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Branches of positive and free boundary solutions for some singular quasilinear elliptic problems

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Abstract

We study the existence and multiplicity of solutions, strictly positive or nonnegative having a *free boundary* (the boundary of the set where the solution vanishes) of some one-dimensional quasilinear problems of eigenvalue type with possibly singular nonlinear terms.

Keywords: Elliptic partial differential equations; Quasilinear equations; Singular problems; Free boundaries

1. Introduction

Semilinear elliptic equations of reaction–diffusion type have been studied extensively during the last forty years as mathematical models for a large variety of problems arising in applications (population dynamics, combustion, chemical reactions, nerve impulses, etc.) and having independent mathematical interest. This gives rise, if we forget problems concerning existence, properties and asymptotic behavior of solutions to parabolic problems, to study existence, multiplicity and other properties (as, e.g., stability as stationary solutions to the evolution problem) of positive solutions for these problems. Positive (and also nonnegative, see below) solutions are often the only physically meaningful (populations, concentrations, etc.) solutions for the corresponding models. A very simple example is given by the boundary value problem

$$-\Delta u = f(u) \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega, \tag{1}$$

where Ω is a smooth bounded domain in the space $\mathbb{R}^N$, Δ is the usual Laplacian modeling (linear) diffusion, and $f(u)$ is a nonlinear reaction term which often depends on some real parameter. This model can be generalized in many possible directions: nonlinearities which also depend on the space variable x and/or the gradient ∇u, other

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linear or even nonlinear boundary conditions, etc. Another very important extension is to consider reaction–diffusion systems, where similar but more involved problems may be studied. Lotka–Volterra systems in mathematical biology have been considered by many authors in these years and interesting problems are still open (for all these matters the reader can consult the book [49] and its references).

Note that when $f(0)=0$ then $u(x)\equiv 0$ is already a solution (the "trivial one"). Obviously, the relevant interest is the study of positive or nonnegative solutions (different of the trivial solution). In this paper we shall pay a special attention to the case in which $f\in C^0((0,\infty))\cap L^1_{\mathrm{loc}}(0,\infty)$, being defined at least on $[0,\infty)$, i.e. with $f(r)$ possibly discontinuous at $r=0$, and such that

$$f(0+):=\limsup_{s\searrow 0} f(s)\leqslant 0 \quad \bigl(\text{where } f(0+)=-\infty \text{ is allowed}\bigr). \tag{2}$$

In the context of chemical reactions this condition means that the reaction is given by an "absorption (or endothermal) process" at low concentrations u. We point out that for the above boundary condition the existence of positive or nontrivial nonnegative solutions require that the reaction term changes sign for some values of u. So, in this paper we always assume that

$$\begin{cases} \text{there exists } r_f>0 \quad \text{such that} \quad f(r_f)=0, \quad \text{and} \\ f(s)<0 \quad \text{if } s\in(0,r_f) \quad \text{and} \quad f(s)>0 \quad \text{if } s\in(r_f,\infty). \end{cases} \tag{3}$$

As above, in the context of chemical reactions this condition means that the reaction term represented is given by a "source (or exothermal process)" for concentrations $u\geqslant r_f$.

If the nonlinearity f is C^1, or at least locally Lipschitz, a simple argument involving the Strong Maximum Principle can be used in order to show that nonnegative solutions (i.e., $u\geqslant 0$ in Ω) are actually positive ($u>0$ in Ω). But if f is not Lipschitz at the origin, then it may happen that nonnegative solutions exhibiting a "free boundary" (i.e., regions of the domain Ω having positive measure where the solution vanishes) arise. These regions are called "dead cores" when we impose, for example, the nonhomogeneous Dirichlet condition $u=1$ on $\partial\Omega$ (notice that, in that case, nontrivial nonpositive solutions exists even if $r_f=+\infty$). The same situation also arises in problems where (as in the well-known logistic equation) the usual linear diffusion is replaced by a nonlinear diffusion [31]. Then, after a change of unknown, problems with non-Lipschitz nonlinearities again arise. Other kinds of degenerate or singular problems arise if the Laplacian is replaced by the so-called p-Laplacian $\Delta_p u=\operatorname{div}(|\nabla u|^{p-2}\nabla u)$. In fact, in that case the Strong Maximum Principle is characterized (see [53] and the recent monograph [47]) in terms of a suitable balance, near $u=0$, between p and the primitive $F(u)$ of $f(u)$. In fact, the main goal of this paper is to study the simplified one-dimensional case given by the quasilinear elliptic problem. More precisely, in Section 2 we consider the quasilinear elliptic one-dimensional problem

$$\mathcal{P}(L) \quad \begin{cases} -\bigl(|u'|^{p-2}u'\bigr)'=f(u) \quad \text{in } (-L,L), \\ u(\pm L)=0 \end{cases}$$

where $p>1$ (the case $p=2$ corresponds to semilinear problems) and $f(u)$ satisfies (besides (2) and (3)) some other conditions given in terms of its primitive:

$$F(u):=\int_0^u f(s)\,ds>-\infty \quad \text{for any } u>0, \tag{4}$$

$$\limsup_{s\searrow 0}\bigl(pF(s)-sf(s)\bigr)=0, \tag{5}$$

and

$$\begin{cases} \text{there exists } r_F>r_f \quad \text{such that} \quad F(r_F)=0, \quad \text{and} \\ F(s)<0 \quad \text{if } s\in(0,r_F) \quad \text{and} \quad F(s)>0 \quad \text{if } s\in(r_F,\infty). \end{cases} \tag{6}$$

Since we are interested in possible nonnegative solutions we shall also assume that

$$\int_0^\varepsilon \frac{dr}{(-F(r))^{1/p}}<\infty, \quad \text{for any } \varepsilon>0 \text{ small enough}, \tag{7}$$

which, at least formally, means that the Strong Maximum Principle fails at the level $u = 0$. As a matter of fact, the correct definition of nonnegative solutions (see Definition 2 below) requires a suitable extension to the origin of the function $f \in C^0(0,\infty) \cap L^1_{\mathrm{loc}}(0,\infty)$: in that case we shall assume that

$$f(0-) = 0. \tag{8}$$

Obviously, this condition is not needed when dealing with positive solutions.

To simplify the exposition of our results we shall also assume that

$$f(r)/r^{p-1} \to 0, \quad \text{as } r \to \infty, \tag{9}$$

but some other growth conditions could be considered (leading then to different bifurcation diagrams).

