolic type in a global ocean with a dynamic and diffusive nonlinear boundary condition. This boundary condition is obtained through a global energy balance for the atmosphere surface temperature. In this section, we recall some mathematical properties of the so-called climate energy balance models.

Climate energy balance models (EBMs) were introduced by M. Budyko and W. Sellers in 1969, independently. EBMs are diagnostic models. They are trying to understand the evolution of the climate for relatively long time scales. One of the main characteristics is its high sensitivity with respect to variation of parameters.

Several aspects of the mathematical treatment of different versions of climate EBMs have been studied by many authors, among them, Díaz [8], Díaz - Tello [12], Ghil - Childress [17], Hetzer [18], North [20], Drazin- Griffel [16] Díaz - Hetzer - Tello [10].

A first classification of the EBMs can be given according to the dimension of the space domain. From the mathematical point of view, two dimensional EBMs (latitude – longitude) have an spatial domain given by a Riemannian manifold without boundary $\mathcal{M}$ simulating the Earth surface, as follows

$$\begin{cases} c(x)u_t - div(k(x)|\nabla u|^{p-2}\nabla u) + R_e(x,u) \in R_a(x,u) & (0,T) \times \mathcal{M}, \\ u(x,0) = u_0(x) & \mathcal{M}, \end{cases} \tag{1}$$

where u represents the mean surface temperature, R_e and R_a the emitted and absorbed energy, respectively. R_a depends on the planetary coalbedo β (the fraction

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of the incoming radiation flux which is absorbed by the surface). The coalbedo function is possibly discontinuous on u (as in the model proposed by Budyko). One dimensional models (early proposed) assume uniform temperature over each latitude. We denote x the sine of the latitude and introducing the spherical coordinates, we obtain

$$\begin{cases} c(x)u_t - (k(x)(1-x^2)^{\frac{p}{2}}|u_x|^{p-2}u_x)_x + R_e(x,u) \in R_a(x,u) & (0,T) \times (-1,1), \\ (1-x^2)^{\frac{p}{2}}|u_x|^{p-2}u_x = 0 & x \in \{-1,1\}, \\ u(x,0) = u_0(x) & (-1,1). \end{cases}$$
$$(2)$$

In these models (EBMs), the effect of the oceans is only considered in an implicit and empirical way in the spatial dependence of the coefficients. However, rapid climate changes during Glacial-Holocene transition could have been the results of variations in the rate of deep water formation (see Berger et al [4]). In this paper we study a model which involves the coupling atmosphere – deep ocean, that is, the equation (2) with some interaction terms and a parabolic equation for the temperature in the inner ocean.

In Watts - Morantine [23], some numerical experiences are shown for a one dimensional version of the problem that we study in this paper. However, the existence, uniqueness, multiplicity and regularity of solutions are not studied, in spite of the boundary condition does not appear often in the literature. Some examples can be found in Bejenaru, Díaz, Vrabie [3] and the references therein.

The goal of this work is to study the stationary solutions of the model including the coupling surface / deep ocean, with the diffusion at the top boundary proposed by Stone and coalbedo feedback of Budyko and Sellers type. The number of stationary solutions of (1) was studied in [9] and [19] with multivalued right hand side (Budyko coalbedo) and in [18] with Lipschitz coalbedo (Sellers type). The proof of the existence of an unbounded connected S-shaped set $\{(Q,u)\}$, where u is the solution, can be found in [2].

2. **The model.** In this work we study a model which is based on one proposed by Watts - Morantine [23] including the coupling surface / deep ocean. The model represents the evolution of the temperature in a global ocean with depth H. The spatial variables x and z represent the sine of the latitude and the depth, respectively. The space domain, Ω, is the rectangle $(-1,1) \times (-H,0)$. We write the boundary of Ω as $\Gamma_H \cup \Gamma_0 \cup \Gamma_1$, where $\Gamma_H = \{(x,z) \in \overline{\Omega} : z = -H\}$, $\Gamma_0 = \{(x,z) \in \overline{\Omega} : z = 0\}$, $\Gamma_1 = \{(x,z) \in \overline{\Omega} : x = 1 \text{ or } x = -1\}$. U represents the ocean temperature. The governing equation for the ocean interior is giving by

$$U_t - (\frac{K_H}{R^2}(1-x^2)U_x)_x - K_V U_{zz} + wU_z = 0 \quad (0,T) \times \Omega,$$

where K_V and K_H are the vertical thermal diffusivity and the horizontal thermal diffusivity, respectively. w is the vertical velocity and R is the Earth radius.

