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BLOW-UP IN SOME ORDINARY AND PARTIAL DIFFERENTIAL EQUATIONS WITH TIME-DELAY

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ABSTRACT. Blow-up phenomena are analyzed for both the delay-differential equation

$$\text{(DDE)} \quad u'(t) = B'(t)u(t-\tau),$$

and the associated parabolic PDE

$$\text{(PDDE)} \quad \partial_t u = \Delta u + B'(t)u(t-\tau, x),$$

where $B : [0,\tau] \to \mathbb{R}$ is a positive L^1 function which behaves like $1/\left|t-t^*\right|^\alpha$, for some $\alpha \in (0,1)$ and $t^* \in (0,\tau)$. Here B' represents its distributional derivative. For initial functions satisfying $u(t^*-\tau) > 0$, blow up takes place as $t \nearrow t^*$ and the behavior of the solution near t^* is given by $u(t) \simeq B(t)u(t-\tau)$, and a similar result holds for the PDDE. The extension to some nonlinear equations is also studied: we use the Alekseev's formula (case of nonlinear (DDE)) and comparison arguments (case of nonlinear (PDDE)). The existence of solutions in some generalized sense, beyond $t = t^*$is also addressed. This results is connected with a similar question raised by A. Friedman and J.B. McLeod in 1985 for the case of semilinear parabolic equations.

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1. **Introduction**

Blow-up or explosion phenomena is a general term that refers to the fact that some solutions of an evolution equation in a Banach space tend to infinity in norm as t approaches some finite explosion time t^* which depends on the solution. This behavior has been extensively studied in the last few decades. In the ODE case (when the Banach space is finite-dimensional) it is more commonly referred to with expressions like non existence of global solutions, since the theory of continuation of solutions shows that, under very general hypotheses, blow-up is the *only* possibility for a maximal solution to

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be defined only on a finite time interval. In the PDE framework, however, more diversity of results is found: for instance, the solution might explode in some norm but not on others (shock waves), or might explode at some points of the spatial domain but not on others, and so on. An extensive bibliography is available. Let us just mention [1] and [14], and the references indicated for semilinear equations at the end of this section.

The relation of blow-up and time delay has not been studied in detail, and the purpose of this paper is to give a class of delay equations (both ODE and PDE) in which this phenomenon appears. The authors (see [4]) have done a similar analysis for the "opposite" situation, namely, finite-time extinction, in which some (nonzero) solutions vanish identically after some finite "extinction time". Some of the techniques used are similar in both cases, but there are important differences. The first of these comes from the very nature of blow-up and how "infinity" is involved, which requires analyzing some technical aspects of the regularity properties of the solution and which is not necessary in finite-time extinction processes, except for the fact that non-Lipschitz functions are usually involved. The second difference is that the structure of the equation enables us to apply delay-PDE comparison techniques which are not usually available in the extinction phenomenon.

We first analyze the delay-differential equation

$$(1.1) \qquad \text{(DDE)} \quad \begin{cases} u'(t) = B'(t)u(t-\tau), & 0 < t, \\ u(\theta) = \xi(\theta) \quad \text{given}, & -\tau \leq \theta \leq 0, \end{cases}$$

and then some delay-PDEs of the type

$$(1.2) \qquad (P_N) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = B'(t)u(t-\tau, \mathbf{x}), & (t, \mathbf{x}) \in (0, +\infty) \times \Omega, \\ \frac{\partial u}{\partial n}(t, \mathbf{x}) = 0, & (t, \mathbf{x}) \in (0, +\infty) \times \partial\Omega, \\ u(\theta, \mathbf{x}) = \xi(\theta, \mathbf{x}), & (\theta, \mathbf{x}) \in (-\tau, 0) \times \Omega, \end{cases}$$

where $B : [0, \tau] \to \mathbb{R}$ is a positive L^1 function which behaves like $1/|t - t^*|^\alpha$, for some $\alpha \in (0, 1)$ and $t^* \in (0, \tau)$, and B' represents its distributional derivative. In (DDE), initial functions with $u(t^* - \tau) > 0$ blow up like $B(t)u(t - \tau)$as $t \nearrow t^*$, and a similar result holds for the PDDE due to the applicability of comparison arguments. Other boundary conditions can be studied in the same way, but have been omitted for simplicity.

The extension to some nonlinear equations is also studied: we use the Alekseev's formula for

$$(1.3) \qquad (NLDDE) \begin{cases} u'(t) = f(t, u(t)) + B'(t)g(t, u(t-\tau)), & 0 < t \\ u(\theta) = \xi(\theta), \quad \tau \leq \theta \leq 0, \end{cases}$$

where f is C^2 and g is C^1 (see subsection 2.7) and some comparison arguments, for the case of

$$(NLP_N) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = f(t, u(t,x), u(t-\tau,x)), & (t,x) \in (0,+\infty) \times \Omega, \\ \frac{\partial u}{\partial n}(t,x) = 0, & (t,x) \in (0,+\infty) \times \partial\Omega, \\ u(\theta, x) = \xi(\theta, x), & (\theta, x) \in (-\tau, 0) \times \Omega, \end{cases}$$

(see subsection 3.2).

The possibility of extending these blow-up solutions beyond the explosion time t^*, that is, the question of existence of solutions in some generalized sense in the whole interval $[0,T]$, when $T > t^*$, is much more delicate.

