On gradient estimates and other qualitative properties of solutions of nonlinear non autonomous parabolic systems.

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February 3, 2009

Abstract

We prove several uniforme L^1-estimates on solutions of a general class of one-dimensional parabolic systems, mainly coupled in the diffusion term, which, in fact, can be of degenerate type. They are uniform in the sense that they don't depend on the coefficients, nor on the size of the spatial domain. The estimates concern the own solution or/and its spatial gradient. This paper extends some previous results by the authors to the case of nonautonomous coefficients and possibly non homogeneous boundary conditions. Moreover, an application to the asymptotic decay of the L^1-norm of solutions, as $t \to +\infty$, is also given.

Presentado por Amable Liñán.
Recibido: 13 de enero de 2009.
Aceptado: ¿¿¿¿¿¿¿¿¿¿¿¿¿¿.
Palabras clave / Keywords: uniform gradient estimates, quasilinear parabolic onedimensional systems, uniform L^1-estimates independent on the spatial domain
Mathematics Subject Classifications: 35K45, 35K65

Estimaciones sobre el gradiente y otras propiedades cualitativas de las soluciones de sistemas parabólicos no lineales no autónomos

Resumen: En este artículo se obtienen varias estimaciones uniformes en L^1 para las soluciones de ciertos sistemas parabólicos no lineales que pueden estar acoplados en los términos de difusión y que, de hecho, puede ser de tipo degenerado.

Tales estimaciones son uniformes en el sentido de que no dependen de los coeficientes del sistema, ni del tamaño del dominio espacial. Las estimaciones se refieren a la norma L^1 de la propia solución o/y de su gradiente espacial. Este trabajo extiende, al caso de coeficientes no autónomos y a posibles condiciones de contorno no homogéneas, ciertos resultados previos de los autores. Además, se ofrece una aplicación al estudio del decaimiento de la norma L^1 de la solución, cuando $t \to +\infty$.

1 Introduction

The main goal of this paper is double: by one hand, we shall extend, to the case of nonautonomus coefficients and with possibly nonhomogeneous boundary conditions, a previous gradient estimate ([11]) for solutions of nonlinear parabolic systems. by the authors. On the other hand, we shall obtain here some L^1-estimates on the norm of the own vectorial solution and to give some application to the decay of to this norm, as $t \to +\infty$.

To be more precise, we consider the following boundary value problem for quasilinear parabolic systems of the type

$$\mathbf{u}_t = (\mathbf{A}(t,x,\mathbf{u})\mathbf{u}_x)_x + \mathbf{B}(t,x,\mathbf{u},\mathbf{u}_x) + \mathbf{f}(t,x), \tag{1}$$

on $Q = \Omega \times (0,T)$ where $\Omega = (0,l)$. We add an initial condition

$$\mathbf{u}(0,x) = \mathbf{u}_0(x),\ x \in \Omega, \tag{2}$$

and some boundary conditions, which, in order to fix ideas, we assume, for instance,

$$\mathbf{u}_x(t,0) = \mathbf{0}, \mathbf{u}(t,l) = \mathbf{h},\ 0 < t < T. \tag{3}$$

Here $\mathbf{u} = (u_1, ..., u_n)$ is the unknown vector valued function and $\mathbf{h} = (h_1, ..., h_n)$ is, for instance, a given constant vector. To simplify the exposition, we assume that the matrix $\mathbf{A}$ is diagonal $\mathbf{A} = (A_{ii})$ but that it may depend on all the components of $\mathbf{u}$ and thus coupling the different scalar equations. We assume that the vectors $\mathbf{f}$ and $\mathbf{u}_0$ are given and satisfy some suitable regularity conditions.

As in ([11]), we shall pay a special attention to the case of degenerate systems: i.e. to the case in which the corresponding matrix $\mathbf{A}$ is semidefinite positive and in fact vanishes at certain critical values of the sought vector solution $\mathbf{u}_{cr}$, i.e. $\mathbf{A}(t,x,\mathbf{u}_{cr}) = \mathbf{0}$. This contrast with the *uniformly parabolic* case in which the matrix $\mathbf{A}$ is definite positive (and that some times is denoted as *regular* case).

Systems the same as (1) frequently appear in many different contexts arising in biology, chemistry or filtration problems. Our particular motivation corresponds to the case in which system (1) describes the discharge of a laminar hot gas in a stagnant colder atmosphere of the same gas under assumptions the boundary layer approximation (see [4]-[9], [29]-[33]). In that case $n = 2$, u_1 is the gas temperature, and u_2 is the horizontal component of the velocity. The existence and uniqueness of weak solutions for this spacial case were proved in [4, 6, 7] under suitable conditions on the data (which guarantee strong positiveness of the matrix $\mathbf{A}$ on the considered vector solution).

We shall not discuss here any question related with the existence and regularity of solutions of boundary initial problems for system (1) under the structural conditions on the coefficients which will be indicated in the next section. By the contrary, we shall always assume that the considered problem has a solution, *regular enough*. We recall that some existence and uniqueness results for the uniformly parabolic case can be find, for instance, in [26, 27] and [2] where the reader can find a long list of references under the essential condition that $\mathbf{A}$ is a positive definite matrix.

The degeneration of the matrix $\mathbf{A}$ ($\mathbf{A}(\mathbf{u}_{cr}) = \mathbf{0}$ at certain critical values $\mathbf{u}_{cr}$ of the sought vector solution) causes the necessity to understand the solution in a weaker form as well as several *localization properties* of solutions such as the *finite speed of propagations*, *the waiting time effect*, etc. (see, e. g. [12]). This creates some significant difficulties in order to get some existence and uniqueness results on weaker classes of solutions. Nevertheless, many results are known to this respect (see, for instance, the treatment of systems of degenerate equations made in [1]). We point out, that, as in the scalar case, most of the times the weak solutions of degenerate systems (1) are found by means of passing to the limit on classical solutions of suitable uniformly parabolic auxiliary systems obtained by some approximation arguments. In this way, some regularity properties of the weak solutions are obtained by proving it first for the approximate systems and then by passing to the limit. That was also the philosophy proposed already by E. Hopf [23] in 1950 to study the Burgers equation by passing to the limit, as $\mu \to 0$ on the solutions of $u_t + uu_x = \mu u_{xx}$.