The existence and additional properties of the above mentioned free boundary solutions (for $p = 2$) were extensively studied in the early eighties by several authors, starting with the pioneering paper by H. Brezis [10] proving that the support of the solution of some variational inequalities (of obstacle type) for a general second order elliptic operator on an unbounded domain can be compact. The case of semilinear equations $-\Delta u + \beta(u) \ni g$ in $\mathbb{R}^N$ (for $g(x)$ with compact support) was studied by Benilan, Brezis and Crandall [6], proving that the necessary and sufficient condition on β in order to get a solution with compact support is that

$$\int_0 \frac{ds}{\sqrt{j(s)}} < +\infty, \tag{10}$$

where j is the convex primitive of β (i.e., such that $\partial j = \beta$). This criterion was extended to the case of quasilinear problems of the type $-\Delta_p u + \beta(u) \ni g$ in $\mathbb{R}^N$, in Díaz and Herrero [23,24] to the criterion

$$\int_0 \frac{ds}{\sqrt[p]{j(s)}} < +\infty \tag{11}$$

which, for instance, now applies to Lipschitz functions $\beta(u)$ if $p > 2$. The above results were extended in many directions in the literature. For instance, the study of the semilinear elliptic equation with $p = 2$, but now on a bounded domain Ω and with $g = 0$ on Ω and $u = 1$ on the boundary, was carried out by Bandle, Sperb and Stakgold [4] showing that condition (10) is, again, the necessary and sufficient condition on β (assumed Ω large enough) for the formation of an internal free boundary (the boundary of the *dead core*). The location of the free boundary for this case was obtained in Díaz [18] and Díaz and Hernández [20] by means of a local comparison argument (which improves some previous arguments used in Bensoussan, Brezis and Friedman [7] for some quasi-variational inequalities) by using local radial supersolutions. The same kind of techniques were used in [27] obtaining also “exterior” estimates and qualitative properties (regularity, convexity, etc.) under suitable conditions for nonmonotone functions $f(u)$. Applications to mathematical biology were given in, e.g., [33] or [45]. Related results for the associated parabolic problems were obtained in [5,21]. Most of the results for the stationary case were collected in the book [19] (see also the monograph by Pucci and Serrin [47] collecting many other recent references), where the case of the degenerate diffusion provided by the p-Laplacian was considered as well. Other physical motivation and references can be found in the above monograph. Concerning solutions with compact support, some results in $\mathbb{R}^N$ with non-Lipschitz nonlinearities were obtained by Cortázar, Elgueta and Felmer [11] and in [32] in the case of a bounded domain with non-Lipschitz terms and a singular (near to the boundary) coefficient $K(x)$ (see [36,37] for similar problems with singular nonlinearities). For a complementary exposition on problems of the $\mathcal{P}(L)$ type and its parabolic version (containing many other references) but this time concentrated on energy type methods, see the monograph [1]. The study of the bifurcation curve of nonnegative solutions for problem $\mathcal{P}(L)$ was considered in [22] when we replace (2) by the special condition $f(0) = 0$. These results were extended by in [39] to some related nonlinearities also in the case of the p-Laplacian.

Concerning the case in which $f(u)$ is singular at $u = 0$ we point out that, if usually nonlinearities arising in the models mentioned at the begining of this Introduction are smooth (C^1 or at least Lipschitz continuous), functions not defined at zero and going to infinite when $u > 0$ goes to 0 also appear in some situations (suitable chemical reactions, non-Newtonian fluids, thin films, etc.; see, e.g., [25,35] and [19]). A typical example is the “exothermal” function $f(u) = u^{-\alpha}$, where $\alpha > 0$; in particular in the case $0 < \alpha < 1$. Problems of this kind have been studied in the last thirty years, starting from the papers by Fulks and Maybee [28], Stuart [51] and Crandall, Rabinowitz and Tartar [14].

For later work, dealing also with "endothermal" cases, see [8,25,36,48], the survey [35] and the book [29] and its references.

Most of these works concern existence (and additional properties) of positive solutions, but free boundary solutions have been studied by Dávila and Montenegro [15–17], providing some interesting examples. In [44] radial positive solutions for the problem $-\Delta u + \frac{1}{u^\alpha} = \lambda u^p$, with $0 < \alpha < 1$, $0 < p < 1$ are studied exhibiting existence of exactly one "upper branch" of positive solutions for $\lambda > \lambda_0 > 0$ (a result obtained for general domains Ω in [48] and [37]), and a "lower branch" stopping at $\lambda = \bar{\lambda} > \lambda_0$, an intriguing feature. A somewhat similar situation was raised in [12], where it is shown that the one-dimensional problem $-u'' + \frac{1}{u^\gamma} = 1$ in $(-L, L)$, $u(\pm L) = 0$, $0 < \gamma < 1$, has a unique positive solution for $\gamma = \frac{1}{2}$ and at least two if $0 < \gamma < \frac{1}{3}$; again an intriguing feature. This matter has been clarified in [38], showing that there is a unique solution for $\gamma \in [\frac{1}{2}, 1[$, and the same two branches structure as above for $\gamma \in]0, \frac{1}{2}[$ again using phase plane arguments (see [41] and [34] for related matters).

Roughly speaking, in this article we extend the results in [22] and show that the same phenomena exhibited in this paper still happen in the case of f singular at the origin, giving in this way a continuation, under the form of continua of free boundary solutions to the "lower" branches "stopping" somewhere in the above examples, and we extend most of these results to the case of $p \neq 2$. So, in Section 2 we generalize Theorem 2 of [22] but now incorporating the peculiar phenomenon on the number of positive solutions arising for suitable singular functions $f(u)$, in the line of the works [12] and [38] mentioned above.

We use phase plane arguments for the associated ODE and get a complete picture of the solution set. We show how these methods also work in our case and get results close to [2,49,50,54]. There is, however, a remarkable difference, since the "lower branch" of positive solutions "stops" at some critical value L_1 of the length interval L, giving rise to "free boundary" solutions.

As mentioned above, our main generalization of Theorem 2 of [22] deals with the case when $f(u)$ is singular at $u = 0$. In that case a sharper study is required and, as we shall show, the number of positive solutions must be studied differently according that the condition

$$\int_0^\delta \frac{dr}{(-F(r))^{(p+1)/p}} = +\infty \quad \text{for } \delta \text{ small enough,} \tag{12}$$

holds or on the contrary

$$\int_0^\delta \frac{dr}{(-F(r))^{(p+1)/p}} < +\infty \quad \text{for } \delta \text{ small enough.} \tag{13}$$

Notice that when $f(u)$ is not singular at $u = 0$, then the rest of conditions on $f(u)$ imply (12). The study of the case (12) still requires some additional conditions on $f(u)$. To state them, we introduce the auxiliary function $\theta(t) := pF(t) - tf(t)$ and assume that:

$$\begin{cases} \text{A.} & \text{There is a } \mu_1 \in (r_F, \infty) \text{ such that } \theta(r) < 0 \text{ on } (0, \mu_1) \text{ and } \theta(r) > 0 \text{ on } (\mu_1, \infty). \\ \text{B.} & \text{There is a } \mu_2 \in (0, \mu_1) \text{ such that } \theta'(r) < 0 \text{ on } (0, \mu_2) \text{ and } \theta'(r) > 0 \text{ on } (\mu_2, \infty). \\ \text{C.} & \text{There exists a } \mu_3 \in (0, \mu_1) \text{ such that } \mu\theta'(\mu) - r\theta'(r) > 0 \text{ on } (0, \mu) \text{ for } \mu > \mu_3. \end{cases} \tag{14}$$

If we assume (13) then we shall show that the number of positive solutions depends of the sign of a suitable limit (see part (vi) of Theorem 1). We illustrate it in some special examples which are considered in other sections of this paper.