It seems that this type of questions was raised by first time in Friedman and McLeod [8] when analyzing blow-up properties of solutions of the semilinear equation

$$(1.4) \qquad (SP) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = |u|^{p-1} u, & \text{in } (0,+\infty) \times \Omega, \\ u = 0, & \text{on } (0,+\infty) \times \partial\Omega, \\ u(0, \mathbf{x}) = u_0(\mathbf{x}), & \text{on } \Omega. \end{cases}$$

They consider the case in which $p > 1$ and $u_0 \in L^\infty(\Omega)$ such that

$$lim_{t \uparrow T_{\max}} \|u(t,\cdot)\|_{L^\infty} = +\infty$$

for some $T_{\max} < +\infty$. The open question raised in [8] is to know if $\|u(t,\cdot)\|_{L^q}$ remains bounded as $t \uparrow T_{\max}$ for q in the "subcritical case", i.e. for q such that $1 \leq q \leq \frac{N(p-1)}{2}$ (it is known that the answer is negative if $q > \frac{N(p-1)}{2}$: [8], [3] and its references). For a very complete survey on the blow-up phenomenon results for the semilinear problem (SP) until 1995 we send the reader to the monograph [16]. Many other more recent results are available today in the literature (see, e.g. the papers [18], [5] and their references). We also mention that many sufficient conditions on Ω, u_0, p and N implying that the explosion region $\{x \in \Omega : u(x,t) \uparrow +\infty \text{ when } t \uparrow T_{\max}\}$ is confined to a proper subset of Ω are well-known for the semilinear problem (SP) (see the above indicated references and specially the extension to quasilinear equations made in [16]).

We define generalized solution by means of the following integral identity in a suitable space of functions on Ω

(1.5)
$$u(t) = e^{At}\xi(0) + B(t)\xi(t-\tau) + \int_0^t e^{A(t-s)} B(s) \left[-A\xi(s-\tau) + \xi'(s-\tau)\right] ds,$$

where A is the abstract operator associated to $-\Delta$ with Neumann boundary conditions and e^{At} is the associated semigroup, and give sufficient conditions for the integral to exist beyond $t = t^*$(for instance on $[0,\tau]$).

2. Setting of the problem

2.1. **Preliminary analysis.** Let $t^* > 0$, let $b : [0, t^*) \to \mathbb{R}$ be a continuous function such that $b(t) \geq 0$ on $[0, t^*)$ and assume that b *"blows up"* at t^*, that is, $b(t) \to \infty$ as $t \nearrow t^*$. Consider the delay differential equation (DDE)

$$\begin{cases} u'(t) = b(t)u(t-\tau), & \text{for } 0 \leq t < t^*, \\ u(\theta) = \xi(\theta), & \text{for } -\tau \leq \theta \leq 0, \end{cases} \tag{2.1}$$

where $\tau > t^*$ is a given delay and ξ represents the "history" or "initial function", which is usually assumed to be continuous on $[-\tau, 0]$, although other function spaces can also also be considered. For a general study of this type of equations see [11].

If $\xi(t) \equiv \xi \in \mathbb{R}$ is a nonzero constant, then direct integration of both sides of equation (2.1) gives

$$u(t) = u(t, \xi) = u(0) + \int_0^t b(s)\xi ds = \xi(1 + B(t)), \quad 0 \leq t < t^*, \tag{2.2}$$

where we have denoted $B(t) = \int_0^t b(s)ds$. If b is integrable on $(0, t^*)$ then $B(t^*) = \lim_{t \to t^*} B(t)$ exists. Otherwise, u blows up at t^* but the singularity of the solution is weaker than that of b, a fact that reminds the "smoothing effect" usually found on delay equations.

If the initial condition ξ is not constant and if $\xi(t^* - \tau) > 0$, then $\xi(t) \geq \xi(t^* - \tau)/2$ on some interval $[t^* - \tau - \delta, t^* - \tau]$ (where $0 < \delta < t^*$) and we may write for $t \in [0, t^*)$:

$$\begin{aligned} u(t) = u(t, \xi) = u(t^* - \delta) + \int_{t^*-\delta}^t b(s)\xi(s-\tau)ds \geq \\ \geq u(t^* - \delta) + \frac{\xi(t^* - \tau)}{2}\left[B(t) - B(t^* - \delta)\right] \end{aligned} \tag{2.3}$$

which implies that, $u(t, \xi)$ blows up at t^* like $B(t)$ as before. Obviously, if $\xi(t^* - \tau) = 0$, the product $b(t)\xi(t - \tau)$ may be integrable or not on $(0, t^*)$, depending on the (fractional) order of $t^* - \tau$ as a zero of ξ. If ξ is C^1, for instance, the order will be an integer and the product will certainly be integrable.

If the function b is also defined and is continuous for $t > t^*$, it is natural to ask whether the solution itself can be continued beyond t^* in some sense. In other words, can the formal integral expression

$$u(t, \xi) = \xi(0) + \int_0^t b(s)\xi(s - \tau)ds, \quad 0 \leq t \leq \tau,$$

be considered a as an "integralsolution" of some kind, defined on the whole interval $[0, \tau]$? This is the real difficulty, since continuation beyond τ is always possible as long as b remains continuous.

Let us start again with constant initial functions $\xi(t) \equiv \xi$. If $B \in L^p(0,\tau)$ for some $p \in [1,\infty]$, the function

$$u(t,\xi) = \xi(1 + B(t)),$$

is a well-defined L^p function. For a general continuous initial ξ, the function

$$u(t,\xi) = \xi(0) + \int_0^t b(s)\xi(s-\tau)ds, \quad 0 \le t \le \tau,$$

is also well defined and belongs to the same L^p class as B does, it is also C^1 except a $t = t^*$ and satisfies the differential equation for all $t \in [0,\tau]$ except for t^*.

Of course, one could *define* an integral solution to be just that, but it is clear that further analysis is necessary in order to justify such a procedure. This is the purpose of the next section, which deals with primitives $B(t)$ only assumed to be in $L^p(0,\tau)$, thus allowing for infinitely many singularities and other more complicated situations.

2.2. **The basic equation.** Let $B \in L^p(0,\tau)$ such that $B' \notin L^1_{\mathrm{loc}}(0,\tau)$, where B' is to be understood in the sense of distributions. Without loss of generality we will assume that $B(0) = 0$.