The boundary condition at $z = 0$ comes from the following energy balance:

$$DU_t - \frac{DK_{H_0}}{R^2}((1-x^2)^{\frac{p}{2}}|U_x|^{p-2}U_x)_x + \mathcal{G}(U) + K_V\frac{\partial U}{\partial n} + wxU_x \in QS(x)\beta(U) + f$$

where $\mathcal{G}(U) - f$ is the emitted energy by cooling, D is the depth of the mixed layer, K_{H_0} is the horizontal thermal diffusivity in the mixed layer. The constants ρ and c represent the density and the specific heat of water.

In the pioneering models, the diffusion operator at the boundary was linear in U. Later, Stone [22] proposed a nonlinear diffusion considering the eddy fluxes in a more realistic way (diffusion coefficient must be dependent on the temperature gradient). We consider $\frac{DK_{H_0}}{R^2}\frac{\partial}{\partial x}\left((1-x^2)^{\frac{p}{2}}|\frac{\partial U}{\partial x}|^{p-2}\frac{\partial U}{\partial x}\right)$, which for $p=2$ is linear (as in [23]) and for $p=3$ corresponds to Stone [22]. The coalbedo feedback effect appears in this diffusive boundary condition, that is, β depends on the temperature. $S(x)$ is the insolation function. Q is the Solar constant (a significant positive parameter of the model).

At the ocean bottom, Γ_H, U satisfies

$$wx\frac{\partial U}{\partial x} + K_V\frac{\partial U}{\partial z} = 0 \qquad \text{on } (0,T)\times\Gamma_H.$$

The unknowns functions are the surface temperature and the ocean temperature. So, the initial conditions:

$$U(x,z,0) = U_0(x,z) \qquad \text{on } \Omega,$$
$$U(x,0,0) = u_0(x) \qquad \text{on } (-1,1).$$

The resultant problem is:

$$\begin{cases} \frac{\partial U}{\partial t} - \frac{K_H}{R^2}\frac{\partial}{\partial x}((1-x^2)\frac{\partial U}{\partial x}) - K_V\frac{\partial^2 U}{\partial z^2} + w\frac{\partial U}{\partial z} = 0 & (0,T)\times\Omega, \\[2mm] wx\frac{\partial U}{\partial x} + K_V\frac{\partial U}{\partial z} = 0 & (0,T)\times\Gamma_H \\[2mm] D\frac{\partial U}{\partial t} - \frac{DK_{H_0}}{R^2}\frac{\partial}{\partial x}\left((1-x^2)^{\frac{p}{2}}|\frac{\partial U}{\partial x}|^{p-2}\frac{\partial U}{\partial x}\right) + \mathcal{G}(U) + K_V\frac{\partial U}{\partial n} + \quad wx\frac{\partial U}{\partial x} \\[2mm] \in \frac{1}{\rho c}QS(x)\beta(x,U) + f(x) & (0,T)\times\Gamma_0 \\[2mm] (1-x^2)^{\frac{p}{2}}|\frac{\partial U}{\partial x}|^{p-2}\frac{\partial U}{\partial x} = 0 & (0,T)\times\Gamma_1 \\[2mm] U(0,x,z) = U_0(x,z) & \Omega, \\ U(0,x,0) = u_0(x) & (-1,1). \end{cases}$$

Structural Hypotheses:

(H_1) β is a bounded maximal monotone graph , that is, $|v| \leq M$ $\forall v \in \beta(s)$, $\forall s \in D(\beta) = \mathbb{R}$.

(H_2) $\mathcal{G}:\mathbb{R} \to \mathbb{R}$ is a continuous and strictly increasing function such that $\mathcal{G}(0) = 0$ and $|\mathcal{G}(\sigma)| \geq C|\sigma|^r$ for some $r > 0$.

(H_3) $S : (-1,1) \to \mathbb{R}$, $s_1 \geq S(x) \geq s_0 > 0$ a.e. $x \in (-1,1)$.

(H_4) $f \in L^{\infty}(\Omega \times (0,T))$.

(H_w) $w \in C^1(\overline{\Omega})$.

