

On a problem of slender, slightly hyperbolic, shells suggested by Torroja's structures

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Received 20 November 2008; accepted after revision 4 February 2009

Presented by Evariste Sanchez-Palencia

Abstract

A rigidification phenomenon for certain thin slender parabolic shells, with curvature in the transversal direction to the main length was addressed in a previous work. The main results are generalized here to the case when the shape of the middle surface is no longer cylindrical, but incorporates a very small curvature in the longitudinal direction (it is “slightly hyperbolic”). The structure of the basic equations changes, but (due to a coupling with the flexion rigidity) the main results hold true independently of the location of the characteristics. This problem constitutes a tentative modelling of certain remarkably rigid structures built by Torroja in the 1930s. *To cite this article: J.I. Díaz, E. Sanchez-Palencia, C. R. Mecanique 337 (2009).*

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Résumé

Sur un problème de coques élancées, légèrement hyperboliques, suggéré par les structures de Torroja. Un phénomène de rigidification pour un type de coques élancées cylindriques avec courbure dans la direction transversale à celle de la longueur principale a été considéré dans un travail précédent. Nous généralisons ici les principaux résultats au cas où la surface moyenne n'est plus cylindrique, mais a une petite courbure dans la direction longitudinale (elle est « légèrement hyperbolique »). Le type des équations change, mais (grâce à un couplage avec la rigidité de flexion) les principaux résultats restent valables indépendamment de la disposition des caractéristiques. Ce problème constitue une tentative de modélisation de certaines structures remarquablement rigides construites par Torroja dans les années 1930. *Pour citer cet article : J.I. Díaz, E. Sanchez-Palencia, C. R. Mecanique 337 (2009).*

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Keywords: Computational solid mechanics; Slender thin shells; Asymptotic behavior; Hyperbolic shells; V-shaped structures

Mots-clés: Mécanique des solides numérique; Coques minces élancées; Comportement asymptotique; Coques hyperboliques; Structures en V

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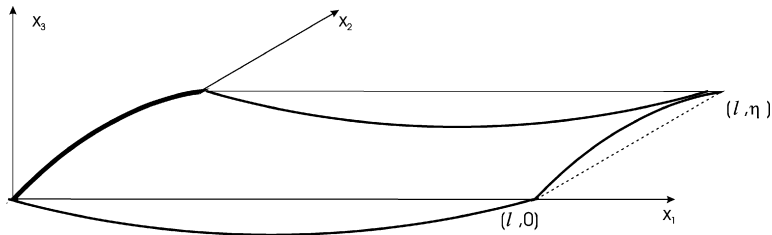


Fig. 1. Geometry under consideration.

1. Introduction

We consider the same problem of [1], Sections 1 and 2 with, in addition, a small curvature in the longitudinal direction. This we proceed to describe, but certain details will only be given in [1]. Let the small parameter ε denote the thickness of the shell; the shape of the middle surface is nearly a slender rectangle, $\Omega_\varepsilon = (0, l) \times (0, \eta)$ in the “plane of the parameters” $x = (x_1, x_2)$ which is assimilated to the middle surface (see Fig. 1).

We take $\eta = \varepsilon^{1/4}$. The curvature in the x_2 direction is equal to the constant b , whereas there is also a small curvature of order η^2 and sign opposite to that of b in the x_1 direction. This small curvature constitutes the difference with [1]; when it is not present, the shell is cylindrical. When it does, the shell is hyperbolic, with asymptotic curves forming a small angle of order $O(\eta)$. The shell is fixed by the boundary $x_1 = 0$. In the cylindrical case, the fixation ensures the geometric rigidity of the middle surface, which enters in the class of “inhibited shells” (see [2] or [3]). On the other hand, when the small longitudinal curvature is present, geometrical rigidity does no longer hold true as in the hyperbolic case it is only ensured in the “determinacy” triangle defined by the characteristics (i.e. the asymptotic curves) issued from the fixed boundary; this is obviously a part of Ω , which may not arrive to $x_1 = l$. The aim of this Note is to show that some coupling of the asymptotic (for small ε) expressions of the membrane and flexion rigidities allows one to prove that the asymptotic scheme and estimates hold true in the slightly hyperbolic case (at least concerning magnitude orders).