We consider the retarded functional differential equation

$$\begin{cases} u'(t) = B'(t)u(t-\tau), \quad 0 < t < \tau, \\ u(\theta) = \xi(\theta), \quad \tau \le \theta \le 0, \end{cases} \tag{2.4}$$

where ξ is a given initial function whose smoothness properties will be discussed below. For the time being we will concentrate on the initial "basic interval" $[0,\tau]$.

As discussed in the previous section, if B is C^1 except for a singularity $t^* \in (0,\tau)$, for instance

$$B(t) = 1/\left|t - t^*\right|^\alpha, \quad \text{where} \quad 0 < \alpha < 1, \tag{2.5}$$

we can integrate both sides, thus obtaining

$$u(t) = \xi(0) + \int_0^t B'(s)\xi(s-\tau)ds, \tag{2.6}$$

but, in general, this formula will make sense only for $t \in [0,t^*)$ because the product $B'(t)\xi(t-\tau)$ need not be integrable. In fact, it will *never* be integrable for nonzero constants ξ. As mentioned above, in order to get a better understanding of the problem and check whether the solution can be continued "beyond" the singular point t^* in a meaningful way we need to give a more precise meaning to the right-hand side of (2.4). A standard strategy in the theory of differential equations with discontinuous right-hand sides (see Filippov [9]) is to try to transform the equation into another with integrable discontinuities, that is, a "Carathéodory form", as follows.

2.3. **Equivalent neutral equation.** By writing

$$B'(t)u(t-\tau) = [B(t)u(t-\tau)]' - B(t)u'(t-\tau), \tag{2.7}$$

equation (2.4) becomes

$$\begin{cases} \dfrac{d}{dt}\left[u(t) - B(t)u(t-\tau)\right] = -B(t)u'(t-\tau), \quad t > 0, \\ u(\theta) = \xi(\theta), \quad \tau \leq \theta \leq 0, \end{cases} \tag{2.8}$$

which is a **neutral** differential-delay equation. Integrating (formally) both sides of (2.4) on $[0,\tau]$ and taking into account the (nonessential) assumption $B(0) = 0$ we obtain

$$u(t) = \xi(0) + B(t)\xi(t-\tau) - \int_0^t B(s)\xi'(s-\tau)ds, \quad 0 \leq t \leq \tau \tag{2.9}$$

which gives an explicit representation of the solution in terms of the initial function. Of course, this is just the standard "method of steps" as long as the integral in the right-hand side is defined. As is usual in neutral FDE's, more smoothness in the initial function is required than in the retarded case. Since $B \in L^p(0,\tau)$, the hypothesis $\xi' \in L^q(-\tau, 0)$ $(1/p + 1/q = 1)$ will be enough. We have just proved the following result:

Theorem 2.1. 1. *Let $B \in L^p(0,\tau)$. Then, for every $\xi \in W^{1,q}(0,\tau)$ (where $1/p + 1/q = 1$) the Cauchy problem (2.8) has a unique solution given by the identity*

$$u(t) = \xi(0) + B(t)\xi(t-\tau) - \int_0^t B(s)\xi'(s-\tau)ds, \quad 0 \leq t \leq \tau, \tag{2.10}$$

Therefore $u \in L^p(0,\tau)$ and $u(t) - B(t)\xi(t-\tau)$ is an absolutely continuous function and we may write symbolically

$$u(t) = B(t)\xi(t-\tau) + AC, \tag{2.11}$$

where "AC" means "an absolutely continuous function". As a consequence, the singularities of the solution on $[0,\tau]$ are also singularities of B

2. *In particular, let $t^* \in (0,\tau)$, $0 < \alpha < 1$, let m be continuous on $[0,\tau]$ and let*

$$B(t) = \frac{a}{|t-t^*|^\alpha} + m(t), \tag{2.12}$$

If the initial function ξ satisfies $\xi(t^* - \tau) \neq 0$, then t^* is also a singularity of u and

$$u(t) \simeq \frac{a}{|t-t^*|^\alpha}\xi(t^*-\tau), \quad \text{as } t \to t^*, \tag{2.13}$$

is an asymptotic expansion of u near t^*.

3. *If $|\xi(t^* - \tau - t)| \leq C\,|t^* - \tau - t|^\alpha$ near t^*, then u is bounded near t^*.*

2.4. **Solutions without Singularities.** Following on point 3 of the previous theorem, let us concentrate again on the single-singularity case as above:

$$B(t) = \frac{a}{|t-t^*|^\alpha} + m(t), \quad t \in [0,\tau], \tag{2.14}$$

with $t^* \in (-\tau, 0)$, $\alpha \in (0,1)$ and m continuous. For any $\gamma > \alpha$ let us consider the following class of initial values:

$$\begin{aligned} E_\gamma = \{\xi \in & W^{1,q}(-\tau,0) : \text{There exists } C>0 \text{ such that} \\ & |\xi(t^*-\tau-\theta)| \leq C\,|t^*-\tau-\theta|^\gamma \text{ for all } \theta \in [-\tau,0]\}. \end{aligned} \tag{2.15}$$

Because of the Sobolev embedding $W^{1,q}(-\tau,0) \subset C[-\tau,0]$, E_γ is a closed subspace of $W^{1,q}(-\tau,0)$. We have thus the following immediate consequence of representation (2.10):

Proposition 2.2. *If $\xi \in E_\gamma$, the solution u of (2.8) is absolutely continuous on $[-\tau,0]$.*

Remark 2.3. If we restrict ourselves to C^1 initial functions (a very standard procedure in neutral delay-differential equations), the hypothesis that the exponent γ be strictly larger than α means that the condition $|\xi(t^*-\tau-t)| \leq C\,|t^*-\tau-t|^\gamma$ is automatically satisfied if $t^*-\tau$ is simply *a zero* of ξ, and the definition of E_γ is much easier:

$$E_\gamma \cap C^1([-\tau,0]) = \{\xi \in C^1([-\tau,0]) : \xi(t^*-\tau) = 0\},$$

and its geometrical structure is clearer too: $E \cap C^1([-\tau,0])$ is just a *closed hyperplane* in $C^1([-\tau,0])$. These are the initial functions which generate solutions without discontinuities in $[0,\tau]$, and any other $\xi \in C^1([-\tau,0])$ may be written as $\xi(t^*) + [\xi(t) - \xi(t^*)]$, which means that the asymptotic expansion of u near t^* can be further simplified to $u(t) \simeq \xi(t^*)/\,|t-t^*| + AC$.