The mathematical treatment leads us to introduce the following function spaces

$$V(\Omega) = \{U \in L^2(\Omega) : (1-x^2)^{\frac{1}{2}}\frac{\partial U}{\partial x} \in L^2(\Omega), \frac{\partial U}{\partial z} \in L^2(\Omega)\},$$

$$V_p(\Gamma_0) = \{u \in L^2(\Gamma_0) : (1-x^2)^{\frac{1}{2}}\frac{\partial U}{\partial x} \in L^p(\Gamma_0)\}.$$

Definition 1. (U,u) is a bounded weak solution of (P) if $(U,u) \in L^2(0,T : V(\Omega) \times V_p(\Gamma_0)) \cap W^{1,2}(0,T : L^2(\Omega) \times L^p(\Gamma_0))$,

$$\int_0^T \int_\Omega \frac{\partial U}{\partial t} \psi dA dt + \int_0^T \int_\Omega \frac{K_H}{R^2}(1-x^2)\frac{\partial U}{\partial x}\frac{\partial \psi}{\partial x} dA dt + \int_0^T \int_\Omega K_v \frac{\partial U}{\partial z}\frac{\partial \psi}{\partial z} dA dt$$

$$+ \int_0^T \int_\Omega w \frac{\partial U}{\partial z}\psi dA dt$$

$$- \int_0^T \int_{-1}^1 wx \frac{\partial U}{\partial x}(x,-H)\psi(x,-H)dx dt - \int_0^T \int_{-1}^1 K_v \frac{\partial U}{\partial z}(x,0)\psi(x,0)dx dt = 0,$$

$$\int_0^T \int_{-1}^1 D\frac{\partial u}{\partial t}\zeta dx dt + \int_0^T \int_{-1}^1 \frac{DK_{H_0}}{R^2}(1-x^2)^{\frac{p}{2}}\left|\frac{\partial u}{\partial x}\right|^{\frac{p}{2}}\frac{\partial u}{\partial x}\frac{\partial \zeta}{\partial x}$$

$$+ \int_0^T \int_{-1}^1 K_v \frac{\partial u}{\partial z}(x,0)\zeta dx dt + \int_0^T \int_{-1}^1 wx \frac{\partial u}{\partial x}\zeta dx dt + \int_0^T \int_{-1}^1 \mathcal{G}(u)\zeta dx dt$$

$$= \int_0^T \int_{-1}^1 \frac{1}{\rho c}QS(x)h\zeta dx dt, \qquad U_{|\Gamma_0} = u,$$

for some $h \in L^\infty(0,T:L^\infty(\Gamma_0))$, $h \in \beta(\cdot,u)$ and $\forall(\psi,\zeta)$ test functions.

Existence results as well as uniqueness and non-uniqueness results for the time dependent problem are given in [13] and [15].

3. **The stationary problem.** We consider the problem

$$(P_Q)\begin{cases} -\frac{K_H}{R^2}\frac{\partial}{\partial x}((1-x^2)\frac{\partial U}{\partial x})-K_V\frac{\partial^2 U}{\partial z^2}+w\frac{\partial U}{\partial z}=0 & \Omega, \\[2mm] wx\frac{\partial U}{\partial x}+K_V\frac{\partial U}{\partial z}=0 & \Gamma_H \\[2mm] -\frac{DK_{H_0}}{R^2}\frac{\partial}{\partial x}\left((1-x^2)^{\frac{p}{2}}|\frac{\partial U}{\partial x}|^{p-2}\frac{\partial U}{\partial x}\right)+\mathcal{G}(U)+K_V\frac{\partial U}{\partial n}+wx\frac{\partial U}{\partial x} \\[2mm] \in \frac{1}{\rho c}QS(x)\beta(x,U)+f(x) & \Gamma_0 \\[2mm] (1-x^2)^{\frac{p}{2}}|\frac{\partial U}{\partial x}|^{p-2}\frac{\partial U}{\partial x}=0 & \Gamma_1. \end{cases}$$

We assume

(H_S) $S:\Omega \to \mathbb{R}$, $S \in L^\infty(-1,1)$, $S_1 \geq S(x) \geq S_0 > 0$ for some $S_1 > S_0$.

($H_\mathcal{G}$) $\mathcal{G}:\mathbb{R}\to\mathbb{R}$ is a continuous strictly increasing function such that $\mathcal{G}(0)=0$ and $\lim_{|s|\to\infty}|\mathcal{G}(s)|=+\infty$.