As mentioned, we shall obtain here a $L^1(\Omega)$ estimate on the (spatial) gradient $\mathbf{u}_x$ which is, in some sense "uniform" since it will not depend on the matrix $\mathbf{A}$ (nor on the vector $\mathbf{B}$). In fact, our main result (see Theorem 2.3) if firstly obtained for a component u_i (for some $i = 1, ..., n$) of vector solution $\mathbf{u}$ and shows that, under suitable conditions on the associated component of the data, we get the estimate

$$\int_0^l |u_{ix}(t,x)|dx \leq \int_0^l |u_{ix}(0,t)|dx + \int_0^t \int_0^l |f_{ix}(\tau,x)|dxd\tau, \tag{4}$$

for any $t \in (0, T]$. Obviously, if the conditions holds for any $i = 1, ..., n$ then we conclude that

$$\|\mathbf{u}_x\|_{L^\infty(0,T:\ \mathbf{L}^1(\Omega))} \leq \|\mathbf{u}_{0x}\|_{\mathbf{L}^1(\Omega)} + \|\mathbf{f}_x\|_{L^1(0,T:\ \mathbf{L}^1(\Omega))},$$

but notice that this property does not imply (4).

This estimate remains true for other boundary conditions. The extension to higher dimensions and to other quasilinear equations including the case of diffusions with nonlinear terms on the gradient, under suitable conditions on the coefficients, will be the object of a separate work by the authors ([10]). We prove that this property implies a compactness criteria, for a sequence of solutions leading to the limit (in a suitable weak topology) satisfies, at least that

$$TV(u_i(t,.)) \leq TV(u_{0i}(.)) + \int_0^t TV(f_i(\tau,.))d\tau, \tag{5}$$

where $TV(\varphi(t,.))$ means the total variation of the Radon measure $\varphi_x(t,.)$. This kind of arguments can be used, for instance, to show that (5) or (4) remains true for the case of the Cauchy problem associated to (1) (i.e. when the spatial domain Ω is $(-\infty,+\infty)$). We also present here an application to the study of some stationary systems associated to (1) (for instance by applying the implicit Euler scheme to (1)).

The main idea to prove estimate (4) consists in multiplying the $i-$equation by $-\frac{\partial\phi(u_{ix},\theta)}{\partial x}$, where $\phi(r,\theta) \to sign_0(r)$ as $\theta \to 0$, and to show that other contributions different to the ones arising in (4) converge to zero when $\delta \to 0$ under suitable conditions on the coefficients. It should be noted that the trick of using a sequence of test functions converging to the sign functions has been very familiar in the realm of elliptic, parabolic and first order hyperbolic equations for several decades in order to prove simpler estimates of the type that we shall get for systems in the next Section)(see, for instance, [25] and [13]). Nevertheless, as far as we know, the use of such test function seems to be new in the literature. Although a formal integration by parts links our method with the one already used in the literature consisting of differentiating the equation and applying L^1-techniques to u_{ix} (see, for instance [19] and [35]), our method has several advantages: it needs less regularity on the coefficients of the diffusion operator, it applies to Dirichlet boundary conditions (avowing some complicated arguments on the value of the second order operator on the boundary) and it does not involve any constant in the final estimate (4).

We send the reader to ([11]) for some comments on this property and the study of some exact solutions of the scalar case (as, for instance, the so-called, *Barenblatt solutions* of the porous media equation), and for some illustrative applications of these results to some special problems as it the case of the discharge of a hot gas mentioned before, the case of one dimensional two phase filtration described by the degenerate parabolic equation $s_t - (a(s)s_x)_x = V(t)b(s)_x$, and the (possibly degenerate) p-Laplacian type equation $u_t - (\Phi(u_x))_x = f$ where Φ is a continuous increasing real function. The reader can find also in ([11]) some applications to some first order Hamilton-Jacobi type system $\mathbf{u}_t + \mathbf{C}(\mathbf{u},\mathbf{u}_x) = \mathbf{f}(t,x), x \in \mathbb{R}$ (including some conservation laws).

A second goal of this paper is to show that an analogous uniform estimate (but simpler) holds for the vector solutions $\mathbf{u}$ of system (1). More precisely, we shall prove that

$$\int_0^l |u_i(t,x) - h_i|dx \leq \int_0^l |u_i(0,x) - h_i|dx + \int_0^t \int_0^l |f_i|\, dxds, i = 1,...,n. \tag{6}$$

In contrast with the above gradient estimates, some previous results on it (by other authors) can be found in the literature although not exactly under our general formulation (see, e.g. [25, 28]) and mostly only concerning the case of scalar equations. Moreover, we shall show that this estimate can be very useful to get some *uniform* asymptotic convergence, as $t \to +\infty$, of $\mathbf{u}(t,x)$ to the stationary solution $\mathbf{u} \equiv \mathbf{h}$, i.e. with a convergence rate which is independent on l.