2.5. **Continuation beyond τ.** Assume that B is defined on a larger interval $[0,T)$, where $T > \tau$. As can be easily seen from the explicit formula (2.9)

$$u(t) = \xi(0) + B(t)\xi(t-\tau) - \int_0^t B(s)u'(s-\tau)ds, \quad 0 \leq t \leq \tau, \tag{2.16}$$

even for very smooth ξ, if B also contains singularities on the interval $[\tau, 2\tau]$ (for instance, if B is τ-periodic, a very important case), the function $B(s)u'(s-\tau)$ may not be integrable beyond τ.

Take, for instance, $\xi \equiv$ constant, $1/2 \leq \alpha < 1$, $B(t) = 1/\,|t-\tau/2|^\alpha$ on $[0,\tau]$ and extended periodically to all of $\mathbb{R}$. Then $u(t) = \xi(1+B(t)) = \xi$ on $[0,\tau]$, on $[\tau, 3\tau/2)$ the equality $u'(t) = B'(t)u(t-\tau)$ does hold and then

$$\begin{aligned} u(t) &= \xi(1+B(\tau)) + \xi \textstyle\int_\tau^t B(s)B'(s-\tau)ds \\ &= \xi\left(1 + B(\tau) + \frac{1}{2}\left[B(t)^2 - B(\tau)\right]\right), \quad \tau \leq t < \frac{3\tau}{2}, \end{aligned} \tag{2.17}$$

because of the periodicity of B. Since B^2 is not integrable, the solution cannot be extended beyond $3\tau/2$ in a meaningful way.

On the other hand, standard results of the general theory of functional differential equations imply that if B is differentiable on $[0,T)$ except at a unique singularity t^*, the solution can be extended to all $[0,T)$. The following theorem is stated in a simplified situation which enables us to give a direct proof.

Theorem 2.4. *Let $T > \tau$ (including $+\infty$), $0 < \alpha < 1$, let B_1 be given by (2.12) and let $m : [0,T) \to \mathbb{R}$ be continuously differentiable and let*

$$B(t) = B_1(t) + m(t), \quad 0 \leq t < T. \tag{2.18}$$

Let $\xi \in C^1([-\tau,0])$. Then the initial value problem

$$\begin{cases} \dfrac{d}{dt}\left[u(t) - B(t)u(t-\tau)\right] = -B(t)u'(t-\tau), \\ u(\theta) = \xi(\theta), \quad \tau \leq \theta \leq 0, \end{cases} \tag{2.19}$$

has a unique solution on $[0,T)$ belonging to L^p for every $p < 1/\alpha$, continuous on $[0,T)$ except at t^* and continuously differentiable at every $t \in [0,T)$ except t^* and $\tau + t^*$.

Proof. We already know that the expression (2.9) gives us an L^p solution $[0,\tau]$. Since $\xi \in C^1$, it is also continuously differentiable except at t^*. In order to extend it beyond τ, we go back to the original retarded presentation

$$u'(t) = B'(t)u(t-\tau),$$

which does not give any trouble for values $t \geq \tau$, since the "coefficient" $B'(t)$ is continuous on $[\tau,T)$. On $[\tau,2\tau]$ we can write

$$u(t) = u(\tau) + \int_\tau^t B'(s)u(s-\tau)ds, \quad \tau \leq t \leq 2\tau,$$

which is absolutely continuous on $(\tau,2\tau)$ and continuously differentiable except at $t = \tau + t^*$. $\square$

Remark 2.5. Since linear retarded functional differential equations are well-posed on L^p spaces (see [21]) and these equations have a well-known "smoothing effect" ([11]), the above result can be extended in a number of ways. For instance, if $B : [0,T) \to \mathbb{R}$ is L^p on $[0,\tau]$ and continuously differentiable on $[\tau,T)$, then the solution belongs to $L^p_{\text{loc}}(0,T)$, belongs to $W^{1,p}(\tau,2\tau)$, to $W^{2,p}(2\tau,3\tau)$ and so on.

2.6. **Linear perturbations.** The above analysis is easily adapted to the case

$$\begin{cases} u'(t) = \lambda u(t) + B'(t)u(t-\tau), \quad t > 0 \\ u(\theta) = \xi(\theta), \quad \tau \leq \theta \leq 0, \end{cases} \tag{2.20}$$

by first applying the Euler change of variables $v(t) = e^{-\lambda t}u(t)$, which gives

$$\begin{aligned} v'(t) &= -\lambda e^{-\lambda t}u(t) + e^{-\lambda t}\left[\lambda u(t) + B'(t)u(t-\tau)\right] \\ &= e^{-\lambda\tau}B'(t)v(t-\tau), \end{aligned} \tag{2.21}$$

and successively obtaining the *equivalent neutral formulation*

$$\begin{cases} \dfrac{d}{dt}\left[v(t) - e^{-\lambda\tau}B(t)v(t-\tau)\right] = -e^{-\lambda\tau}B(t)v'(t-\tau), \quad t > 0, \\ v(\theta) = e^{-\lambda\theta}\xi(\theta), \quad \tau \le \theta \le 0, \end{cases} \tag{2.22}$$