(H_f) $f \in L^\infty(\Omega)$ and there exist $C_f > 0$ such that $-\|f\|_\infty \leq f(x) \leq -C_f$ a.e. $x \in \Omega$.

(H_β) β is a bounded maximal monotone graph of $\mathbb{R}^2$ and there exists two real numbers $0 < m < M$ and $\epsilon > 0$ such that
$\beta(r)=\{m\}$ for any $r \in (-\infty,-10-\epsilon)$ and
$\beta(r)=\{M\}$ for any $r \in (-10+\epsilon,+\infty)$.

(H_{C_f}) $\mathcal{G}(-10-\epsilon)+C_f > 0$ and $\dfrac{\mathcal{G}(-10+\epsilon)+\|f\|_\infty}{\mathcal{G}(-10-\epsilon)+C_f} \leq \dfrac{S_0 M}{S_1 m}$.

(H_w) $w \in C^1(\overline{\Omega})$ (by simplicity).

(H_K) The constants K_H, K_V, K_H, K_{H_0}, D, R, ρ, c and Q are positive.

Definition 2. A bounded weak solution of the stationary problem is a pair $(U, u) \in (V(\Omega) \times V_p(\Gamma_0)) \cap (L^\infty(\Omega) \times L^\infty(\Gamma_0))$ such that $U_{|\Gamma_0} = u$ and

$$\int_\Omega \frac{K_H}{R^2}(1-x^2)\frac{\partial U}{\partial x}\frac{\partial \psi}{\partial x}dA + \int_\Omega K_v \frac{\partial U}{\partial z}\frac{\partial \psi}{\partial z}dA + \int_\Omega w\frac{\partial U}{\partial z}\psi dAdt$$

$$- \int_{-1}^1 wx\frac{\partial U}{\partial x}(x,-H)\psi(x,-H)dx - \int_{-1}^1 K_v\frac{\partial U}{\partial z}(x,0)\psi(x,0)dx = 0,$$

$$\int_{-1}^1 \frac{DK_{H_0}}{R^2}(1-x^2)^{\frac{p}{2}}\left|\frac{\partial u}{\partial x}\right|^{\frac{p}{2}}\frac{\partial u}{\partial x}\frac{\partial \zeta}{\partial x}$$

$$+ \int_{-1}^1 K_v\frac{\partial u}{\partial z}(x,0)\zeta dx + \int_{-1}^1 wx\frac{\partial u}{\partial x}\zeta dx + \int_{-1}^1 \mathcal{G}(u)\zeta dx$$

$$= \int_{-1}^1 \frac{1}{\rho c}QS(x)h\zeta dx$$

for some $h \in L^\infty(\Gamma_0)$, $h \in \beta(\cdot, u)$ and $\forall (\psi, \zeta)$ test functions.

The main result of this work is the following.

Theorem 1. *Let $(H_S), (H_\mathcal{G}), (H_f), (H_w), (H_K)$ and (H_β) be satisfied. Then*

  i) *for any $Q > 0$ there is a minimal solution $(\underline{U}, \underline{u})$ (resp. a maximal solution $(\overline{U}, \overline{u})$) of problem (P_Q).*

Moreover, if (H_{C_f}) holds, then there exist $Q_1 < Q_2 < Q_3 < Q_4$ such that

  ii) *if $0 < Q < Q_1$, then (P_Q) has a unique solution.*
  iii) *if $Q_2 < Q < Q_3$, then (P_Q) has at least three solutions.*
  iv) *if $Q_4 < Q$, then (P_Q) has a unique solution;*

where

$$Q_1 = \frac{(\mathcal{G}(-10-\epsilon)+C_f)\rho c}{S_1 M} \qquad Q_2 = \frac{(\mathcal{G}(-10+\epsilon)+\|f\|_\infty)\rho c}{S_0 M}$$

$$Q_3 = \frac{(\mathcal{G}(-10-\epsilon)+C_f)\rho c}{S_1 m} \qquad Q_4 = \frac{(\mathcal{G}(-10+\epsilon)+\|f\|_\infty)\rho c}{S_0 m}.$$