2 Structural assumptions and main results

In this section we deal with (regular enough) solutions of the system

$$\mathbf{u}_t = (\mathbf{A}(t,x,\mathbf{u})\mathbf{u}_x)_x + \mathbf{B}(t,x,u,u_x) + \mathbf{f}(t,x), \tag{7}$$

on $Q = \Omega \times (0,T)$ where $\Omega = (0,l)$ with the auxiliary conditions

$$\mathbf{u}_x(t,0) = \mathbf{0},\ \ \mathbf{u}(t,l) = \mathbf{h},\ \ t \in (0,T), \tag{8}$$

$$\mathbf{u}(0,x) = \mathbf{u}_0(x),\ \ x \in \Omega, \tag{9}$$

where $\mathbf{u} = (u_1, ..., u_n)$ is the unknown vector valued function, $\mathbf{h} = (h_1, ..., h_n)$ is a constant given vector. On the data we assume at least that

$$\mathbf{f} \in \mathbf{L}^1(Q) \text{ and } \mathbf{u}_0 \in \mathbf{L}^1(\Omega). \tag{10}$$

Note that our formulation is slightly different than the one presented in ([11]): here the coefficients may depend on t and/or x and $\mathbf{h}$ is not necessarily zero, among other details.

We assume that

$$(H_{\mathbf{A}}) \begin{cases} |\frac{\partial A_{ii}}{\partial u_k}(t,x,\mathbf{u})| \leq C < \infty,\ \ i,k = 1,...,n \text{ for any } (t,x,\mathbf{u}) \in [0,T] \times \Omega \times \mathbb{R}^n, \\ \left\| \frac{\partial A_{ii}}{\partial x}(.,.,\mathbf{u}) \right\|_{L^2((0,T)\times\Omega)} \leq C(T,l) < \infty, i = 1,...,n \text{ for any } \mathbf{u} \in \mathbb{R}^n. \end{cases} \tag{11}$$

Notice that

$$(\mathbf{A}(t,x,\mathbf{u})\mathbf{u}_x)_x \equiv \mathbf{A}\mathbf{u}_{xx} + (\frac{\partial \mathbf{A}}{\partial \mathbf{u}}\mathbf{u}_x)\mathbf{u}_x + \frac{\partial \mathbf{A}}{\partial x}\mathbf{u}_x,$$

and thus, vectorial equation (7) can be rewritten as a system of scalar equations

$$u_{it} = A_{ii}u_{ixx} + \sum_{k=1}^{n} \frac{\partial A_{ii}}{\partial u_k} u_{kx}u_{ix} + \frac{\partial A_{ii}}{\partial x} u_{ix} + B_i + f_i,\ \ i = 1,...,n. \tag{12}$$

In most of the cases we shall assume that $\mathbf{A} = (A_{ij})$ is diagonal $\mathbf{A} = (A_{ii})$ and that $\mathbf{A}$ is a semi-definite positive matrix, i.e. such that

$$a_0|\vec{\xi}|^2 \leq (\mathbf{A}(t,x,\mathbf{p})\vec{\xi},\vec{\xi}) = \sum_{i=1}^{n} A_{ii}\xi_i^2 \leq a_1|\vec{\xi}|^2 < \infty, \tag{13}$$

for any $(\vec{\xi}, \mathbf{p}) \in \mathbb{R}^{2n}, (t,x) \in Q$ and for some $0 \le a_0 \le a_1$. We recall that the case $0 < a_0$ corresponds to uniformly parabolic systems and that $0 = a_0$ arises for degenerate systems.

Since the proof of our main estimate (4) will be obtained firstly for a single component u_i (for some $i = 1, ..., n$) of the vector solution $\mathbf{u}$ we shall assume that the main coupling among the equations of the system comes from the diffusion terms and that, by the contrary, the vectorial lower order term $\mathbf{B}(t, x, \mathbf{u}, \mathbf{u}_x)$ is, in some sense, weakly coupled. The concrete structural assumptions on $\mathbf{B}(t, x, \mathbf{u}, \mathbf{u}_x)$ we shall assume start by the decomposition in a purely absorption part and a convective part

$$\mathbf{B}(t, x, \mathbf{u}, \mathbf{u}_x) = \mathbf{B}^a(t, x, \mathbf{u}) + \mathbf{B}^c(t, x, \mathbf{u}, \mathbf{u}_x).$$

On the components of the absorption part of $\mathbf{B}$ we shall assume that

$$B_i^a(t, x, \mathbf{r}) sign(r_i - h_i) \le 0, \text{ for any } (t,x) \in Q \text{ and } \mathbf{r} \in \mathbb{R}^n, \tag{14}$$

or / and

$$B_i^a(t, x, \mathbf{r}) = B_i^a(t, r_i), B_i^a(t, h_i) = 0, B_i^a \in C^1 \text{ and } \frac{\partial B_i^a(t, r_i)}{\partial r_i} \le 0, \tag{15}$$

$$\text{for any } t \in (0, T) \text{ and any } r_i \in \mathbb{R}.$$

On the components of the convective part of $\mathbf{B}$ we shall assume that

$$\begin{cases} |B_i^c(t, x, \mathbf{u}, \mathbf{p})| \le B_0(t,x)|p_i|^{\gamma_i} \text{ for some } \gamma_i > 1/2 \text{ and some } B_0 \in L^2(Q), \\ \text{for any } (t,x) \in Q \text{ and any } (\mathbf{u}, \mathbf{p}) \in \mathbb{R}^{2n}. \end{cases} \tag{16}$$

or

$$\begin{cases} B_i^c(t, x, \mathbf{u}, \mathbf{u}_x) = -\frac{\partial \mathbf{C}(\mathbf{u})}{\partial \mathbf{u}} \mathbf{u}_x, \ \ \mathbf{C} \in C^2, \\ \mathbf{C}(\mathbf{u}) = (C_1(u_1), C_2(u_2), ..., C_n(u_n)). \end{cases} \tag{17}$$

2.1 L^1- estimate: uniform asymptotic behavior (as $t \to +\infty$) with respect to l, when $a_0 > 0$.

Our first result shows an uniform L^1 estimate on the own vectorial solution $\mathbf{u}$.