the *representation* for $v(t)$

$$\begin{aligned} v(t) = {} & e^{-\lambda\tau}B(t)e^{-\lambda(t-\tau)}\xi(t-\tau) + \xi(0) \\ & - \textstyle\int_0^t e^{-\lambda\tau}B(s)e^{-\lambda(s-\tau)}\left[-\lambda\xi(s-\tau) + \xi'(s-\tau)\right] ds \\ = {} & \xi(0) + e^{-\lambda t}B(t)\xi(t-\tau) - \textstyle\int_0^t e^{-\lambda s}B(s)\left[-\lambda\xi(s-\tau) + \xi'(s-\tau)\right] ds, \end{aligned} \tag{2.23}$$

and the *representation* for $u(t) = e^{\lambda t}v(t)$

$$\begin{cases} u(t) = e^{\lambda t}\xi(0) + B(t)\xi(t-\tau) \\ \qquad\quad + \int_0^t e^{\lambda(t-s)}B(s)\left[-\lambda\xi(s-\tau) + \xi'(s-\tau)\right] ds, \end{cases} \tag{2.24}$$

which is very similar to (2.9). The qualitative statements of theorem 1 and the asymptotic expansion near t^* are translated to this case without change.

Similar results can be written for non-autonomous versions of the above equation

$$\begin{cases} u'(t) = \lambda(t)u(t) + B'(t)u(t-\tau), \quad t > 0, \\ u(\theta) = \xi(\theta), \quad \tau \le \theta \le 0, \end{cases} \tag{2.25}$$

obtaining the representation

$$\begin{cases} u(t) = B(t)\xi(t-\tau) \\ \qquad\quad + \int_0^t e^{\Lambda(t)-\Lambda(s)}B(s)\left[-\lambda(s)\xi(s-\tau) + \xi'(s-\tau)\right] ds, \end{cases} \tag{2.26}$$

where $\Lambda(t)$ is a primitive of $\lambda(t)$ on $[0, \tau]$. It suffices that $\lambda \in L^1(-\tau, 0)$, thus allowing for singularities on the coefficient λ which give rise to very interesting interactions with the singularities of B.

2.7. **The nonlinear case. Using Alekseev's variation-of-constants formula.**

2.7.1. *A first nonlinear case.* We now generalize the results presented above to the "partially nonlinear" case, that is

$$\begin{cases} u'(t) = B'(t)g(t, u(t-\tau)), \quad 0 < t < \tau, \\ u(\theta) = \xi(\theta), \quad \tau \le \theta \le 0, \end{cases} \tag{2.27}$$

where g is C^1. By formally writing

$$B'(t)g(t,u(t-\tau)) = \frac{d}{dt}\left[B(t)g(t,u(t-\tau))\right] - B(t)\frac{d}{dt}\left[g(t,u(t-\tau))\right], \tag{2.28}$$

we see that the equivalent neutral equation is completely similar to those obtained in the previous section, that is

$$\begin{cases} \dfrac{d}{dt}\left[u(t) - B(t)g(t,u(t-\tau))\right] = -B(t)\dfrac{d}{dt}\left[g(t,u(t-\tau))\right], \quad t>0, \\ u(t) = \xi(t), \quad \tau \le t \le 0. \end{cases} \tag{2.29}$$

On $[0,\tau]$ we have (formally)

$$\begin{cases} u(t) = B(t)g(t,\xi(t-\tau)) + \xi(0) \\ \qquad\qquad - \int_0^t B(s)\dfrac{d}{ds}\left[g(s,\xi(s-\tau))\right] ds, \quad 0 \le t \le \tau. \end{cases} \tag{2.30}$$

But if $\xi \in W^{1,q}(-\tau,0)$ and g is C^1, $s \mapsto g(s,\xi(s-\tau))$ is also in $W^{1,q}(-\tau,0)$ and the integral actually is an absolutely continuous function. Therefore, the representation or "asymptotic expansion" $u(t) = B(t)g(t,\xi(t-\tau)) + AC$ is still valid.

If an additive term $\lambda u(t)$ appears in the right-hand side, a similar analysis can be performed by means of the change of variable $v(t) = e^{-\lambda t}u(t)$, although the nonlinearity $g(t,u(t-\tau))$ makes the integral representation much more complicated than (2.30).

2.8. **The fully nonlinear case.** Let us now analyze the "fully nonlinear" case, that is

$$\begin{cases} u'(t) = f(t,u(t)) + B'(t)g(t,u(t-\tau)), \quad 0<t<\tau \\ u(\theta) = \xi(\theta), \quad \tau \le \theta \le 0 \end{cases} \tag{2.31}$$

where f is C^2 and g is C^1. Its reduction to a "neutral form" is still possible:

$$\begin{cases} \dfrac{d}{dt}\left[u(t) - B(t)g(t,u(t-\tau))\right] \\ \qquad\qquad = f(t,u(t)) - B(t)\dfrac{d}{dt}\left[g(t,u(t-\tau))\right], t>0, \\ u(\theta) = \xi(\theta), \quad \tau \le \theta \le 0. \end{cases} \tag{2.32}$$

However, the presence of the term $f(t,u(t))$ makes the (formal) integration of both sides of the equation hard to deal with: instead of an explicit expression of u, it becomes an *integral equation* with u as the unknown, and it would be necessary to choose the right function space in which the equation not only made sense but had a unique fixed point as well. In any case, the neutral formulation can be used to give a precise meaning to the equation, but we will not follow this approach here.

Instead, we will change our strategy and make use of a very useful, but little-known mathematical device: *Alekseev's nonlinear variation of constants formula*. We now briefly recall this result in a very simple setting, which will suffice for our purposes. For more general statements and the proofs, see [12]:

Proposition 2.6 (Alekseev's formula)**.** *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be C^2 and $G : \mathbb{R} \to \mathbb{R}$ be L^1_{loc}. Let $y = \phi(t, t_0, \xi)$ represent the unique solution of the ODE*

$$\begin{cases} y' = f(t, y(t)), \\ y(t_0) = \xi, \end{cases} \tag{2.33}$$

and let $\Phi(t, t_0, \xi) = \partial_\xi \phi(t, t_0, \xi)$, where ∂_ξ denotes partial differentiation. Then ϕ is C^2, Φ is C^1 and the solution $u(t)$ of the so-called "perturbed problem"

$$\begin{cases} u' = f(t, u(t)) + G(t), \\ u(t_0) = \xi, \end{cases} \tag{2.34}$$

has the integral representation

$$u(t) = y(t) + \int_{t_0}^{t} \Phi(t, s, y(s))G(s)ds, \tag{2.35}$$

where $y(t) = \phi(t, t_0, \xi)$ is the "unperturbed" or "reference" solution.