In Section 2.2 we consider the case of internal *dead core solutions*

$$\mathcal{P}_\mu(L) \quad \begin{cases} -\left(|u'|^{p-2}u'\right)' = f(u) & \text{in } (-L, L), \\ u(\pm L) = \mu, \end{cases}$$

for $\mu = r_F$ generalizing previous results due to Friedman and Phillips [27]. By defining $w = \mu - u$ and $f^*(w) = -f(\mu - w)$ we get solutions to problems of the type

$$\mathcal{P}^*(L) \quad \begin{cases} -\left(|u'|^{p-2}u'\right)' = f^*(u) & \text{in } (-L, L), \\ u(\pm L) = 0, \end{cases}$$

whose solutions attain its maximum on positively measured intervals. If we extend $f^*(w)$ to the half-line $[0, +\infty)$ by any function growing slower than w^p as $w \to +\infty$, then we show the existence of a second positive solution of $\mathcal{P}^*(L)$ for L large enough and so we get, again, some nonuniqueness results.

In Section 3 we particularize the results of Section 2 to the special problem

$$\mathcal{P}_{\lambda,\alpha} \quad \begin{cases} -\left(|u'|^{p-2}u'\right)' + \alpha u^m = \lambda u^q & \text{in } (-1,1), \\ u(\pm 1) = 0, \end{cases} \tag{15}$$

where $p > 1$ and m, q are such that

$$-1 < m < q < p - 1, \tag{16}$$

which corresponds to the case of strong absorption with respect to the diffusion (see [19]). We first consider, in Section 3.1, the case in which $m \neq 0$ and $q \neq 0$. Hence this includes semilinear equations with non-Lipschitz nonlinearities, as $-u'' + u^\alpha = \lambda u^\beta$, $-1 < \alpha < \beta < 1$, and quasilinear degenerate equations as, for instance, $-(|u'|^2 u')' + \alpha u^{-1/2} = \lambda u$. As a matter of fact, when studying nonnegative solutions for some negative exponents we replace the power function (only defined for positive values) by the extended function by zero at the origin, i.e.

$$g_s(r) = \begin{cases} r^s & \text{if } r > 0, \\ 0 & \text{if } r = 0, \end{cases}$$

if $s \in (-1, 0)$.

In the singular case we shall prove that if $f(w) = w^q - w^m$ with $-1 < m < q < p - 1$, then (12) holds if and only if condition $-\frac{1}{p+1} \leqslant m$ is satisfied and that the study of the number of positive solutions in the opposite case ($-\frac{1}{p+1} > m > -1$) needs some additional study. Although the complete generalization of the results by Horváth and Simon [38] to our framework (for instance, to the cases $p \neq 2$ and $q \neq 0$) seems to be hard to be obtained, we improve them by showing that if $-\frac{1}{p+1} > m > -1$ (and for any $q \neq 0$ satisfying $m < q < p-1$), then there exists a small $\delta > 0$ such that there are exactly two positive solutions if $m \in (-\frac{1}{1+p} - \delta, -\frac{1}{1+p})$ and $\lambda \in (\lambda_0, \lambda_1)$ (for some suitable λ_0 and λ_1, see Theorem 2 below) but there is a unique positive solution if $m \in (-1, -1+\delta)$ and $\lambda \in (\lambda_0, \lambda_1)$. Moreover, we get a sharper result for the special case of

$$q = 1 + 2m$$

(assumed $-\frac{1}{p+1} > m > -1$) since we prove that the exact number of positive solutions, for $\lambda \in (\lambda_0, \lambda_1)$, is

$$\begin{cases} 2 & \text{if } p > \frac{2(1+m)}{(-m)}, \\ 1 & \text{if } p = \frac{2(1+m)}{(-m)}, \\ 1 & \text{if } p < \frac{2(1+m)}{(-m)}. \end{cases}$$

Our proof is reduced to the application of some properties of the Euler Beta function $\mathrm{B}(\alpha, \beta)$ defined as

$$\mathrm{B}(\alpha, \beta) := \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\, du. \tag{17}$$

A difference between the cases $p = \frac{2(1+m)}{(-m)}$ and $p < \frac{2(1+m)}{(-m)}$ seems to be associated to the stability of the stationary solution (see Remark 14). As mentioned above, the multiplicity study of positive solutions must be completed linking it with the study of the multiplicity of nonnegative solutions, by continuation of the “lower” branches of positive solutions where they “stop.”

Finally, in Sections 3.2 and 3.3 we consider the problem $\mathcal{P}_{\lambda,\alpha}$ in the discontinuous (or multivalued) cases corresponding to the choice of the exponents $0 = m < q < p - 1$ (Section 3.2) and $-1 < m < q = 0$ (Section 3.3). In that case the equivalent extended function becomes the maximal monotone graph of $\mathbb{R}^2$ associated to the Heaviside function

$$H(r) = \begin{cases} 1 & \text{if } r > 0, \\ [0,1] & \text{if } r = 0, \\ 0 & \text{if } r < 0. \end{cases}$$

For instance, in Section 3.3 ($-1 < m < q = 0$) our result complements the results in the literature for $p = 2$. In that case (12) becomes $-\frac{1}{3} \leqslant m$, and the particularization of our result of Section 3.1 (valid for any $p > 1$) for positive solutions (and so, replacing λu^q by the positive λ in $\mathcal{P}_{\lambda,\alpha}$) and $p = 2$ implies that $m = -\frac{1}{2}$ corresponds to a change in the monotonicity of the bifurcation branch of the lower positive solution (as shown in [38]).

2. The problem $\mathcal{P}(L)$

Due to the possible singularity at $u = 0$ we should be careful with the notion of weak solution.

Definition 1. We say that $u \in W_0^{1,p}(-L, L)$ is a *positive strong solution* of problem $\mathcal{P}(L)$ if $u > 0$ on $(-L, L)$, $f(u) \in L^1(-L, L)$, $(|u'|^{p-2}u')' \in L^1(-L, L)$ and $-(|u'|^{p-2}u')'(x) = f(u(x))$ for a.e. $x \in (-L, L)$.

Definition 2. Assume that (8) is satisfied, we say $u \in W_0^{1,p}(-L, L)$ is a *nonnegative strong solution* of problem $\mathcal{P}(L)$ if $u \geqslant 0$ on $(-L, L)$, $f(u) \in L^1(-L, L)$, $(|u'|^{p-2}u')' \in L^1(-L, L)$, $-(|u'|^{p-2}u')'(x) = f(u(x))$ for a.e. $x \in (-L, L)$ such that $u(x) > 0$ and the set of points $x \in (-L, L)$ such that $u(x) = 0$ is not empty.

Remark 1. Definition 2 is not the only possible one. A notion of weak solution in $H_0^1(\Omega)$ for $p = 2$ was introduced by Dávila and Montenegro in [15,16]. Another definition was given in a paper by P. Pucci et al. [46] for $f \in C(0, +\infty) \cap L^1(0, 1)$ and allowing singular weights. This last paper deals with problems in $\mathbb{R}^N$ and mostly with uniqueness of solutions.

As a matter of fact, all weak solutions which will be obtained satisfy some additional regularity properties.