Theorem 2.1 *Assume that there exists a component $i_0 \in \{1, ..., n\}$ for which (14) and $B_{i_0}^c$=0 hold. Let $\mathbf{u} = (u_1, ..., u_n)$ be a weak solution of the problem (7) -(9). Then, for any $t \in [0, T]$ we have the estimate*

$$\int_0^l |u_{i_0}(t,x) - h_{i_0}| dx \le \int_0^l |u_{i_0}(0,x) - h_{i_0}| dx + \int_0^t \int_0^l |f_{i_0}| \, dx ds. \tag{18}$$

Remark 2.1 *We recall that in the scalar case ($n = 1$), and when $h_i \equiv 0$, the L^p-estimate*

$$\int_0^l |u(t,x)|^p dx \le \int_0^l |u(0,x)|^p dx + \int_0^t \left(\int_0^l |f(\tau,x)|^p dx \right)^{\frac{1}{p}} d\tau, \tag{19}$$

are well known for any p, $1 \le p \le \infty$ (see, for instance [25], [13] and [14]).

Proof of Theorem 2.1. We denote i_0 simply by i. For fixed $\theta > 0$, we introduce the real function

$$\phi(u_i, \theta) = \frac{u_i - h_i}{\sqrt{\theta^2 + (u_i - h_i)^2}}. \tag{20}$$

Then $\phi'(u_i, \theta) = \frac{\theta^2}{(\theta^2 + (u_i - h_i)^2)^{\frac{3}{2}}} > 0$ and we have that

$$\phi(u_i, \theta) \to sign(u_i - h_i). \tag{21}$$

Moreover, if we define

$$F(u_i, \theta) = \int_{h_i}^{u_i} \frac{s - h_i}{\sqrt{\theta^2 + (s - h_i)^2}} ds$$

then $F(u_i, \theta) \to |u_i - h_i|$, when $\theta \to 0$. Now, by multiplying equation (12) by $\phi(u_i, \theta)$ (which is a good test function), by well known results (see, e.g. Lions [27]) we get that

$$\int_0^l F(u_i(t,x), \theta) dx + \int_0^t \int_0^l A_{ii} \phi'(u_i, \theta) u_{ix}^2 dx ds + \int_0^t \int_0^l B_i^a \phi(u_i, \theta) dx ds \tag{22}$$

$$= \int_0^l F(u_i(0,x), \theta)) dx + \int_0^t \int_0^l f_i \phi(u_i, \theta) dx ds.$$

Thanks to $A_{ii}\phi' \ge 0$ and assumptions (14), (10) we can pass to the limit, as $\theta \to 0$, we come to the desired estimate (18).■

Let us use estimate (18) to study the asymptotic convergence of the vectorial solution $\mathbf{u}(t,x)$ to a stationary solution $\mathbf{u}(t,x) \equiv \mathbf{h}$ when $t \to \infty$ (uniformly on $l \in (0, \infty]$) under assumption $a_0 > 0$. We introduce the *energy* function

$$Y(t) := \left(\int_0^l |\mathbf{u}(t,x) - \mathbf{h}|^2 dx \right)^{\frac{1}{2}}. \tag{23}$$

Theorem 2.2 *Assume $a_0 > 0$ and that, for any $i = 1, ..., n$ (14) and B_i^c=0 hold. Let $\mathbf{u} = (u_1, ..., u_n)$ be a weak solution of the problem (7) -(9). Let $\mathbf{u}_0 \in \mathbf{L}^2(\Omega)$ with*

$$\int_0^l |\mathbf{u}_0(x) - \mathbf{h}| dx \le U_0 \tag{24}$$

for some $U_0 > 0$. *Then, if* $\mathbf{f} \equiv 0$, *the solution* $\mathbf{u}(t,x)$ *satisfies the decay estimate*

$$Y(t) \leq \frac{1}{\sqrt[4]{t}} \frac{\sqrt[4]{C_0(n)}U_0}{\sqrt[4]{a_0}} \tag{25}$$

and, in particular, $Y(t) \to 0$, *as* $t \to \infty$. *Moreover, if* $\mathbf{f} \neq 0$ *and it satisfies* $\mathbf{f} \in \mathbf{L}^2(Q)$ *for any* $T > 0$ *with*

$$\|\mathbf{f}(t,\cdot)\|_2 \leq f_0 \, (1+t)^{-\frac{5}{4}} \,, \tag{26}$$

for some $f_0 > 0$ *then there exists a constant* $K\left(f_0, Y(0), a_0, n, U_0\right)$, *and thus independent on* l, *such that*

$$Y(t) \leq K \, (1+t)^{-\frac{1}{4}} \to 0, \ \text{as } t \to \infty. \tag{27}$$

Proof. First we note that the solution $\mathbf{u}(t,x)$ satisfies the energy relation

$$\frac{1}{2}\frac{d}{dt}\int_0^l |\mathbf{u}(t,x) - \mathbf{h}|^2 dx + \int_0^l (\mathbf{A}\mathbf{u}_x, \mathbf{u}_x) dx = \int_0^l (\mathbf{f}, \ \mathbf{u}(t,x) - \mathbf{h}) dx. \tag{28}$$

Next we use the known interpolation inequality ([12], Appendix, Lemma 3.2)

$$\left(\int_0^l |\mathbf{u}(t,x) - \mathbf{h}|^2 dx\right)^3 \leq 4C_0(n) \left(\int_0^l |\mathbf{u}(t,x) - \mathbf{h}| dx\right)^4 \left(\int_0^l |\mathbf{u}_x|^2 dx\right), \tag{29}$$

$C_0(n) = 9n^3/16$. Then taking into account (18), we can rewrite last inequality in the form

$$Y^6(t) \leq 4C_0 U_0^4 \left(\int_0^l |\mathbf{u}_x|^2 dx\right) .... \tag{30}$$

Using (13), (28) and applying Cauchy's inequality to the right side of (28), we arrive to the ordinary differential inequality

$$Y(t)\frac{dY(t)}{dt} + \frac{a_0}{4C_0 U_0^4} Y^6(t) \leq \int_0^l |(\mathbf{f}(t,x), \ \mathbf{u}(t,x) - \mathbf{h})| \, dx. \tag{31}$$