Remark 2.7. $\Phi(t, t_0, \xi)$ satisfies $\Phi(t, t, \xi) = 1$.

Remark 2.8. Alekseev's formula is usually stated under stronger regularity conditions on G. However, it is very simple to check by direct differentiation that the function $u(t)$ defined by (2.35) is an absolutely continuous solution of the (Carathéodory) equation (2.34). Alekseev's formula is usually applied to the more ambitious setting of having G depending on t *and* u, which is typical of control theory. (2.35) then becomes an integral equation and a more delicate analysis is required.

Fortunately, we can consider the retarded term as an external "forcing"

$$G(t) = B'(t)g(t, \xi(t - \tau)), \tag{2.36}$$

and by setting $t_0 = 0$, $\xi = u(0) = \xi(0)$, $y(t) = \phi(t, 0, \xi)$, write (formally):

$$u(t) = y(t) + \int_0^t \Phi(t, s, y(s))B'(s)g(s, \xi(s - \tau))ds, \tag{2.37}$$

and integrate by parts:

$$\begin{aligned} u(t) &= y(t) + [\Phi(t, s, y(s))B(s)g(s, \xi(s - \tau))]_{s=0}^{s=t} \\ &\quad - \textstyle\int_0^t B(s)\dfrac{d}{ds}\left[\Phi(t, s, y(s))g(s, \xi(s - \tau))\right] ds \\ &= y(t) + \Phi(t, t, y(t))B(t)g(t, \xi(t - \tau)) \\ &\quad - \textstyle\int_0^t B(s)\dfrac{d}{ds}\left[\Phi(t, s, y(s))g(s, \xi(s - \tau))\right] ds. \end{aligned} \tag{2.38}$$

By the remark above, $\Phi(t, t, y(t)) = 1$. On the other hand, as we saw before, for $\xi \in W^{1,q}(-\tau, 0)$ and $g \in C^1$ the composite function $s \mapsto g(s, \xi(s-\tau))$ is also $W^{1,q}(-\tau, 0)$ and so is its product by the C^1 function $\Phi(t, s, y(s))$. Therefore, its derivative belongs to $L^q(-\tau, 0)$ and the indefinite integral, as in all the previous cases, is an absolutely continuous function. This means that the integration by parts is legitimate and we may state the following result, which is an extension of the previous ones. We may summarize the previous comments in the following way:

The initial value problem

$$\begin{cases} u'(t) = f(t, u(t)) + B'(t) g(t, u(t-\tau)), & 0 < t < \tau, \\ u(\theta) = \xi(\theta), \quad \tau \leq \theta \leq 0, \end{cases} \tag{2.39}$$

with $F \in C^2(\mathbb{R}^2)$, $g \in C^1(\mathbb{R}^2)$ and initial function ξ in $W^{1,q}(-\tau, 0)$ can be given a precise integral sense in $[0, \tau]$ by means of the neutral equivalent equation (2.32) and its unique solution u admits the integral representation (2.40)

$$u(t) = y(t) + B(t) g(t, \xi(t-\tau)) - \int_0^t B(s) \frac{d}{ds} \left[\Phi(t, s, y(s)) g(s, \xi(s-\tau)) \right] ds,$$

(where $y(t) = \phi(t, 0, \xi(0))$) as well as the "asymptotic expansion"

$$u(t) = B(t) g(t, \xi(t-\tau)) + AC, \tag{2.41}$$

which gives the qualitative picture of the behavior of the solution near singularities of B.

3. **The PDE case**

In order to avoid technicalities, let us consider the delayed linear heat equation with Neumann boundary conditions

$$(3.1) \quad (P_N) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = B'(t) u(t-\tau, x), & \text{for } (t, x) \in (0, +\infty) \times \Omega, \\ \frac{\partial u}{\partial n}(t, x) = 0, & \text{for } (t, x) \in (0, +\infty) \times \partial\Omega, \\ u(\theta, x) = \xi(\theta, x), & \text{for } (\theta, x) \in (-\tau, 0) \times \Omega, \end{cases}$$

where Ω is a connected domain of $\mathbb{R}^N, N \geq 1$, with smooth boundary, and concentrate on a simplified version of the single-singularity case

$$B(t) = \frac{a}{|t - t^*|^\alpha} + m(t), \tag{3.2}$$

with $m \in C^1([0, \tau])$, $\alpha \in (0, 1)$ and $t^* \in (0, \tau)$. It is well known (see, for instance, [10] or [22]) that on $[0, t^*)$ the initial value problem is well defined for continuous initial functions ξ and has a unique solution. The possibility of extending the solution beyond t^* will be discussed later. Also, other function spaces and boundary conditions are easily treated by these methods.

We will also assume that $B'(t) \geq 0$ (so that $a > 0$) on $[0, t^*)$. The reason for this restriction will be explained below.