We shall prove our main result.

Theorem 1. *We define*

$$\gamma(\mu) := \frac{1}{[p/(p-1)]^{1/p}} \int_0^{\mu} \frac{dr}{(F(\mu) - F(r))^{1/p}} \tag{18}$$

and assume (12). *Then $\gamma'(\mu) = 0$ has a unique root $\mu_0 \in (r_F, \infty)$. We introduce the numbers $L_0 = \gamma(\mu_0)$ and $L_1 = \gamma(r_F)$. For $L \geqslant L_0$ we denote by $\mu_+(L)$ the largest positive number such that $L = \gamma(\mu)$, and for $L_1 \geqslant L \geqslant L_0$ let $\mu_-(L)$ be the smallest solution of $L = \gamma(\mu)$. Then we have the following cases*:

(i) *if $L \in (0, L_0)$ there is no positive* (*in fact, nonnegative*) *solution*;

(ii) *if $L = L_0$, there is a unique positive solution $u(\cdot, \mu_+(L_0))$, and*

$$u'(\pm L_0, \mu_+(L_0)) = \mp A^{-1}(F(\mu_+(L_0))) \ (\lessgtr 0) \quad \text{where } A(r) := [(p-1)/p]r^p;$$

(iii) *if $L \in (L_0, L_1]$, there are two positive solutions $P(\cdot, L) = u(\cdot, \mu_-(L))$ and $Q(\cdot, L) = u(\cdot, \mu_+(L))$, Moreover, $P(\cdot, L) < Q(\cdot, L)$ on $(-L, L)$. In addition, if $L \in (L_0, L_1)$ then*

$$u'(\pm L, \mu_\pm(L)) = \mp A^{-1}(F(\mu_\pm(L))) \ (\lessgtr 0),$$

but $u'(\pm L_1, \mu_-(L_1)) = 0$. In fact

$$u(|x|, \mu_-(L_1)) = \psi^{-1}(L_1 - |x|), \quad \text{for any } x \in (-L_1, L_1),$$

where $\psi(.)$ is defined by

$$\psi(\tau) = \frac{1}{[p/(p-1)]^{1/p}} \int_0^{\tau} \frac{dr}{(-F(r))^{1/p}};$$

(iv) *if $L \geqslant L_1$ there is one positive solution $Q(\cdot, L) = u(\cdot, \mu_+(L))$, and*

$$u'(\pm L, \mu_+(L)) = \mp A^{-1}(F(\mu_+(L)));$$

(v) *if we assume* (8)*, for any* $L > L_1$ *there is a family of nonnegative solutions which is generated by* $u(\cdot, \mu_-(L_1))$*. In fact, for any* h*,* $|h| \leqslant L - L_1$*, the function*

$$s(x,h) = \begin{cases} u(x-h, \mu_-(L_1)) & \text{for } |x-h| \leqslant L_1, \\ 0 & \text{for } |x-h| > L_1, \end{cases}$$

is also a nonnegative solution. If N *is a positive integer and* $L \geqslant NL_1$*, given a vector* $\mathbf{y} = (y_1, y_2, \ldots, y_N)$ *with* $-L \leqslant y_i - L_1$*,* $y_i + L_1 \leqslant y_{i+1} - L_1$*,* $i = 1, \ldots, N-1$*,* $y_N + L_1 \leqslant L$ *the function*

$$s(x,\mathbf{y}) = \begin{cases} u(x-y_i, \mu_-(L_1)) & \text{for } |x-y_i| \leqslant L_1, \\ 0 & \text{for } |x-y_i| > L_1, \text{ for } i = 1, \ldots, N, \end{cases}$$

is a nonnegative solution. We call $\mathcal{S}_N(L)$ *the set of such solutions* $s(x,\mathbf{y})$*. Finally, for* $L > L_1$ *let* N *be the integral part of* L/L_1 *and let* $\mathcal{S}(L) = \bigcup_{j=1}^{N} \mathcal{S}_j(L)$*. Then the set of nontrivial solutions of* $\mathcal{P}(L)$ *is formed by* $\mathcal{S}(L)$ *jointly with* $Q(\cdot, L)$*;*

(vi) *if we replace assumption* (12) *by* (13)*, then the conclusions* (iv) *and* (v) *remain true but the number of positive solutions depends on the sign of* $\gamma'(r_F)$*: conclusions* (i)–(iii) *remain valid if* $-\infty < \gamma'(r_F) < 0$*, the unique root* μ_0 *of* $\gamma'(\mu) = 0$ *satisfying that* $\mu_0 = r_F$ *when* $\gamma'(r_F) = 0$ *and there is a unique positive solution* (*for* $L \geqslant L_1$) *if* $0 < \gamma'(r_F) < +\infty$*.*

Remark 2. If we assume (13) then we can get that $\gamma'(\mu) \to \xi$ as $\mu \downarrow r_F$, for some $\xi \in \mathbb{R}$ without any prescribed sign. Indeed, we shall prove that if $f(w) = w^q - w^m$ with $-1 < m < q < p-1$ then (13) holds if and only if $-\frac{1}{p+1} > m > -1$ and that for any $q \neq 0$ satisfying $m < q < p-1$, then there exists a small $\delta > 0$ such that $\xi < 0$ if $m \in (-\frac{1}{1+p} - \delta, -\frac{1}{1+p})$ but $\xi > 0$ if $m \in (-1, -1+\delta)$. Moreover, we get a sharper result for the special case of $q = 1 + 2m$ (assumed $-\frac{1}{p+1} > m > -1$) since we shall prove that

$$\begin{cases} \xi < 0 & \text{if } p > \frac{2(1+m)}{(-m)}, \\ \xi = 0 & \text{if } p = \frac{2(1+m)}{(-m)}, \\ \xi > 0 & \text{if } p < \frac{2(1+m)}{(-m)}. \end{cases}$$

In the special case $-1 < m < q = 0 < p - 1$, then

$$-\infty < \gamma'(r_F) < 0 \quad \text{if } m \text{ is close to } 0,$$
$$0 < \gamma'(r_F) < +\infty \quad \text{if } m \text{ is close to } -1.$$

If, in addition, $p = 2$, it was shown in [38] that

$$-\infty < \gamma'(r_F) < 0 \quad \text{if } m \in \left(-\frac{1}{2}, 0\right),$$
$$\gamma'(r_F) = 0 \quad \text{if } m = -\frac{1}{2},$$
$$0 < \gamma'(r_F) < +\infty \quad \text{if } m \in \left(-1, -\frac{1}{2}\right).$$

2.1. Proof of Theorem 1

Lemma 1. *A function* u *is a strong positive solution of problem* $\mathcal{P}(L)$ *if and only if*

$$\frac{1}{[p/(p-1)]^{1/p}} \int_{u(x)}^{\mu} \frac{dr}{(F(\mu) - F(r))^{1/p}} = |x|, \quad \text{for } |x| \leqslant L,$$

where $\mu := \|u\|_{L^\infty}$ (*such that* $\mu \in (r_F, \infty)$) *and* $L > 0$ *are related by the equation* $\gamma(\mu) = L$ *with* $\gamma(\mu)$ *given by* (18)*. Moreover*

$$u'(\pm L) = \mp A^{-1}\big(F(\mu)\big)\ (\lessgtr 0) \quad \text{where } A(r) := \big[(p-1)/p\big] r^p. \tag{19}$$