Denoting by $\tilde{u} = (\tilde{u}_1, \tilde{u}_2, \tilde{u}_3)$ the displacement (with components in the directions of x_1, x_2 and the normal to the surface, the kinematic boundary conditions are

$$0 = \tilde{u}_1 = \tilde{u}_2 = \tilde{u}_3 = \tilde{\partial}_1 \tilde{u}_3 \quad \text{on } \{0\} \times (0, \eta) \quad (1)$$

and the components of the strain tensor are

$$\begin{aligned} \tilde{\gamma}_{11}(\tilde{\mathbf{v}}) &= \tilde{\partial}_1 \tilde{v}_1 - \eta^2 \tilde{v}_3 \\ \tilde{\gamma}_{22}(\tilde{\mathbf{v}}) &= \tilde{\partial}_2 \tilde{v}_2 + b \tilde{v}_3 \\ \tilde{\gamma}_{12}(\tilde{\mathbf{v}}) &= \tilde{\gamma}_{21}(\tilde{\mathbf{v}}) = \frac{1}{2} (\tilde{\partial}_2 \tilde{v}_1 + \tilde{\partial}_1 \tilde{v}_2) \end{aligned} \quad (2)$$

Here we used the notation $\tilde{\partial}_\alpha = \frac{\partial}{\partial x_\alpha}$. As the shell is shallow, the tensor of variations of curvature has the components

$$\tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{v}}) = \tilde{\partial}_{\alpha\beta} \tilde{v}_3 \quad (3)$$

The variational formulation of the shell problem can be written

$$\varepsilon a(\mathbf{u}^\varepsilon, \mathbf{v}) + \varepsilon^3 b(\mathbf{u}^\varepsilon, \mathbf{v}) = \langle \mathbf{f}^\varepsilon, \mathbf{v} \rangle \quad (4)$$

The two bilinear forms on the space $\mathbf{V} = H^1(\Omega_\varepsilon) \times H^1(\Omega_\varepsilon) \times H^2(\Omega_\varepsilon)$ are then defined by:

$$a(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \int_{\Omega_\varepsilon} A^{\alpha\beta\lambda\mu} \tilde{\gamma}_{\alpha\beta}(\tilde{\mathbf{u}}) \tilde{\gamma}_{\lambda\mu}(\tilde{\mathbf{v}}) \, dx \quad (5)$$

$$b(\tilde{\mathbf{u}}, \tilde{\mathbf{v}}) = \int_{\Omega_\varepsilon} B^{\alpha\beta\lambda\mu} \tilde{\rho}_{\alpha\beta}(\tilde{\mathbf{u}}) \tilde{\rho}_{\lambda\mu}(\tilde{\mathbf{v}}) \, dx \quad (6)$$

where A and B denote the membrane and flexion coefficients. Moreover

$$\langle \mathbf{f}, \mathbf{v} \rangle = \varepsilon^3 \int_{\Omega_\varepsilon} F_3(x_1, x_2/\eta) \tilde{v}_3(x_1, x_2) \, d\mathbf{x} \tag{7}$$

Following [1] or [4], we perform the change of variables:

$$\begin{cases} \mathbf{x} = (x_1, x_2) \Rightarrow \mathbf{y} = (y_1, y_2) \\ y_1 = x_1, \quad y_2 = \eta^{-1}x_2 \end{cases} \tag{8}$$

so, the domain Ω_ε is transformed into $\Omega = (0, l) \times (0, 1)$ and

$$\partial_1 = \tilde{\partial}_1, \quad \partial_2 = \eta \tilde{\partial}_2; \quad \partial_\alpha = \frac{\partial}{\partial y_\alpha} \tag{9}$$

The change of unknowns

$$\begin{cases} \tilde{u}_1(\mathbf{x}) = \eta^6 u_1(\mathbf{y}) \\ \tilde{u}_2(\mathbf{x}) = \eta^5 u_2(\mathbf{y}) \\ \tilde{u}_3(\mathbf{x}) = \eta^4 b^{-1} u_3(\mathbf{y}) \end{cases} \tag{10}$$

allows one to write the variational problem under the form:

Problem Π_ε . Find $\mathbf{u}^\varepsilon \in \mathbf{V}$ satisfying

$$a^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}) = \int_{\Omega} F_3(y_1, y_2) v_3(y_1, y_2) \, d\mathbf{y} \tag{11}$$