Definition 3. We define the vectorial operator $\mathcal{A} : L^2(\Omega) \times L^2(\Gamma_0) \to L^2(\Omega) \times L^2(\Gamma_0)$, $\mathcal{A}(U, u) := (AU, Bu)$ and the domain,

$$D(\mathcal{A}) = \{(U, u) \in L^2(\Omega) \times L^2(\Gamma_0) : AU \in L^2(\Omega), Bu \in L^2(\Gamma_0), U_{|\Gamma_0} = u\},$$

where,

$$AU = -\frac{K_H}{R^2}\frac{\partial}{\partial x}((1-x^2)\frac{\partial U}{\partial x})-K_V\frac{\partial^2 U}{\partial z^2} + w\frac{\partial U}{\partial z}$$

and

$$Bu = -\frac{DK_{H_0}}{R^2}\frac{\partial}{\partial x}\left((1-x^2)^{\frac{p}{2}}|\frac{\partial U}{\partial x}|^{p-2}\frac{\partial U}{\partial x}\right) + K_V\frac{\partial U}{\partial n} + wx\frac{\partial U}{\partial x} + \mathcal{G}(U).$$

Now, we can rewrite the problem as a system of pdes:

$$\begin{cases} AU = 0 & \Omega \\ Bu \in \frac{1}{\rho c}QS(x)\beta(u) + f & \Gamma_0 \\ U_{|\Gamma_0} = u & \\ wxU_x + K_V U_z = 0 & \Gamma_H \\ (1-x^2)^{\frac{p}{2}}|U_x|^{p-2}U_x = 0 & \Gamma_1. \end{cases} \tag{3}$$

We need the following lemmas:

Lemma 1. $\mathcal{A} + \omega I$ *is T-acretive in* $L^2(\Omega) \times L^2(\Gamma_0)$, *where* $\omega > \frac{1}{2}$. $\quad\square$

To prove Lemma 1, we denote $\mathbf{u} = (U, u)$ and $\mathbf{u}_+ = (U_+, u_+)$, with s_+ the positive part of s. Assume $p = 2$ then we have

$$(\omega\mathbf{u} + \mathcal{A}\mathbf{u}, \mathbf{u}_+)_{(L^2(\Omega)\times L^2(\Gamma_0))\times(L^2(\Omega)\times L^2(\Gamma_0))}$$
$$= (\omega U, U_+) + (AU, U_+) + (\omega u, u_+) + (Bu, u_+)$$

$$= \int_\Omega \omega|U_+|^2 dxdz + \int_{\Gamma_0} \omega|u_+|^2 dx + \int_\Omega \frac{K_H}{R^2}(1-x^2)|\frac{\partial U_+}{\partial x}|^2 dxdz$$

$$+ \int_\Omega K_V|\frac{\partial U_+}{\partial z}|^2 dxdz - \int_\Omega w\frac{\partial U}{\partial z}U_+ dxdz + \int_0^1 K_V U_z(x,-H)U_+(x,-H)dx$$

$$+ \int_{\Gamma_0} \frac{DK_{H_0}}{R^2}(1-x^2)^{\frac{p}{2}}|\frac{\partial u_+}{\partial x}|^2 dx - \frac{DK_{H_0}}{R^2}\frac{\partial U}{\partial x}(0,0)U_+(0,0) + \int_{\Gamma_0} wx\frac{\partial u}{\partial x}u_+ dx$$

$$+ \int_{\Gamma_0} \mathcal{G}(u)u_+ dx.$$

By using Young inequality and the monotonicity of $\mathcal{G}$, for all $\omega > \frac{1}{2}$ we obtain $0 \le (\omega\mathbf{u} + \mathcal{A}\mathbf{u}, \mathbf{u}_+)_{(L^2(\Omega)\times L^2(\Gamma_0))^2}$. In the quasilinear case $(p \neq 2)$, we get that $0 \le (\omega(\mathbf{u}-\mathbf{v}) + \mathcal{A}\mathbf{u}-\mathcal{A}\mathbf{v}, (\mathbf{u}-\mathbf{v})_+)_{L^2(\Omega)\times L^2(\Gamma_0)}$.