Assume now that $\mathbf{f} \equiv 0$. Integrating the last inequality, we obtain that

$$Y(t) \leq \frac{Y(0)}{\sqrt[4]{1 + t\frac{a_0}{C_0 U_0^4} Y^4(0)}} \leq \frac{1}{\sqrt[4]{t}} \frac{\sqrt[4]{C_0} U_0}{\sqrt[4]{a_0}} \to 0, \ \text{as } t \to \infty. \tag{32}$$

Assume now that $\mathbf{f} \neq 0$. By applying Cauchy's inequality we get the estimate

$$\int_0^l |(\mathbf{f}(t,x), \ \mathbf{u}(t,x) - \mathbf{h})| \, dx \leq \|\mathbf{f}(t,\cdot)\|_2 \, Y(t). \tag{33}$$

Then condition (26) leads to the following nonhomogeneous ordinary differential inequality

$$\frac{dY(t)}{dt} + \frac{a_0}{4C_0U_0^4}Y^5(t) \leq \|\mathbf{f}(t,\cdot)\|_2 \leq f_0\,(1+t)^{-\frac{5}{4}}\,. \tag{34}$$

It is easy to verify that there is some constant $K\,(f_0, Y(0), a_0, C_0, U_0^4)$ such that the function

$$W(t) = K\,(1+t)^{-\frac{1}{4}} \tag{35}$$

satisfies that $W(0) \geq Y(0)$ and that

$$\frac{dW(t)}{dt} + \frac{a_0}{4C_0U_0^4}W^5(t) \geq f_0\,(1+t)^{-\frac{5}{4}}\,.$$

Then, by the comparison principle for ordinary differential equations we get to

$$Y(t) \leq W(t) \tag{36}$$

and the result is proved.$\blacksquare$

Remark 2.2 *Notice that the above result shows that* $\mathbf{u}(\cdot,x) \to \mathbf{h}$ *in* $C^0([0,\infty);\mathbf{L}^2(0,l))$ *as* $t \to \infty$, *uniformly with respect to* l. *We also point out that there is a "regularizing effect" since the coefficient* U_0 *only takes into account the* L^1*-integral although the conclusion is stated in terms of the* L^2*-integral.*

Remark 2.3 *We send the reader to the book Chipot [17] for a collection of results concerning the case* $l \to +\infty$. *For very general convergence results for systems of parabolic equations see, for instance, [20] and their references.*

2.2 A L^1-gradient estimate: the possibly degenerate system

The other goal of this paper is to derive a L^1-gradient universal estimate (i.e., which doesn't depend on the constants l, a_0, C nor on $B_{0i}(x,t)$) generalizing the previous result of ([11]). In fact, we shall give a direct proof of it without splitting it in a series of lemmas (as presented in ([11])).

Theorem 2.3 *Assume (10), (11), (13). Assume that there exists a component* $i_0 \in \{1,...,n\}$ *for which*

$$f_{i_0} \in L^1(0,T:W^{1,1}(\Omega)),\ f_{i_0}(t,l){=}0 \textit{ for } t \in (0,T) \textit{ and } u_{0i_0} \in W^{1,1}(\Omega). \tag{37}$$

and that (15) and (16) or (17) hold. Let $\mathbf{u} = (u_1,...,u_n)$ *be a classical solution of problem (7) -(9). Then*

$$\int_0^l |u_{i_0x}(t,x)|dx \leq \int_0^l |u_{0i_0x}(x)|dx + \int_0^t\int_0^l |f_{i_0x}(\tau,x)|dxd\tau, \tag{38}$$

for any $t \in [0,T]$.

Proof. We denote i_0 simply by i. For fixed $\theta > 0$ we introduce the functions

$$\phi(r,\theta) = \frac{r}{\sqrt{\theta^2 + r^2}}, \tag{39}$$

$$F(r,\theta) = \int_0^r \phi(s,\theta)ds = \int_0^r \frac{s}{\sqrt{\theta^2 + s^2}}ds, \tag{40}$$

so that

$$\phi(r,\theta) \to sign\ r,\ \ F(r,\theta) \to |r|,\ \ if\ \ \ \theta \to 0. \tag{41}$$

Notice that

$$\frac{d\phi}{dr} = \frac{\theta^2}{(\theta^2 + r^2)^{\frac{3}{2}}} \text{ and that } \frac{\partial \phi(u_{ix},\theta)}{\partial x} = \frac{\theta^2 u_{ixx}}{(\theta^2 + u_{ix}^2)^{\frac{3}{2}}}.$$

Multiplying i-th equation (12) by $\frac{\partial \phi(u_{ix},\theta)}{\partial x}$ and integrating over Ω, we get

$$\int_0^l F(u_{ix}(t,x),\theta)dx + I_1 = \int_0^l F(u_{ix}(0,x),\theta)dx + I_2 + I_3 + I_4, \tag{42}$$

where

$$I_1 := \int_0^t \int_0^l A_{ii} \frac{\theta^2 u_{ixx}^2}{(\theta^2 + u_{ix}^2)^{\frac{3}{2}}} dx d\tau \geq a_0 \int_0^t \int_0^l \frac{\theta^2 (u_{ixx})^2}{(\theta^2 + u_{ix}^2)^{\frac{3}{2}}} dx d\tau := I_0$$

$$I_2 := -\int_0^t \int_0^l \frac{\partial A_{ii}}{\partial u_k} u_{kx} u_{ix} \frac{\theta^2 u_{ixx}}{(\theta^2 + u_{ix}^2)^{\frac{3}{2}}} dx d\tau,$$

$$I_3 := -\int_0^t \int_0^l \frac{\partial A_{ii}}{\partial x} u_{ix} \frac{\theta^2 u_{ixx}}{(\theta^2 + u_{ix}^2)^{\frac{3}{2}}} dx d\tau.$$

$$I_4 := -\int_0^t \int_0^l f_i \frac{\partial \phi(u_{ix},\theta)}{\partial x} dx d\tau.$$

$$I_5 := -\int_0^t \int_0^l B_i^c \frac{\theta^2 u_{ixx}}{(\theta^2 + u_{ix}^2)^{\frac{3}{2}}} dx d\tau.$$

$$I_6 := -\int_0^t \int_0^l B_i^a \frac{\partial \phi(u_{ix},\theta)}{\partial x} dx d\tau$$