$\forall v \in \mathbf{V}$, where

$$a^\varepsilon(\mathbf{u}^\varepsilon, \mathbf{v}) \stackrel{\text{def}}{=} \int_{\Omega} A^{\alpha\beta\lambda\mu} \gamma_{\alpha\beta}^\varepsilon(\mathbf{u}^\varepsilon) \gamma_{\lambda\mu}^\varepsilon(\mathbf{v}) \, d\mathbf{y} + \int_{\Omega} B^{\alpha\beta\lambda\mu} \rho_{\alpha\beta}^\varepsilon(\mathbf{u}^\varepsilon) \rho_{\lambda\mu}^\varepsilon(\mathbf{v}) \, d\mathbf{y} \tag{12}$$

and

$$\gamma_{11}^\varepsilon(\mathbf{v}) = \partial_1 v_1 - v_3 \tag{13}$$

$$\gamma_{12}^\varepsilon(\mathbf{v}) = \gamma_{21}^\varepsilon(\mathbf{v}) = \eta^{-1} \frac{1}{2} (\partial_2 v_1 + \partial_1 v_2) \tag{14}$$

$$\gamma_{22}^\varepsilon(\mathbf{v}) = \eta^{-2} (\partial_2 v_2 + v_3) \tag{15}$$

$$\rho_{11}^\varepsilon(\mathbf{v}) = \eta^2 \partial_1^2 v_3 \tag{16}$$

$$\rho_{12}^\varepsilon(\mathbf{v}) = \rho_{21}^\varepsilon(\mathbf{v}) = \eta \partial_1 \partial_2 v_3 \tag{17}$$

$$\rho_{22}^\varepsilon(\tilde{\mathbf{v}}) = \partial_2^2 v_3 \tag{18}$$

Defining asymptotic expansions in powers of η for u^ε , γ^ε and the corresponding membrane stresses T^ε (see [1]), we see that the terms tending to infinity should vanish. This lead to

$$\partial_2 u_1 + \partial_1 u_2 = 0 \tag{19}$$

for the limit (leading order term) and

$$\partial_2 u_2 + u_3 = 0 \tag{20}$$

for the two leading order terms. Then, proceeding as in [1] (end of Section 2), the limit problem is

$$a^0(\mathbf{u}, \mathbf{v}) \stackrel{\text{def}}{=} \int_{\Omega} \frac{1}{C_{1111}} (\partial_1 u_1 - u_3) (\partial_1 v_1 - v_3) \, d\mathbf{y} + \int_{\Omega} B^{2222} \partial_2^2 u_3 \partial_2^2 v_3 \, d\mathbf{y} \tag{21}$$

in a space of functions including the constraints (19) and (20). Here, and in the sequel, irrelevant constants will be taken equal to 1.

2. Main result

Because of the constraints (19) and (20), the limit may be described in terms of a potential φ :

$$v_1 = \partial_1 \psi, \quad v_2 = -\partial_2 \psi, \quad v_3 = -\partial_2^2 \psi \quad (22)$$

with

$$\psi(0, y_2) = \partial_1 \psi(0, y_2) = 0 \quad (23)$$

We denote by G the completion of the set of the smooth functions defined on Ω satisfying (23) with the norm

$$\|\psi\|_G^2 = \int_{\Omega} |\square\psi|^2 dy + \int_{\Omega} |\partial_2^4 \psi|^2 dy \quad (24)$$

where \square denotes the D'Alembertian

$$\square = \partial_1^2 - \partial_2^2 \quad (25)$$

Obviously, because of the completion process, the exact definition of the set is irrelevant; “smooth” may be understood in the sense “of class C^2 ”, for instance. On the other hand, the fact that $\|\psi\|_G$ is a norm is not so evident; the proof will be given later, at the same time as the proof of Theorem 1 below. The limit problem (21) then becomes

Problem Π_0 . Find $\varphi \in G$ such that, $\forall \psi \in G$

$$\int_{\Omega} \square\varphi \square\psi dy + \int_{\Omega} \partial_2^4 \varphi \partial_2^4 \psi dy = - \int_{\Omega} F \partial_2^2 \psi dy. \quad (26)$$

Our main result is the following:

Theorem 2.1. *There exist $c > 0$ such that*

$$\|\psi\|_G \geq c \|\psi\|_{L^2(0,l;H^4(0,1))} \quad \text{for any } \psi \in G \quad (27)$$

This ensures the existence and uniqueness of the solution with (for instance) $F \in L^2(\Omega)$. Because of the presence of $\partial_2^4 \psi$ in the norm of G , obvious equivalence of norms in $H^4(0, 1)$ show that it is sufficient to prove:

Lemma 2.2. *There exist $c > 0$ such that*

$$\|\psi\|_G \geq c \|\psi\|_{L^2(0,l;L^2(0,1))} \quad (28)$$

for any $\psi \in G$.