Notice that these inequalities allow us to prove a comparison principle for the system

$$(P_{F,f}) \quad \begin{cases} \omega U + AU = F & \text{in } L^2(\Omega) \\ \omega u + Bu = f & \text{in } L^2(\Gamma_0) \\ U_{|\Gamma_0} = u & \\ wx\frac{\partial U}{\partial x} + K_V\frac{\partial U}{\partial z} = 0 & \Gamma_H \\ (1-x^2)^{\frac{p}{2}}|\frac{\partial U}{\partial x}|^{p-2}\frac{\partial U}{\partial x} = 0 & \Gamma_1. \end{cases}$$

In fact, if $F_1 \le F_2$ and $f_1 \le f_2$ then the solutions of (P_{F_1,f_1}) and (P_{F_2,f_2}) satisfy

$$U_1 \le U_2,$$
$$u_1 \le u_2.$$

Moreover, we have

Lemma 2. $R(\mathcal{A} + \lambda I) = L^2(\Omega) \times L^2(\Gamma_0)$ *for all* $\lambda > \frac{1}{2}$.

To prove Lemma 2 we notice that the operator B can be expressed as $B_1 + B_2 + B_3$, where B_1 and B_2 are maximal monotone operators in $L^2(\Gamma_0)$,

$$B_1 u = -\frac{DK_{H_0}}{R^2}\frac{\partial}{\partial x}\left((1-x^2)^{\frac{p}{2}}|\frac{\partial U}{\partial x}|^{p-2}\frac{\partial U}{\partial x}\right) + \mathcal{G}(U)$$

and the pseudo-differential operator

$$B_2 u = K_V \frac{\partial U}{\partial n},$$

where U is the solution of the problem

$$\omega U + AU = F \quad \text{in } L^2(\Omega)$$
$$U_{|\Gamma_0} = u.$$

The operator B_3 is defined by

$$B_3 u = wx \frac{\partial U}{\partial x}.$$

B_3 is not necessarily monotone but it is dominated (in some sens) by the operators B_1 and B_2.

Proof. (i) To prove the existence of maximal and minimal solutions, we use the comparison for this auxiliary system:

$$\begin{cases} \omega U + AU = H \quad \Omega \\ \omega u + Bu = h. \end{cases} \tag{4}$$

If $H_1 \leq H_2$ and $h_1 \leq h_2$ then $U_1 \leq U_2$ and $u_1 \leq u_2$.

We find constant functions $(\underline{V}, \underline{v})$ and $(\overline{U}, \overline{u})$ verifying

$$\begin{cases} \omega \underline{V} + A\underline{V} = \omega \underline{V} \quad \Omega \\ \omega \underline{v} + B\underline{v} = \omega \underline{v} + \frac{1}{\rho c} Q S_0 m - \|f\|_\infty \leq \omega \underline{v} + \frac{1}{\rho c} Q S(x) \underline{\beta}(\underline{v}) + f, \end{cases}$$

$$\begin{cases} \omega \overline{U} + A\overline{U} = \omega \overline{U} \quad \Omega \\ \omega \overline{u} + B\overline{u} = \omega \overline{u} + \frac{1}{\rho c} Q S_1 M - C_f \geq \omega \overline{u} + \frac{1}{\rho c} Q S(x) \overline{\beta}(\overline{u}) + f. \end{cases}$$

Define the sequence $\{(\underline{V}_n, \underline{v}_n)\}$ as follows,

$$(\underline{P}_n) \begin{cases} \omega \underline{V}_n + A\underline{V}_n = \omega \underline{V}_{n-1} \\ \omega \underline{v}_n + B\underline{v}_n = \omega \underline{v}_{n-1} + Q S(x) \underline{\beta}(\underline{v}_{n-1}) + f \\ \text{Boundary Cond. on } \Gamma_H \cup \Gamma_1 \end{cases}$$

and $(\underline{V}_0, \underline{v}_0) := (\underline{V}, \underline{v})$. From the comparison principle for the auxiliary problem (4), the sequences $\{\underline{V}_n\}$ and $\{\underline{v}_n\}$ are monotone. Estimates on $\{(\underline{V}_n, \underline{v}_n)\}$, allow us to pass to the limit in the weak formulation and to obtain

$$(\underline{V}_n, \underline{v}_n) \to (V_*, v_*),$$

where the limit (V_*, v_*) is a solution of (P_Q) and every solution (W, w) verifies $V_* \leq W$ and $v_* \leq w$, that is, (V_*, v_*) is a minimal solution. Analogously, we get the maximal solution (U^*, u^*).