Proof. If a positive solution exists then necessarily it will have a maximum $\mu > 0$ at some point $\zeta \in (-L, L)$. So, let us consider

$$\mathcal{CP} \quad \begin{cases} -\left(|u'|^{p-2}u'\right)' = f(u), \\ u(\zeta) = \mu, \qquad u'(\zeta) = 0. \end{cases}$$

Hence, we have to consider only the case $\mu \in [r_F, \infty)$. (See Remark 1 below.) Notice that, if u is a strong positive solution of $\mathcal{P}(L)$ then $(|u'|^{p-2}u')' \in L^1(-L, L)$ implies that $u' \in L^\infty(-L, L)$. Then, since by definition $f \in L^1(-L, L)$, the formula

$$-\int_\zeta^x \left(|u'|^{p-2}u'\right)'u'\,d\tau = -\int_\zeta^x A\left(|u'|\right)'(\tau)\,d\tau = \int_\zeta^x f\left(u(\tau)\right)u'(\tau)\,d\tau = \int_\zeta^x F'\left(u(\tau)\right)d\tau$$

is well justified. Since $F'(s) = f(s) > 0$ if $s \in (r_f, \infty)$ and $A : [0, \infty) \to [0, \infty)$ is strictly increasing we have that for any $x \in (-L, L)$

$$\left|u'(x)\right| = A^{-1}\left(F(\mu) - F\left(u(x)\right)\right). \tag{20}$$

Notice that, in fact $u'(x) \leqslant 0$ for $x \in (-L, L)$ near ζ with $x > \zeta$ and that $u'(x) \geqslant 0$ for x near ζ and $x < \zeta$.

The solution of this equation is implicitly defined by

$$\int_{u(x)}^{\mu} \frac{dr}{A^{-1}(F(\mu) - F(r))} = |\zeta - x|, \tag{21}$$

since the singularity at $r = \mu$ (recall that $A^{-1}(0) = 0$) is integrable, i.e., that

$$\int_s^\mu \frac{dr}{(F(\mu) - F(r))^{1/p}} < \infty, \quad \text{for any } s \in (\mu - \varepsilon, \mu), \text{ for any } \varepsilon \text{ small enough.} \tag{22}$$

Indeed, it is easy to check that (22) always holds since $F(\mu) - F(r) \geqslant \delta(\mu - r)$ for some $\delta > 0$ and for any r near μ. Moreover, for a strong positive solution u of problem $\mathcal{CP}$, $u = 0$ only at $r = \pm L$. Therefore $\zeta = 0$. The proof of (19) is obvious from (20).

To finish the proof we only must justify that the function $u(x)$ defined implicitly by (21) satisfies that $f(u) \in L^1(-L,L)$. But we know that for $x \in (-L, -L + \delta)$, $u(x) \in [0, \varepsilon)$ for some $\varepsilon > 0$ small enough and we get that

$$-C \int_{-L}^{-L+\delta} f\left(u(x)\right)dx \leqslant -\int_{-L}^{-L+\delta} f\left(u(x)\right)u'(x)\,dx = \int_{-L}^{-L+\delta} |u'|^p\,dx < \infty$$

for some C near $A^{-1}(F(\mu))$, since $u \in C^1(-L, L)$, and the proof ends. □

Remark 3. If $\mu = r_f$, then $\mu \equiv r_f$ is clearly the only solution to $\mathcal{CP}$. If $\mu < r_f$, then it follows from the equation $\mathcal{P}(L)$ and (3) that $(|u'(x)|^{p-2}u')' > 0$, i.e., that $\omega(x) = |u'(x)|^{p-2}u'(x)$ is increasing. Since $\omega(\zeta) = 0$, this means that $u'(x) > 0$ for $x > \zeta$ and $u'(x) < 0$ for $x < \zeta$, and u cannot have a maximum at $x = \zeta$.

Now if $\mu \in (r_f, r_F)$, then by (6) we have $F(\mu) - F(u(x)) > 0$ for x near ζ. Then it follows from

$$-u'(x) = A^{-1}\left(F(\mu) - F\left(u(x)\right)\right) \tag{23}$$

that $u'(x) < 0$ in an interval containing ζ, again a contradiction.

When $\mu = r_F$ the integral of the function γ may have second singularity at $r = 0$ which is integrable due to assumption (7). The next result shows some general qualitative behavior of the graph of $\gamma(\mu)$.

Proposition 1. *If we assume* (7) *we have*:

(i) $\gamma \in C[r_F, \infty) \cap C^1(r_F, \infty)$.
(ii) *If in addition* (9) *holds then* $\gamma(\mu) \to +\infty$ *as* $\mu \to +\infty$.
(iii) $\gamma'(\mu) \to -\infty$ *as* $\mu \downarrow r_F$ *if*

$$\int_0^\delta \frac{dr}{(-F(r))^{(p+1)/p}} = +\infty \quad \textit{for } \delta \textit{ small enough} \tag{24}$$

and $\gamma'(\mu) \to -\xi$ *as* $\mu \downarrow r_F$, *for some* $\xi > 0$, *if*

$$\int_0^\delta \frac{dr}{(-F(r))^{(p+1)/p}} < +\infty \quad \textit{for } \delta \textit{ small enough.} \tag{25}$$

Proof. We use the change of variables $\tau := r/\mu$. Then

$$\gamma(\mu) = \frac{(p-1)^{1/p}}{p^{1/p}} \int_0^\mu \frac{dr}{(F(\mu) - F(r))^{1/p}} = \mu \int_0^1 \frac{d\tau}{[p/(p-1)]^{1/p}(F(\mu) - F(\tau\mu))^{1/p}}.$$

It is useful to introduce the function

$$\Lambda(\mu) = \frac{p^{1/p}}{(p-1)^{1/p}} \gamma(\mu) = \mu \int_0^1 \frac{d\tau}{(F(\mu) - F(\tau\mu))^{1/p}}.$$

Then

$$\Lambda'(\mu) = \frac{\Lambda(\mu)}{\mu} - \frac{\mu}{p} \int_0^1 \frac{F'(\mu) - \tau F'(\tau\mu)\, d\tau}{(F(\mu) - F(\tau\mu))^{(p+1)/p}}. \tag{26}$$

For $\mu \in (r_F, \infty)$ we have that $F'(\mu) \neq 0$ and it is not difficult to verify that the integral in (26) is convergent and thus $\Lambda'(\mu) \in C(r_F, \infty)$. To study the behavior of $\Lambda(\mu)$ as $\mu \downarrow r_F$, write

$$\Lambda(\mu) = \left(\int_0^{r_F} + \int_{r_F}^{\mu} \right) \frac{dr}{(F(\mu) - F(r))^{1/p}} \equiv I_1(\mu; \lambda) + I_2(\mu; \lambda).$$

For $r < r_F$ (since $F(\mu) > 0$),

$$\frac{1}{(F(\mu) - F(r))^{1/p}} \leqslant \frac{1}{(-F(r))^{1/p}}$$

and, using assumption (7), we conclude that

$$I_1(\mu) \to \int_0^{r_F} \frac{dr}{(-F(r))^{1/p}} = \Lambda(r_F) \quad \text{as } \mu \downarrow r_F,$$

and in conclusion $\Lambda \in C([r_F, +\infty))$.