Here we used the formulas

$$\int_0^l u_{it} \frac{\partial \phi(u_{ix},\theta)}{\partial x} dx = -\int_0^l u_{ixt} \phi(u_{ix},\theta)dx + u_{it}\phi(u_{ix},\theta)|_{x=0}^{x=l} \tag{43}$$

$$= -\int_0^l u_{ixt}\phi(u_{ix},\theta)dx = -\frac{d}{dt}\int_0^l \left[\int_0^{u_{ix}} \phi(\theta,s)ds\right] dx = -\frac{d}{dt}\int_0^l F(u_{ix},\theta)dx,$$

(notice that, under the assumed regularity, $u_{it}(t,l)=0$ for any $t\in(0,T)$),

$$\int_0^l A_{ii}\frac{\partial\phi(u_{ix},\theta)}{\partial x}u_{ixx}dx=\int_0^l A_{ii}\frac{\theta^2(u_{ixx})^2}{(\theta^2+u_{ix}^2)^{\frac{3}{2}}}dx, \tag{44}$$

and

$$\int_0^l \frac{\partial A_{ii}}{\partial u_k}u_{kx}u_{ix}\frac{\partial\phi(u_{ix},\theta)}{\partial x}dx=\int_0^l \frac{\partial A_{ii}}{\partial u_k}\frac{u_{kx}u_{ix}\theta^2 u_{ixx}}{(\theta^2+u_{ix}^2)^{\frac{3}{2}}}dx. \tag{45}$$

Using the Cauchy's inequality we can estimate the terms $I_i,\ i=2,3,4$ in (42) as follows:

$$|I_2|\le\int_0^t\int_0^l\left[\frac{a_0}{3}\frac{\theta^2(u_{ixx})^2}{(\theta^2+u_{ix})^{\frac{3}{2}}}+\frac{3\theta C^2}{4a_0}\left(\sum_{k=1}^n|u_{kx}|\right)^2\frac{\theta}{(\theta^2+u_{ix}^2)^{\frac{1}{2}}}\frac{u_{ix}^2}{(\theta^2+u_{ix}^2)}\right]dxd\tau$$

$$\le\frac{1}{3}I_0+\theta\frac{3C^2}{4a_0}\int_0^t\int_0^l\left(\sum_{k=1}^n|u_{kx}|\right)^2dxd\tau,$$

$$|I_3|\le\frac{1}{3}I_0+\frac{3\theta}{4a_0}\int_0^t\int_0^l\left|\frac{\partial A_{ii}}{\partial x}\right|^2\frac{\theta}{(\theta^2+u_{ix}^2)^{\frac{1}{2}}}\frac{u_{ix}^2}{(\theta^2+u_{ix}^2)}dxd\tau$$

$$\le\frac{1}{3}I_0+\frac{3\theta}{4a_0}\int_0^t\int_0^l\left|\frac{\partial A_{ii}}{\partial x}\right|^2dxd\tau\le\frac{1}{3}I_0+\theta\frac{3C(T,l)}{4a_0},$$

$$|I_4|=\left|\int_0^t\int_0^l f_i\frac{\partial\phi(u_{ix},\theta)}{\partial x}dxd\tau\right|=\left|\int_0^t\int_0^l f_{ix}\phi(u_{ix},\theta)\right|dxd\tau\le\int_0^t\int_0^l|f_{ix}|\,dxd\tau,$$

Moreover if (16) holds then

$$|I_5|=\left|\int_0^t\int_0^l B_i^c\frac{\theta^2u_{ixx}}{(\theta^2+u_{ix}^2)^{\frac{3}{2}}}dxd\tau\right|\le\int_0^t\int_0^l\frac{\theta|u_{ixx}|}{(\theta^2+u_{ix}^2)^{\frac{3}{4}}}\frac{\theta B_0|u_{ix}|^{\gamma_i}}{(\theta^2+u_{ix}^2)^{\frac{3}{4}}}dxd\tau$$

$$\le\int_0^t\int_0^l\left(\frac{a_0}{3}\frac{\theta^2u_{ixx}^2}{(\theta^2+u_{ix}^2)^{\frac{3}{2}}}+\frac{3}{4a_0}B_0^2\frac{\theta^2\,|u_{ix}|^{2\gamma_i}}{(\theta^2+u_{ix}^2)^{\frac{3}{2}}}\right)dxd\tau\le\frac{1}{3}I_0+\frac{3\theta^{2\gamma_i-1}}{4a_0}\int_0^t\int_0^l B_0^2dxd\tau.$$

In the case in which (17) holds

$$I_5=\int_0^t\int_0^l\frac{dC_i}{du_i}\frac{u_{ix}\theta^2u_{ixx}}{(\theta^2+u_{ix}^2)^{\frac{3}{2}}}dxd\tau=-\theta^2\int_0^t\int_0^l\frac{dC_i}{du_i}\frac{\partial}{\partial x_i}\left(\frac{1}{(\theta^2+u_{ix}^2)^{\frac{1}{2}}}\right)dxd\tau$$

$$=\theta^2\left[\int_0^t\int_0^l\frac{d^2C_i}{d^2u_i}\left(\frac{u_{xi}}{(\theta^2+u_{ix}^2)^{\frac{1}{2}}}\right)dxd\tau-\frac{dC_i}{du_i}\frac{1}{(\theta^2+u_{ix}^2)^{\frac{1}{2}}}\Bigg|_{x=0}^{x=l}\right].$$