(ii) If $Q < Q_1$ then $\underline{V} \leq \overline{U} \leq -10 - \epsilon$. So, every solution (U, u) of (P_Q) verifies $u < -10 - \epsilon$ and it is a solution of the problem

$$(P_Q^m) \begin{cases} AU = 0 & \Omega \\ Bu = \frac{1}{\rho c} Q S(x) m + f & \Gamma_0 \\ U_{|\Gamma_0} = u \\ wx \frac{\partial U}{\partial x} + K_V \frac{\partial U}{\partial z} = 0 & \Gamma_H \\ (1 - x^2)^{\frac{p}{2}} |\frac{\partial U}{\partial x}|^{p-2} \frac{\partial U}{\partial x} = 0 & \Gamma_1, \end{cases}$$

which has a unique solution. To prove it, we assume there exist two solutions, (U_1, u_1) and (U_2, u_2) and we take the difference $U_1 - U_2$ as a test function in the weak formulation. Gronwall Lemma allows us to conclude the uniqueness.

(iii) If $Q_4 < Q$ then $-10 + \epsilon \leq \underline{V} \leq \overline{U}$. So, every solution (U, u) verifies $-10 + \epsilon \leq u$ and $\beta(u) = M$.

$$(P_Q^M) \begin{cases} AU = 0 & \Omega \\ Bu = \frac{1}{\rho c} QS(x)M + f & \Gamma_0 \\ U_{|\Gamma_0} = u \\ wx\frac{\partial U}{\partial x} + K_V \frac{\partial U}{\partial z} = 0 & \Gamma_H \\ (1 - x^2)^{\frac{p}{2}} |\frac{\partial U}{\partial x}|^{p-2} \frac{\partial U}{\partial x} = 0 & \Gamma_1. \end{cases}$$

As in (ii), this problem has a unique solution.

(iv) The proof of multiplicity of solutions is based in Díaz - Hernandez - Tello [9], where we found at least three solutions to the problem

$$-\Delta_p u + \mathcal{G}(u) \in QS(x)\beta(u) + f \text{ on } \mathcal{M}$$

if $0 < Q_2 < Q < Q_3$.

The proof consists of three steps
Step 1. Construction of upper and lower solutions. If $Q_2 < Q < Q_3$ then,

$$\begin{aligned}
\overline{U}_1 &:= \mathcal{G}^{-1}(\frac{1}{\rho c} QS_1 M - C_f) & \text{is an uppersolution of } (P_Q^M) \\
\underline{V}_1 &:= \mathcal{G}^{-1}(\frac{1}{\rho c} QS_0 M - \|f\|_\infty) & \text{is a lowersolution of } (P_Q^M) \\
\overline{U}_2 &:= \mathcal{G}^{-1}(\frac{1}{\rho c} QS_1 m - C_f) & \text{is an uppersolution of } (P_Q^m) \\
\underline{V}_2 &:= \mathcal{G}^{-1}(\frac{1}{\rho c} QS_0 m - \|f\|_\infty) & \text{is a lowersolution of } (P_Q^m).
\end{aligned}$$

Moreover, $\underline{V}_2 < \overline{U}_2 < -10 - \epsilon < -10 + \epsilon < \underline{V}_1 < \overline{U}_1$. Then, there exist two solutions (U_1, u_1) and (U_2, u_2) of (P_Q) such that u_1 and u_2 do not cross the level -10. To find the third solution, we want to apply a result of Amann [1]. This is possible for the case where β is a Lipschitz function. In next step, we will approximate the graph β by Lipschitz functions.
Step 2. Approximate problem.
We define a new family of problems

$$(P_{Q,\lambda}) \begin{cases} AU = 0 & \Omega \\ Bu = QS(x)\beta_\lambda(u) + f(x) & \Gamma_0 \\ B.C. & \Gamma_H \cup \Gamma_1 \end{cases}$$

where β_λ is the Lipschitz function $\beta_\lambda = \frac{1}{\lambda}(I - (I - \lambda\beta)^{-1})$, $\lambda > 0$ (the Yosida approximation of β). Since β verifies (H_β), we get that

β_λ is a bounded and nondecreasing function $\forall \lambda > 0$,

$\beta_\lambda(s) = \beta(s)$ for any $s \notin [-10 - \epsilon, -10 + \epsilon + \lambda M]$, $\forall \lambda > 0$,