On the other hand, from assumption (7), $I_2(\mu) \downarrow 0$ as $\mu \downarrow r_F$, and in conclusion $\Lambda \in C([r_F, \infty))$.

To show that $\Lambda(\mu) \to \infty$ as $\mu \to +\infty$ we use that $f \in C^0((0, \infty))$, hence $F(r)$ is locally Lipschitz continuous (for any $r > 0$ and increasing for large values of r). Then

$$F(\mu) - F(r) \leqslant \delta(\mu - r) \quad \text{for some } \delta = \delta(\mu) > 0 \text{ and for any } r \in (0, \mu),$$

and

$$\Lambda(\mu) \geqslant \frac{1}{[\delta(\mu)]^{1/p}} \int_0^\mu \frac{ds}{s^{1/p}} = \frac{p\mu^{(p-1)/p}}{[\delta(\mu)]^{1/p}(1-p)}.$$

Hence, since

$$\delta(\mu) \leqslant F'(\mu) = f(\mu),$$

from assumption (9) we get that $\Lambda(\mu) \to \infty$.

For the rest of the proof it is useful to introduce the auxiliary function $\theta(t) := pF(t) - tf(t)$. Then we get

$$\Lambda'(\mu) = \frac{1}{\mu p} \int_0^\mu \frac{(\theta(\mu) - \theta(r))\,dr}{(F(\mu) - F(r))^{(p+1)/p}}. \tag{27}$$

Let us show that $\Lambda'(\mu) \to -\infty$ as $\mu \downarrow r_F$. We write (27) as

$$\Lambda'(\mu) = \frac{1}{\mu}\left(\left\{\int_0^\delta + \int_\delta^\mu\right\} \frac{(\theta(\mu) - \theta(r))\,dr}{(F(\mu) - F(r))^{(p+1)/p}}\right) \equiv J_1(\mu) + J_2(\mu).$$

Since $\theta(r_F) = pF(r_F) - f(r_F) = -f(r_F) < 0$, by continuity of $\theta(t)$, we can choose $\sigma > 0$ such that $\theta(t) < \theta(r_F)/2 < 0$ for $t \in [r_F, r_F + \sigma)$. Moreover, since $\theta(0) = 0$ by assumption (5) and continuity we can choose $\delta \in (0, r_F)$ such that $|\theta(t)| < -\theta(r_F)/4$ for $t \in [0, \delta)$. Thus, in particular, if $\mu \in [r_F, r_F + \sigma)$ and $r \in [0, \delta)$ we have

$$\theta(\mu) - \theta(r) < \theta(r_F)/4 < 0.$$

Hence $J_2(\mu)$ remains bounded as $\mu \downarrow r_F$. For $\mu \in [r_F, r_F + \sigma)$ we have

$$\Lambda'(\mu) < \frac{\theta(r_F)}{8\mu} \int_0^\delta \frac{dr}{(F(\mu) - F(r))^{(p+1)/p}} := J^*(\mu).$$

As $\mu \downarrow r_F$, the integrand of $J^*(\mu)$ converges pointwise to $(-F(r))^{-(p+1)/p}$ near $r = 0$ and so, assumed (12), it is not integrable. Thus Fatou's lemma yields $J^*(\mu) \to \infty$ (and so $J_1(\mu)$, $\Lambda'(\mu) \to -\infty$) as $\mu \downarrow r_F$. □

In order to get a more precise information on the number of zeroes of function $\gamma'(\mu)$ we need some arguments of a different kind.

Proposition 2. *Assume the conditions of Proposition* 1 *and* (14). *Then $\gamma'(\mu)$ has a unique root $\mu_0 \in (r_F, \infty)$.*

Proof. We take some ideas from the proof given for the nondegenerate case [50]. It follows from (14), properties A and B, that

$$\Lambda'(\mu) > 0 \quad \text{on } (\mu_1, \infty)$$

and, if $r_F < \mu_2$,

$$\Lambda'(\mu) < 0 \quad \text{on } (r_F, \mu_2).$$

Thus we need only to consider $\Lambda'(\mu)$ for $\max(r_F, \mu_2) \leqslant \mu \leqslant \mu_1$. It is clear that necessarily $\Lambda'(\mu)$ has at least one zero in the interval $J := [\max(r_F, \mu_2), \mu_1]$ (recall that $\Lambda'(r_F) < 0$ by (iii) of Proposition 1). We shall show that, in fact, there can be at most one by proving that

$$\Lambda''(\mu) + C\Lambda'(\mu) > 0 \tag{28}$$

on this interval J, for some $C > 0$ (notice that then $\Lambda''(\mu) > 0$ at any such zero). To prove (28) we need to consider $\Lambda'(\mu)$. We have

$$\Lambda'(\mu) = \frac{1}{\mu p}\int_0^\mu \frac{(\theta(\mu)-\theta(r))\,dr}{(F(\mu)-F(r))^{(p+1)/p}} = \frac{1}{\mu p}\int_0^\mu \frac{\mu(\delta_1\theta)(\delta_1 F)\,dr}{(\delta_1 F)^{(p+1)/p}} = \frac{1}{p}\int_0^1 \frac{(\theta(\mu)-\theta(\mu\tau))\,d\tau}{(F(\mu)-F(\mu\tau))^{(p+1)/p}}$$

where

$$(\delta_1 h)(r) := h(\mu) - h(r), \quad 0 \leqslant r < \mu.$$

Then, differentiating we get that

$$\begin{aligned}\Lambda''(\mu) &= \frac{1}{p}\int_0^1 \bigg[\frac{(\theta'(\mu)-\tau\theta'(\mu\tau))(F(\mu)-F(\mu\tau))^{(p+1)/p}}{(F(\mu)-F(\mu\tau))^{2(p+1)/p}} \\ &\quad - \frac{(\theta(\mu)-\theta(\mu\tau))((p+1)/p)(F(\mu)-F(\mu\tau))^{1/p}(f(\mu)-\tau f(\tau\mu))}{(F(\mu)-F(\mu\tau))^{2(p+1)/p}}\bigg]\,d\tau \\ &= \frac{1}{\mu^2 p}\int_0^\mu \frac{\{(\delta_2\theta')(\delta_1 F)^{(p+1)/p} - ((p+1)/p)(\delta_1\theta)(\delta_1 F)^{1/p}(\delta_2 f)\}\,dr}{(\delta_1 F)^{2(p+1)/p}}\end{aligned}$$

where

$$(\delta_2 h)(r) := \mu h(\mu) - r h(r), \quad 0 \leqslant r < \mu.$$

Thus

$$\Lambda''(\mu) + K\Lambda'(\mu) = \frac{1}{\mu^2 p}\int_0^\mu \frac{\{(\delta_2\theta')(\delta_1 F)^{(p+1)/p} + (\delta_1\theta)(\delta_1 F)^{1/p}[\mu K(\delta_1 F) - ((p+1)/p)(\delta_2 f)]\}\,dr}{(\delta_1 F)^{2(p+1)/p}}.$$