Remark 2.3. The above lemma shows a remarkable fact: if $\square\psi = 0$ in the rectangle Ω we can extend the uniqueness of solution beyond the usual cone of dependence once we know that in the “spatial variable” we have that $\partial_2^4 \psi = 0$. We also point out that, as in [1], the solution $\varphi \in G$ of problem Π_0 satisfies a variational principle which is very implicit for the formulation Π_ε .

3. First decomposition of $L^2(\Omega)$

Let us denote by K the subspace of $L^2(\Omega)$ formed by the functions of the form:

$$\psi(y_1, y_2) = A(y_1) + B(y_1)y_2 + C(y_1)\frac{y_2^2}{2} + D(y_1)\frac{y_2^3}{6} \quad (29)$$

with A, B, C, D in $L^2(0, l)$, which is clearly closed. By identifying $L^2(\Omega)$ with $L^2(0, l; L^2(0, 1))$, the decomposition of $L^2(\Omega)$ as $K \times K^\perp$ is associated with the decomposition of $L^2(0, 1)$ into the subspace of polynomials of degree 3 and its orthogonal.

Classically, $\|\partial_2^4 \psi\|_{L^2(0,1)}$ is the norm of ψ in $H^4(0,1)/\mathbb{R}^4$, i.e. in $H^4(0,1)$ up to polynomials of degree 4; from the expression of the norm in G we have (where c denotes various constants)

$$\|\psi\|_G \geq c \|\psi\|_{L^2(0,l;H^4(0,1)/\mathbb{R}^4)} \geq c \|\psi\|_{L^2(0,l;L^2(0,1)/\mathbb{R}^4)} \tag{30}$$

But this last expression is precisely $\|\psi^{K^\perp}\|_{L^2(\Omega)}$, so that

$$\|\psi\|_G \geq c \|\psi^{K^\perp}\|_{L^2(0,l;L^2(0,1))} \tag{31}$$

for any $\psi \in G$. This is the desired inequality “up to the component in K ”.

4. Second decomposition of $L^2(\Omega)$

Let us now consider another decomposition of $\psi \in G$ (or rather an smooth function of the set which the completion is G). We shall only develop the proof in the case when $1 < l < 2$. This amounts to saying that in the forthcoming explicit construction of the solution of the wave equation with given boundary values, there is only one reflection. The case $l > 2$ with several reflections is analogous but cumbersome, whereas the case $l < 1$ is simpler by the absence of reflections. Let

$$\psi = \psi_1 + \psi_2 \tag{32}$$

where ψ_1 and ψ_2 are defined by:

$$\begin{cases} \square \psi_1 = 0 & \text{in } \Omega = (0, l) \times (0, 1) \\ \psi_1(0, y_2) = \partial_1 \psi_1(0, y_2) = 0 \\ \psi_1(y_1, 0) = \psi(y_1, 0) \\ \psi_1(y_1, 1) = \psi(y_1, 1) \end{cases} \tag{33}$$

and

$$\begin{cases} \square \psi_2 = \square \psi & \text{in } \Omega = (0, l) \times (0, 1) \\ \psi_2(0, y_2) = \partial_1 \psi_2(0, y_2) = 0 \\ \psi_2(y_1, 0) = 0 \\ \psi_2(y_1, 1) = 0 \end{cases} \tag{34}$$

We note that, since $\psi \in G$, functions ψ_1 and ψ_2 are both well defined as solutions of mixed problems for the wave equation (with initial conditions in $y_1 = 0$ and boundary conditions in $y_2 = 0$ and $y_2 = 1$). From classical estimates for mixed problems for the wave equation it follows that

$$\|\psi\|_G \geq c \|\square \psi\|_{L^2(\Omega)} \geq c \|\psi_2\|_{L^2(\Omega)} \tag{35}$$

We are now giving an explicit expression for ψ_1 . Let us extend both ψ and ψ_1 with value 0 for $y_1 < 0$; they are then defined on $(-\infty, l) \times (0, 1)$; see Fig. 2.