$\beta_\lambda(s) \to \beta(s)$ in the sense of maximal monotone graphs when $\lambda \to 0$

(see Brezis [5]). In the case of β is a Lipschitz function, we take $\beta_\lambda = \beta$.
Now, by applying the argument of step 1 to problem $(P_{Q,\lambda})$, there exit λ_0 such that $\underline{V}_2 < \overline{U}_2 < -10 - \epsilon < -10 + \epsilon + \lambda_0 M < \underline{V}_1 < \overline{U}_1$. Then, we have two families of solutions of $\{(P_{Q,\lambda})\}$ such that u_1^λ and u_2^λ do not cross the level -10. We have the third family of solutions by using the following lemma.

Lemma 3. *(Amann* [1]*) Let X be a retract of some Banach space E and let $F :$ $X \to X$ be a compact map. Suppose that X_1 and X_2 are disjoint retracts of X, and let Y_k, $k = 1, 2$ be open subset of X such that $Y_k \subset X_k$. Moreover, suppose that*

$F(X_k) \subset X_k$ and that F has no fixed points on $X_k - Y_k$, $k = 1, 2$. Then F has at least three distinct fixed points x, x_1, x_2 with $x_k \in X_k$ and $x \in X - (X_1 \cup X_2)$.

We establish that the assumptions of this lemma are satisfied. Any solution u of the problem (P_Q^λ) is a fixed point of the equation $u = F(u)$ with $F : L^\infty(\Gamma_0) \to L^\infty(\Gamma_0)$ is defined by

$$u = P_2(\mathcal{A}^{-1}(0, \frac{1}{\rho c} QS(\cdot)\beta_\lambda(u) + f_\infty(\cdot))).$$

$\mathcal{A}$ was given in definition 3 and P_2 is the projection over the second component.

Let $E = L^\infty(\Gamma_0)$ which is an ordered Banach space with respect to the natural ordering whose positive cone is given by

$$L_+^\infty(\Gamma_0) = \{v \in L^\infty(\Gamma_0) : v(x) \geq 0 \text{ a.e. } x \in \Gamma_0\},$$

having a nonempty interior. Let us define the intervals $X = [\underline{V}_2 - \delta , \overline{U}_1 + \delta]$, $X_1 = [\underline{V}_1 - \delta , \overline{U}_1 + \delta]$ and $X_2 = [\underline{V}_2 - \delta , \overline{U}_2 + \delta]$ where $\delta > \lambda_0 M$ is taken such that $\underline{V}_1 > -10 + \epsilon + \delta$, $\overline{U}_2 > -10 - \epsilon - \delta$. So, there exists an open set Y_k of $L^\infty(\Gamma_0)$ containing u_k^λ for $k = 1, 2$ such that $Y_k \subset X_k$.

The sets X, X_1 and X_2 are retracts of $L^\infty(\Gamma_0)$ (resp. X), since they are nonempty closed convex subsets of $L^\infty(\Gamma_0)$ (resp. X). Moreover, $F(X) \subset X$ and $F(X_k) \subset X_k$. Finally, from the properties of β_λ and the compact embeddings $V_p(\Gamma_0) \subset L^\infty(\Gamma_0)$ for $p \geq 2$, we arrive to $F : X \to X$ is a compact map.

So, by the lemma 3 we conclude that F has at least three fixed points, or equivalently, $(P_{Q,\lambda})$ has at least three solutions: $u_1^\lambda \in X_1$, $u_2^\lambda \in X_2$ and $u_3^\lambda \in X - (X_1 \cup X_2)$.

Step 3. The proof ends by observing the convergence of a subsequence of $\{u_3^\lambda\}$ to u_3 such that (U_3, u_3) is a solution of (P_Q). To get this limit we need to use a result of maximal monotone graphs ([6]) which guarantees that the limit of $\beta_\lambda(u_3^\lambda)$ is in the graph $\beta(u_3)$. Finally, the convergence in $L^\infty(\Gamma_0)$ allow us to show that u_3 is different from u_1 and u_2. In particular, u_3 must cross the level -10. $\qquad\square$

Remark 1. The existence of infinitely many solutions for a one dimensional problem for p-laplacian in presence of a graph β of Heaviside type ([14]), suggests us that the problem here studied could have more than three solutions for some values of parameter Q.

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