3.1. **Separable solutions.** Assume that the initial function is separable: $u(x,t) = \xi(t)\phi_0(x)$ for $t \in [-\tau, 0]$ and $x \in \Omega$. It is then natural to look for solutions of the same type $u = w(t)\phi(x)$, thus obtaining

$$w'(t)\phi(x) = w(t)\Delta\phi(x) + B'(t)w(t-\tau)\phi(x).$$

In order to have a separable solution we divide by $w(t)\phi(x)$ and observe that the assumed identity

$$\frac{w'(t)}{w(t)} = \frac{\Delta\phi(x)}{\phi(x)} + B'(t)\frac{w(t-\tau)}{w(t)},$$

can only hold if there exists a real constant λ such that

$$\Delta\phi = \lambda\phi,$$

(that is, ϕ is an *eigenfunction* of Δ with the given boundary conditions, with associated eigenvalue λ) and w satisfies the delay-differential equation

$$\left\{ \begin{array}{l} w'(t) = \lambda w(t) + B'(t)w(t-\tau), \quad \text{for } t \geq 0, \\ w(\theta) = w_0(\theta), \quad \text{for } t \in [-\tau, 0], \end{array} \right. \tag{3.3}$$

which is of the type studied in Section 3.

This obviously requires that $\phi_0(x) = \phi(x)$ be already an eigenfunction. Assuming this is the case, we have an explicit representation of these separable solutions from Section 3, namely (2.24):

$$\begin{aligned} u(t,x) &= w(t)\phi(x) \\ &= B(t)\xi(t-\tau)\phi(x) \\ &+ \phi(x)\textstyle\int_0^t e^{\lambda(t-s)}B(s)\left[-\lambda\xi(s-\tau) + \xi'(s-\tau)\right] ds, \quad t \in [0,\tau]. \end{aligned} \tag{3.4}$$

If $\xi(t^* - t) > 0$ then $w(t) = B(t)\xi(t-\tau) \to \infty$ as $t \to t^*$, and the same will happen for the separable solution on the region $\{\phi > 0\}$, while $u(t,x) = w(t)\phi(x) \to -\infty$ as $t \to t^*$ when $\phi(x) < 0$. Clearly, the opposite behavior takes place when $\xi(t^* - t) < 0$. In any case, we have *instantaneous blow-up* outside the nodal region $\{x \in \Omega : \phi(x) = 0\}$, meaning that the *explosion time* is the same for all the points involved. The most important case from the practical viewpoint is that of $\phi(x) \equiv 1$, the first eigenfunction of Δ with Neumann boundary conditions.

3.2. **More general delay-PDEs with blowing-up solutions via comparison arguments.** Now enters the sign condition $B'(t) \geq 0$ on $[0, t^*)$, whose importance comes from the fact that some *comparison arguments* can be applied in this case, thus enlarging considerably the set of equations for which we get blowing-up solutions. Although our arguments also apply to the case of $(NLDDE)$ here we merely state a simple version of more general results for (NLP_N), which will be enough for our purposes

Proposition 3.1. *For $i = 1, 2$, consider the delayed reaction-diffusion equations*

$$(NLP_N)\quad \begin{cases} \frac{\partial u^i}{\partial t} - \Delta u^i = f^i(t, u^i(t,x), u^i(t-\tau,x)), & (t,x) \in (0,+\infty) \times \Omega, \\ \frac{\partial u^i}{\partial n}(t,x) = 0, & (t,x) \in (0,+\infty) \times \partial\Omega, \\ u^i(\theta,x) = \xi^i(\theta,x), & (\theta,x) \in (-\tau,0) \times \Omega, \end{cases}$$

where f^i are locally Lipschitz in all its arguments and nondecreasing in its third variable, i.e.

$$\begin{gathered} p^1 \leq p^2 \Longrightarrow f^1(t,u^1,p^1) \leq f^2(t,u^2,p^2), \\ \textit{for a.e. } t \geq 0, \textit{ for any } u^i, p^i \in \mathbb{R}. \end{gathered} \tag{3.5}$$

Let ξ^1 and ξ^2 be two initial functions, in $C([-\tau,0] : L^p(\Omega))$ for some $p \in [1,+\infty]$, ordered as follows:

$$0 \leq \xi^1(\theta,x) \leq \xi^2(\theta,x), \quad \textit{for any } \theta \in [-\tau,0] \textit{ and a.e. } x \in \Omega. \tag{3.6}$$

Then there exists the corresponding weak solutions $u^1(t,x)$, $u^2(t,x)$, in $C([-\tau, T^i_{\max}) : L^p(\Omega))$, for some $T^i_{\max} \in (0,+\infty]$, and they satisfy

$$0 \leq u^1(t,x) \leq u^2(t,x), \quad \textit{for all } t \in \left[-\tau, T^i_{\max}\right), \textit{ a.e. } x \in \Omega.$$

Proof. The existence of solutions is consequence of well-known results (see, e.g., the monographs [14], [10] and [22]). Most of the comparison results in the indicated literature are presented for the simper case in which $f^1 \equiv f^2$ (see also other general references such as [20]). The case of (3.5) with $f^1 \neq f^2$ is well known in the literature without delay (see, e.g. [6], [7] (Theorem 4.3) and the references indicated there) and can be easily adapted to the case of delayed equations (see, e.g. [13] Proposition1). $\square$

Other boundary conditions are possible. The condition to be taken into account is that $-\Delta$ with the given boundary condition generates a *positive semigroup* on the usual function spaces.

Applying this result to our case, we have the following

Theorem 3.2. *Assume that the initial function satisfies*

$$\xi(\theta,x) \geq \xi_0(\theta)\phi(x), \quad \textit{for } \theta \in [-\tau,0],\ x \in \Omega, \tag{3.7}$$

where ϕ is an eigenfunction of Δ with the Neumann boundary condition, $\xi_0 \in W^{1,q}(-\tau,0)$ and $\xi_0(t^* - \tau) > 0$. Assume (3.2) and let $f(t,u,p)$ be locally Lipschitz in all its arguments, nondecreasing in its third variable, and such that

$$\begin{gathered} p^1 \leq p^2 \Longrightarrow B'(t)p^1 \leq f(t,u,p^2), \\ \textit{for a.e. } t \geq 0, \textit{ for any } u, p^1, p^2 \in \mathbb{R}. \end{gathered} \tag{3.8}$$