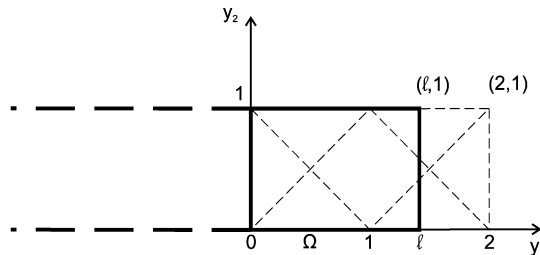


Fig. 2. Extension of ψ and ψ_1 .

It is then easily seen that ψ_1 can be written:

$$\psi_1(y_1, y_2) = \psi(y_1 - y_2, 0) + \psi(y_1 + y_2 - 1, 1) - \psi(y_1 + y_2 - 2, 0) - \psi(y_1 - y_2 - 1, 1) \tag{36}$$

In order to check this expression, we recall that it was supposed that $1 < l < 2$, so that the first line amounts to the “propagation of the boundary data”, and the second one to their reflections (there are not successive reflections as $l < 2$).

Let us now consider the set of functions $U(y_1, y_2)$ defined on Ω of the form:

$$U(y_1, y_2) = f(y_1 - y_2) + g(y_1 + y_2 - 1) - f(y_1 + y_2 - 2) - g(y_1 - y_2 - 1) \quad (37)$$

with f and g smooth functions defined on $(-\infty, l)$ and vanishing for $y_1 < 0$. Let us denote by K' its completion with the norm of $L^2(\Omega)$. It is easily checked that *locally*, convergence in $L^2(\Omega)$ of elements of the form (37) implies L^2 convergence of both f and g in L^2 , but there is some trouble concerning the points $(l, 0)$ and $(l, 1)$. This is concerned with square-integrability in the vicinity of an extremity of a segment versus in a corner of the plane. Anyway, the next lemma is easily checked, and sufficient for our purpose:

Lemma 4.1. *The elements of K' are of the form (33) with f and g of class $L^2(-\infty, \alpha)$ (vanishing for negative y_1), for any $\alpha < l$.*

As $\psi_1 \in K'$, the previous estimate (35) is the desired inequality “up to the component in K' ”.

5. Proof of Theorem 2.1

According to the two previous sections, the desired estimate (28) was proved “either up to the component in K or K' ”. Thus it suffices to prove that the intersection of K and K' reduces to the zero element of $L^2(\Omega)$. This is easily done. Let us consider ψ as an element of K' ; for fixed $0 < y_1 < 1/2$, as the corresponding f and g vanish for negative values of the argument, ψ vanishes in the vicinity of $y_2 = 1/2$ and is a polynomial for $0 < y_2 < 1$. So, ψ vanishes for $0 < y_1 < 1/2$, and f and g vanish for their arguments less than $1/2$. Repeating the reasoning, they also vanish for $0 < y_1 < 1$, and so on. Theorem 1 is proved.

6. Case when the shell has an edge with slight folding

There is an interesting variant of the previous problem, when the shell has a slight folding along a curve $x_2 = c$. This is an attempt to modelling elements of shell which, when pasted one after other in the x_2 direction, constitute the very elemental piece of the “Torroja’s structure” [5,6]. It should be noted that the first essays by Torroja before of constructing his roofs (reported in [5]) were precisely concerned with a unique element of this kind.

Once more, the case when the elements of shell are cylindrical was addressed in [1], Section 3. The modifications with respect to the case without folding apply exactly to the present case of small curvature in the longitudinal direction. The angle of the folding is small of order η . When passing to the variables and unknowns y and u , the kinematic conditions along the folding write:

$$u_1^+ = u_1^-, \quad u_3^+ = u_3^-, \quad u_2^+ - u_2^- = u_3 \quad (38)$$

where the symbols $+$ and $-$ denote the traces on both sides of the folding. In terms of the potential ψ , they become

$$\begin{cases} \psi^+ = \psi^- \\ \partial_2^2 \psi^+ = \partial_2^2 \psi^- \\ 2\partial_2^2 \psi = \partial_2 \psi^+ - \partial_2 \psi^- \end{cases} \quad (39)$$

It then appears that the space for ψ in the case of the folding is a subspace of the product of the spaces for the two parts of the shell. As the inequality of Theorem 2.1 holds true for each part, it clearly holds for the product and then for a part of it.

Acknowledgement

Research partially supported by the project MTM2008-06208 of the Ministerio de Ciencia e Innovacion, Spain.

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