Now we use that $F(\mu) \geqslant f(\mu)$ (since $\mu \geqslant r_F$) and that $F(r) \leqslant rf(r)$ for $r \in [0,\mu]$ and $\mu \geqslant r_F$ (make a separate analysis for $r \in [0, r_F]$ and for $r \in [r_F, \mu]$). Then, taking $K = (p+1)/p\mu$ we get that

$$\Lambda''(\mu) + \big[(p+1)/p\mu\big]\Lambda'(\mu) \geqslant \frac{1}{\mu^2 p}\int_0^\mu \frac{\delta_2\theta'\,dr}{(\delta_1 F)^{(p+1)/p}}.$$

In view of (14), properties B and C, we have that

$$\delta_2\theta' = \mu\theta'(\mu) - r\theta'(r) > 0 \quad \text{for } \max(r_F, \mu_3) < \mu$$

and it follows that

$$\Lambda''(\mu) + \big[(p+1)/p\mu\big]\Lambda'(\mu) > 0 \quad \text{for } \max(r_F, \mu_3) < \mu. \tag{29}$$

As noted above, $\Lambda'(\mu) < 0$ for $\mu < \max(r_F, \mu_2)$ and $\Lambda'(\mu) > 0$ for $\mu > \mu_1$. Therefore Λ' has at least one zero in the interval $[\max(r_F, \mu_2, \mu_3), \mu_1]$. The relation (29) implies that $\Lambda''(\mu) > 0$ at any such zero and therefore there can be at most one, completing the proof. □

Proof of Theorem 1. Parts (i), (ii), (iv) and (iii) with $L \in (L_0, L_1)$ are trivial consequences of Lemma 1 and the description found for the graph of $\gamma(\mu)$. To end part (iii) we recall that. when $\mu = r_F$, the integral of the implicit formula (21), has second singularity at $r = 0$ which is integrable due to assumption (7). So, if we define the function

$$\psi(\tau) = \frac{1}{[p/(p-1)]^{1/p}}\int_0^\tau \frac{dr}{(-F(r))^{1/p}}$$

we get that ψ is a strictly increasing function from $[0, r_F]$ into $[0, +\infty)$ and that $\psi(r_F) = \gamma(r_F) = L_1$. Then we can rewrite the expression (21) as

$$\psi(r_F) - \psi\big(u(x)\big) = |x|$$

(recall that $\zeta = 0$), and then

$$u(x) = \psi^{-1}\big(L_1 - |x|\big), \quad \text{for any } x \in (-L_1, L_1). \tag{30}$$

Since $\psi^{-1}(0) = 0$, to prove that formula (30) gives a positive strong solution $u(., \mu_-(L_1))$ we only have to justify that $f(u) \in L^1(-L_1, L_1)$. The crucial point in the rest of the proof of Theorem 1 is that $u'(\pm L_1, \mu_-(L_1)) = 0$ (notice that expression (23) says now that $-u'(x) = A^{-1}(-F(u(x)))$ since $\mu = \mu_-(L_1) = r_F$). But, for $x \in (-L_1, -L_1 + \delta)$, we have that

$$\begin{aligned} -\int_{-L_1}^{-L_1+\delta} f\big(u(x)\big)\,dx &= \lim_{\varepsilon\to 0} \int_{-L_1+\varepsilon}^{-L_1+\delta} \big(|u'|^{p-2}u'\big)'\,dx \\ &= \lim_{\varepsilon\to 0}\big(|u'|^{p-2}u'(-L_1+\delta) - |u'|^{p-2}u'(-L_1+\varepsilon)\big) \\ &= |u'|^{p-2}u'(-L_1+\delta) = \big(A^{-1}\big(-F\big(u(-L_1+\delta)\big)\big)\big)^{p-1} < +\infty, \end{aligned}$$

since $u \in C^1(-L, L)$, and, by arguing analogously on $(L_1 - \delta, L_1)$ we get that $f(u) \in L^1(-L_1, L_1)$.

To prove part (v) it is enough to observe that, since $u'(\pm L_1, \mu_-(L_1)) = 0$, the function $u(\cdot, \mu_-(L_1))$ can be extended as 0 for $L \geqslant |x| \geqslant L_1$ and the result is also a nonnegative strong solution on $(-L, L)$. In fact, as the equation is autonomous, for any h, $|h| \leqslant L - L_1$ the function

$$r(x,h) = \begin{cases} u(x-h, \mu_-(L_1)) & \text{for } |x-h| \leqslant L_1, \\ 0 & \text{for } |x-h| > L_1 \end{cases}$$

is also a nonnegative strong solution on $(-L, L)$. The rest of the statement of (v) follows easily and the proof is completed. □

Remark 4. Some generalizations to the case of a nonhomogeneous diffusion or a function $f(u)$ with several zeroes are possible (see, e.g., the comments already made in [22] and [42]).

Remark 5. In the case of part (vi) of Theorem 1 with a unique solution, i.e. under the assumptions (13), $\mu_0 = r_F$ and $L = L_1$, a difference appearing between the cases $\gamma'(r_F) = 0$ and $0 < \gamma'(r_F) < +\infty$ seems to be the one concerning the stability of the associate solutions: we conjecture that in the second case the solution is stable (as it is any positive solution in the increasing part of the bifurcation diagram) but this cannot be ensured in the first case. See also Remark 14 of Section 3.2.

2.2. Internal dead core solutions and the problem $\mathcal{P}^*(L)$

For strictly positive Dirichlet boundary conditions we can construct nonnegative solutions of

$$\mathcal{P}(L : r_F) \quad \begin{cases} -\big(|u'|^{p-2}u'\big)' = f(u) & \text{in } (-L, L), \\ u(\pm L) = r_F \end{cases}$$

having a nonempty dead core. Indeed we have

Proposition 3. *Let $L \geqslant L_1$ (with $L_1 = \gamma(r_F)$). Then, the function $u(x)$ defined as*

$$u(x) = \begin{cases} \psi^{-1}(|x| - (L - L_1)) & \text{for any } |x| \in (L - L_1, L), \\ 0 & \text{for any } |x| \in [0, L - L_1], \end{cases} \tag{31}$$

is a nonnegative strong solution of $\mathcal{P}(L : r_F)$.

Proof. We first argue for $L = L_1$ as in Lemma 1 but now putting the minimum at $x = 0$. Then we construct the solution $u(x) = \psi^{-1}(|x|)$ (recall that $\psi^{-1}(L_1) = r_F$ since $F(r_F) = 0$). Moreover, $u'(0) = 0$ since from (23) we have that $-u'(x) = A^{-1}(-F(u(x)))$ (use again that $\mu = r_F$). Since the equation is autonomous, if $L > L_1$, we get that (31) is also a nonnegative strong solution on $(-L, L)$. □

Corollary 1. *Let f satisfying $f \in C^0([0,\infty))$ with conditions from* (1) *to* (6) *and define*

$$f^*(w) = -f(r_F - w) \quad \text{for } w \in [0, r_F].$$

By defining $w(x) = r_F - u(x)$, with $u(x)$ given by (31) *we get a positive strong solution of problem*

$$\mathcal{P}^*(L) \quad \begin{cases} -\left(|z'|^{p-2}z'\right)' = f^*(z) & \text{in } (-L, L), \\ z(\pm L) = 0, \end{cases}$$

which attains its maximum ($\|w\|_\infty = r_F$) on the interval $[-L + L_1, L - L_1]$.