Finally applying the formula of the integration by parts and using (15), we have

$$I_6 = \int_0^t \int_0^l \frac{\partial B_i^a(t,u_i)}{\partial r_i} \frac{u_{ix}^2}{(\theta^2+u_{ix}^2)} dx d\tau \leq 0$$

Joining (42) and the above estimates, we arrive to the inequality

$$\int_0^l F(u_{ix}(t,x),\theta)dx \leq \int_0^l F(u_{0ix}(x),\theta)dx + \int_0^t \int_0^l |f_{ix}| \, dx d\tau \tag{46}$$

$$+\theta\frac{3C^2}{4a_0}\left(\int_0^t \int_0^l \left(\sum_{k=1}^n |u_{kx}|\right)^2 dx d\tau + \frac{C(T,l)}{C^2}\right) + \frac{3\theta^{2\gamma_i-1}}{4a_0}\int_0^t \int_0^l B_0 dx d\tau.$$

(if (16) holds) or

$$\int_0^l F(u_{ix}(t,x),\theta)dx \leq \int_0^l F(u_{0ix}(x),\theta)dx + \int_0^t \int_0^l |f_{ix}| \, dx d\tau \tag{47}$$

$$+\,\theta\frac{3C^2}{4a_0}\left(\int_0^t \int_0^l \left(\sum_{k=1}^n |u_{kx}|\right)^2 dx d\tau + \frac{C(T,l)}{C^2}\right)$$

$$+\,\theta^2\left[\int_0^t \int_0^l \frac{d^2C_i}{d^2u_i}\left(\frac{u_{xi}}{(\theta^2+u_{ix}^2)^{\frac{1}{2}}}\right) dx d\tau - \frac{dC_i}{du_i}\frac{1}{(\theta^2+u_{ix}^2)^{\frac{1}{2}}}\Bigg|_{x=0}^{x=l}\right].$$

(if (17) holds). Passing to the limit as $\theta \to 0$, we get the desired estimate.■

Remark 2.4 *Let us consider more in detail the case $\gamma_i = 1/2$ in the conditions (16),(57). In that case we can evaluate I_5 in the following way*

$$|I_5| \leq \frac{1}{3}\int_0^t \int_0^l A_{ii}\frac{\theta^2 u_{ixx}^2}{(\theta^2+u_{ix}^2)^{\frac{3}{2}}} dx d\tau + \frac{3}{4}\int_0^t \int_0^l \frac{B_{0i}^2}{A_{ii}} dx d\tau.$$

Then, if additionally, we assume the subordination between terms A_i and B_{0i}:i.e.

$$B_{0i}^2 A_{ii}^{-1} \in L^1(Q) \tag{48}$$

arguing as in the Proof of Theorem 2.3 we arrive to a slightly different estimate to (38): more precisely, we get

$$\int_0^l |u_{ix}(x,t)|dx \leq \int_0^l |u_{ix}(x,0)|dx + \int_0^t \int_0^l |f_{ix}(x,\tau)|dx + \lambda_i \int_0^t \int_0^l \frac{B_{0i}^2}{A_{ii}} dx d\tau, \tag{49}$$

for any $t \in [0,T]$, where $\lambda_i = 0$ if $\gamma_i > 1/2$ and $\lambda_i = 3/4$ if $\gamma_i = 1/2$.

Remark 2.5 *The statement of the above theorem remains valid if we replace boundary conditions (8) and assumption (37) by one of the following alternative boundary conditions (and alternative assumptions): either*

$$u_{i_0x}(t,l) = u_{i_0x}(t,0) = 0, 0 < t < T, \tag{50}$$

and we assume merely

$$f_{i_0} \in L^1(0,T : W^{1,1}(\Omega)), \text{ and } u_{0i_0} \in W^{1,1}(\Omega), \tag{51}$$

or

$$u_{i_0}(t,l) = u_{i_0}(t,0) = 0, \quad 0 < t < T, \tag{52}$$

and we assume

$$f_{i_0} \in L^1(0,T : W_0^{1,1}(\Omega)) \text{ and } u_{0i_0} \in W^{1,1}(\Omega). \tag{53}$$

Notice that in this last case Theorem 2.3 remains true if $h_{i_0} = 0$ *and we assume (17) instead* $B^c_{i_0}$=0. *Many combined possibilities can be considered: for instance, we can assume that for a subset of the components the boundary conditions (8) hold but that there are other subsets of the components for which (50) or (52) are prescribed.*

One of the most important consequences of the above uniform L^1-gradient estimate is the following one:

Theorem 2.4 *Let* $\{\mathbf{u}_m\}_{m\in\mathbb{N}}$ *be a sequence of classical solutions of (7) -(9) corresponding to the data*

$$\mathbf{f}_m \in L^1(0,T : \mathbf{W}^{1,1}(\Omega)), \ \mathbf{f}_m(t,l) = \mathbf{0} \text{ for } t \in (0,T) \text{ and } \mathbf{u}_{0,m} \in \mathbf{W}^{1,1}(\Omega) \tag{54}$$

such that

$$\int_0^l |u_{0ix}(x)|dx + \int_0^t \int_0^l |f_{ix}(\tau,x)|dxd\tau \leq C$$

for some C *independent on* m, *for any* $i = 1,...,n$ *and* $t \in [0,T]$. *Assume also that* $\mathbf{u}_{mx} \in L^2(0,T : \mathbf{L}^\infty(\Omega)); \mathbf{u}_{mt}, \mathbf{u}_{mxx} \in L^\infty(0,T : \mathbf{L}^1(\Omega))$ *and that for such solutions the following conditions*

$$0 < a_{0m} \leq (\mathbf{A}(t,x,\mathbf{u}_m)\vec{\xi},\vec{\xi}) \leq a_{1m} < \infty, \forall \vec{\xi} \in R^n, \tag{55}$$

$$\left[\left(\frac{\partial A_{ii}}{\partial u_k}\right)^2 + \left(\frac{\partial A_{ii}}{\partial x}\right)^2\right] A_{ii}^{-1} \leq C_m < \infty, \tag{56}$$