Then, if $u(x,t)$ is the solution of

$$(NLP_N) \quad \begin{cases} \frac{\partial u}{\partial t} - \Delta u = f(t, u(t,x), u(t-\tau, x)), & (t,x) \in (0,+\infty) \times \Omega, \\ \frac{\partial u}{\partial n}(t,x) = 0, & (t,x) \in (0,+\infty) \times \partial\Omega, \\ u(\theta, x) = \xi(\theta, x), & (\theta, x) \in (-\tau, 0) \times \Omega, \end{cases}$$

we have

$$u(x,t) \geq \left[\frac{a}{|t-t^*|^\alpha} \xi_0(t-\tau) + n(t) \right] \phi(x), \tag{3.9}$$

where n is an absolutely continuous function on $[-\tau, 0]$. In particular, u blows up at some finite time $T_{\max} \leq t^*$ in the sense that

$$\lim_{t \to t^*} u(t,x) = \infty, \quad a.e. x \in \{x \in \Omega : \phi(x) > 0\}. \tag{3.10}$$

Proof. Let $w(t)$ denote the solution of the initial value problem (3.3):

$$\begin{cases} w'(t) = \lambda w(t) + B'(t) w(t-\tau), & \text{for } t \geq 0, \\ w(\theta) = \xi_0(\theta), & \text{for } \theta \in [-\tau, 0], \end{cases}$$

where λ is the eigenvalue associated to the eigenfunction ϕ. As before, the hypotheses on ξ_0 imply that $w(t)$ admits the asymptotic expansion

$$w(t) = B(t)\xi(t-\tau) + n(t),$$

where $n(t)$ is absolutely continuous. On the other hand, the comparison result stated above implies that $u(t,x) \geq w(t)\phi(x)$, and the result is proved. □

Remark 3.3. The theorem holds for the Dirichlet boundary condition without any change. For (possibly nonlinear) Robin boundary condition $\partial u/\partial n + k(t,x,u) = 0$, for some nondecreasing function $k(.,.,u)$ of u (a requirement imposed for the applicability of comparison arguments: see, for instance, [6]).

Remark 3.4. For both Dirichlet and Neumann boundary conditions, if the initial function satisfies $\xi(t,x) \geq \mu > 0$ in Ω, we can always choose ϕ to be the *first eigenfunction,* which does not change sign by the Krein-Milman theorem. We have then *instantaneous blow-up on the whole domain* Ω.

Remark 3.5. In the region $\{x \in \Omega : \phi(x) < 0\}$ the comparison argument does not give us any useful information, unless some symmetric condition $u(\theta,x) \leq \tilde{\xi}_0(\theta)\phi(x)$, $\tilde{\xi}_0(t^* - \tau) < 0$ holds.

3.3. **Continuation beyond t^*.** The question of existence of solutions on the whole interval $[0,\tau]$ is more delicate since it involves performing some kind of integration by parts in order to define a suitable notion of generalized solution, as in Sections 2 and 3. The special structure of the right hand side of our equation, however, simplifies the situation, since the method of steps is directly applicable. Using the standard notation of abstract evolution equations in Banach spaces X, our basic equation is written as follows

$$\begin{cases} u'(t) = Au(t) + B'(t)u(t-\tau), & \text{in } X, \text{ for } t \geq 0, \\ u(\theta) = \xi(\theta), & \text{for } \theta \in [-\tau, 0], \end{cases} \tag{3.11}$$

where $u(t)$ is the function $u(t)(x) = u(t,x)$, the same for ξ and A is the abstract operator on X associated to $-\Delta$ with Neumann boundary conditions. On the basic interval $[0, \tau]$ we may express the solution by means of the variation of constants formula:

$$u(t) = e^{At}\xi(0) + \int_0^t e^{A(t-s)}B'(s)\xi(s-\tau)ds, \quad 0 \leq t \leq \tau.$$

Assume, again, (3.2). The fact that B' is not a function (and so $u' \notin L^1(0, \tau : X)$) requires integration by parts, as in Section 3. By proceeding formally we arrive to a direct extension of equation (2.24), by substituting λ by A :

$$\begin{cases} u(t) = e^{At}\xi(0) + B(t)\xi(t-\tau) \\ \qquad + \int_0^t e^{A(t-s)}B(s)\left[-A\xi(s-\tau) + \xi'(s-\tau)\right] ds, \end{cases} \tag{3.12}$$

which we may use as definition of "generalized solution in $W^{-1,p'}(0, \tau : X)$" for some $p = p(\alpha) > 1$ small enough. As an illustration, let us state a simple sufficient condition for a "generalized solution in $W^{-1,p'}(0, \tau : L^2(\Omega))$" to exist:

Theorem 3.6. *Let $\xi \in C^2([-\tau, 0] \times \bar{\Omega})$ satisfying $\partial\xi/\partial n = 0$ on $\partial\Omega$ for all $\theta \in [-\tau, 0]$. Assume (3.2). Then the integral in (3.12) is well defined and the equation (3.11) has a "generalized solution in $W^{-1,p'}(0, \tau : L^2(\Omega))$" for some $p = p(\alpha) > 1$ small enough, and, so defined, at least, on $[0, \tau]$.*

Proof. The hypotheses imply that $\xi(t, .)$ belongs to the domain of A and the function $s \mapsto \Delta\xi(s-\tau, \cdot) + \partial_t\xi(s-\tau, \cdot)$ is continuous from $[0, \tau]$ into $C(\bar{\Omega})$. Therefore, its product by the L^2 function B is in L^2, and the integral is well defined (see Vrabie [19]). □

Remark 3.7. As mentioned at the beginning of this paragraph, a complete treatment of the existence, uniqueness and continuation of generalized solutions of the PDE problem is complicated and it is currently under study by the authors. Our goal in this paper is to present the basic results concerning blow-up results for the main equation and to suggest some of the difficulties involved in its treatment, avoiding the technicalities as much as possible. This is the reason why we have excluded the explicit use of the theory of distributions, although it obviously lies behind many of the arguments we have employed in a more loose way in the text.

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