(15) and

$$(B_i^c)^2 A_{ii}^{-1} \leq B_m^c |u_{ix}|^{2\gamma_i}, 0 < B_m^c \in L^2(Q), \gamma_i > 1/2, \ i = 1,...,n, \tag{57}$$

or (17) are valid for any $i = 1, ..., n$. Then estimate (38) holds for any $\mathbf{u}_m$, $m \in \mathbb{N}$. In particular, there exists a subsequence of $\{\mathbf{u}_m\}_{m\in\mathbb{N}}$ which converges to a function $\mathbf{u}$ of $L^\infty(0,T : \mathbf{BV}(\Omega))$ in the weak-topology and*

$$TV(\mathbf{u}(t,.)) \leq TV(\mathbf{u}_0(.)) + \int_0^t TV(\mathbf{f}(\tau,.))d\tau, \tag{58}$$

where $TV(\varphi(t,.))$ means the total variation of the Radon measure $\varphi_x(t,.)$, i.e.

$$TV(\varphi(t,.)) = \sup\{\int_0^l \varphi(t,x)\phi_x(x)dx : \phi \in C_0^\infty(0,l),\ \ |\phi(x)| \leq 1 \textit{ for } x \in (0,l)\}.$$

Proof. Clearly, assumptions (55), (56) and (56) imply (10), (11), (13) and (16). Then, we can apply Theorem 2.3 and, in particular, we get that

$$\int_0^l |u_{mix}(t,x)|dx \leq C \tag{59}$$

for any $i = 1, ..., n$ and $t \in [0,T]$. Since $L^\infty(0,T : \mathbf{BV}(\Omega))$ is the dual of a separable space (see, e.g. [3] p. 299) we can apply Banach-Alaoglu-Bourbaki compactness theorem (see, e.g. [15]).■

Remark 2.6 *In many cases, by using some supplementary assumptions and arguments it can be shown that, in fact, the limit function $\mathbf{u} \in L^\infty(0,T : \mathbf{BV}(\Omega))$ is more regular and $\mathbf{u} \in L^1_{loc}(0,T : \mathbf{W}^{1,1}(\Omega))$ (see, for instance, [1] for some systems and [22] for some scalar equations, but many other references in the literature contains other proofs of the strong convergence of the gradients). In that case conclusion (58) can be replaced by the stronger one*

$$\int_0^l |\mathbf{u}_x(t,x)|dx \leq \int_0^l |\mathbf{u}_{0x}(x)|dx + \int_0^t \int_0^l |\mathbf{f}_x(\tau,x)|dxd\tau.$$

We shall end this section by giving an application to the case of the associated stationary system

$$\lambda\mathbf{u} = (\mathbf{A}(x,\mathbf{u})\mathbf{u}_x)_x + \mathbf{B}(x,u,u_x) + \mathbf{f}(x), \tag{60}$$

on $\Omega = (0,l)$ with the auxiliary conditions

$$\mathbf{u}_x(0) = \mathbf{0},\ \ \mathbf{u}(l) = \mathbf{h}, \tag{61}$$

where $\mathbf{u} = (u_1, ..., u_n)$ and $\mathbf{h} = (h_1, ..., h_n)$ is a constant given vector. Note that this stationary system is associated to (1), for instance, by applying the implicit Euler scheme to (1)) when developing the semigroup theory to the evolution problem (see, e.g. [13], [18] and [14]).

Theorem 2.5 *Assume (10), (11), (13) but for the stationary case and for a given* $\lambda > 0$. *Assume that there exists a component* $i_0 \in \{1, ..., n\}$ *for which*

$$f_{i_0} \in W^{1,1}(\Omega),\ f_{i_0}(l){=}0, \tag{62}$$

and that (15) and (16) or (17) hold. Let $\mathbf{u} = (u_1, ..., u_n)$ *be a classical solution of problem (60) -(61). Then*

$$\int_0^l |u_{i_0x}(x)|dx \leq \frac{1}{\lambda}\int_0^l |f_{i_0x}(x)|dx. \tag{63}$$

Proof. It is similar to the one of Theorem 2.3 but replacing the "weak time dependence integration by parts formula" (see also Lemma 3.1 of ([11])) by its (even easier) stationary version

$$\int_0^l u_i \frac{\partial\phi(u_{ix},\theta)}{\partial x} dx = -\int_0^l u_{ix}\phi(u_{ix},\theta)dx + u_i\phi(u_{ix},\theta)|_{x=0}^{x=l} \tag{64}$$

which holds by a direct integration by parts (recall that, in fact, $u_i\phi(u_{ix},\theta)|_{x=0}^{x=l} = 0$ thanks to the boundary conditions (61)). Then, it suffices to pass to the limit, as $\theta \to 0$, and to use that $s\phi(s,\theta) \to |s|$ when $\theta \to 0$.■

Remark 2.7 *Estimate (63) can be understood as the gradient estimate similar to the obtained in [16] for the* L^1*-norm of the solution of semilinear stationary scalar equations.*

Acknowledgements

The work of the first author was supported by the research project POCI/MAT/61576/2004, FCT (Portugal) and the one by the second author by the projects MTM2005-03463 of the DGISGPI (Spain) and CCG07-UCM/ESP-2787 of the DGUIC of the CAM and the UCM.

This paper was written when the first author was Sabbatical Professor at the Departamento de Matemática Aplicada of the Universidad Complutense de Madrid during the academic year 2006/2007 (project SAB-2005-0017 of the Secretaria de Estado de Universidades e Investigación of Spain) and during of his visit in 2008 (Programa de Visitantes Distinguidos, Grupo Santander).

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