Remark 6. The application of the comparison principle allows to prove existence of solutions with an internal dead core even if the boundary condition is smaller than r_F but we shall not develop this here.

Remark 7. Function $f^*(w)$ verifies that $f^* \in C^0([0, r_F])$ and

$$f^*(r_F-) := \lim \sup_{s \nearrow r_F} f(s) \geqslant 0.$$

If we extend $f^*(w)$ to the rest of $[0, +\infty)$ by any function growing slower than w^pas $w \to +\infty$, then we can apply the arguments of Proposition 1 above to show the existence of a second positive solution of $\mathcal{P}^*(L)$, for L large enough (notice that assumption (14) is not necessarily satisfied and so we cannot ensure the exact multiplicity of the positive solutions).

Remark 8. The existence of solutions reaching its maximum on a positively measured set is typical of some non-Newtonian flows (see, e.g. the case of Bingham fluids in [30]). In the framework of the p-Laplacian operators, this was pointed out in [19] (see Theorem 1.14) and then extended by other authors to more general settings (see, e.g. [40]).

3. The problem $\mathcal{P}_{\lambda,\alpha}$

3.1. Case of $m \neq 0$ and $q \neq 0$

Our first remark is that, in order to define nonnegative solutions, we extend the nonlinear function by zero at $r = 0$: i.e., in the case of nonnegative solutions, we shall replace the formulation $\mathcal{P}_{\lambda,\alpha}$ by $\widehat{\mathcal{P}}_{\lambda,\alpha}$ given by

$$\widehat{\mathcal{P}}_{\lambda,\alpha} \quad \begin{cases} -\left(|u'|^{p-2}u'\right)' + \alpha g_m(u) = \lambda g_q(u) & \text{in } (-1, 1), \\ u(\pm 1) = 0, \end{cases} \tag{32}$$

with the notation

$$g_s(r) = \begin{cases} r^s & \text{if } r > 0, \\ 0 & \text{if } r = 0, \end{cases}$$

each time that $s \in (-1, 0)$ and $g_s(r) = r^s$ if $s > 0$.

To state our main result for problem (15) it is useful to introduce the notation $f(w) = w^q - w^m$ and

$$F(r) := \int_0^r f(s)\, ds = \frac{1}{1+q} r^{1+q} - \frac{1}{1+m} r^{1+m}.$$

Notice that the assumption (16) $-1 < m < q < p - 1$, $m \neq 0$, $q \neq 0$, implies that $f(0+) = -\infty$ if $m < 0$ and $f(0+) = 0$ if $m > 0$. Moreover, $r_f = 1$ and $r_F = ((1 + q)/(1 + m))^{1/(q-m)}$ (the unique zero of $F(r)$). Condition (7) holds and since

$$\frac{1}{1+m}r^{1+m} \geqslant \frac{1}{1+m}r^{1+m} - \frac{1}{1+q}r^{1+q} \geqslant \frac{(q-m)}{(1+q)(1+m)}r^{1+m} \quad \text{for } r \in (0,1),$$

we get that there exist two positive constants $\underline{C} < \overline{C}$ such that

$$\underline{C}\tau^{(1-\frac{1+m}{p})} \leqslant \psi(\tau) = \frac{1}{[p/(p-1)]^{1/p}} \int_0^\tau \frac{dr}{(-F(r))^{1/p}} \leqslant \overline{C}\tau^{(1-\frac{1+m}{p})} \tag{33}$$

for any $\tau \in (0,1)$.

Let $\lambda_1 > 0$ be given by

$$\lambda_1 := \alpha^{(p-1-q)/(p-1-m)} \left(\frac{(p-1)^{1/p}}{p^{1/p}} \int_0^{r_F} \frac{dr}{(-F(r))^{1/p}} \right)^{(p-1)(q-m)/(p-1-m)}. \tag{34}$$

Notice that $\lambda_1 < \infty$ thanks to the assumption (16). We have

Theorem 2. *Assume* (16) *and that*

$$-\frac{1}{p+1} \leqslant m. \tag{35}$$

There exists $\lambda_0 \in (0, \lambda_1)$ such that:

(a) *If $\lambda \in (0, \lambda_0)$ there is no positive solution.*
(b) *If $\lambda = \lambda_0$, there is a unique positive solution $u(\cdot, \mu_+(\lambda_0))(\mu_+(\lambda_0) := \|u\|_\infty)$ and $u'(\pm 1, \mu_+(\lambda_0)) \leqslant 0$.*
(c) *If $\lambda \in (\lambda_0, \lambda_1]$, there are two positive solutions $p(\cdot,\lambda) = u(\cdot, \mu_-(\lambda))$ and $q(\cdot,\lambda) = u(\cdot, \mu_+(\lambda))$ $(\mu_+(\lambda) := \|q\|_\infty$, $\mu_-(\lambda) := \|p\|_\infty)$. Moreover, $p(\cdot,\lambda) < q(\cdot,\lambda)$ on $(-1,1)$, $u'(\pm 1, \mu_+(\lambda_0)) \leqslant 0$, and $u'(\pm 1, \mu_-(\lambda_1)) = 0$ with*

$$\underline{k}|\pm 1 - x|^{\frac{p}{p-(1+m)}} \leqslant u\big(x, \mu_-(\lambda_1)\big) \leqslant \bar{k}|\pm 1 - x|^{\frac{p}{p-(1+m)}} \tag{36}$$

for some positive constants $\underline{k} < \bar{k}$ and for $|\pm 1 - x|$ small enough.
(d) *If $\lambda \geqslant \lambda_1$ there is one positive solution $q(\cdot,\lambda) = u(\cdot,\mu_+(\lambda))$ and $u'(\pm 1, \mu_+(\lambda)) \leqslant 0$.*
(e) *If $\lambda > \lambda_1$, there is a family of nonnegative solutions which are generated by $u(\cdot, \mu_-(\lambda_1))$ and for $\lambda > \lambda_1$ we have $\mu_-(\lambda) = (\frac{\alpha}{\lambda})^{1/(q-m)}C$ with $C = \|u(\cdot, \mu_-(\lambda_1))\|_\infty$. More precisely, let $u_1$be the function defined on $|z| \leqslant L(\alpha,\lambda_1)$, $L(\alpha,\lambda_1) := \alpha^{-(p-1-q)/(p(q-m))}\lambda_1^{(p-1-m)/(p(q-m))}$ by the formula*

$$u_1(z) = \left(\frac{\lambda_1}{\alpha}\right)^{1/(q-m)} u\big(zL(\alpha,\lambda_1)^{-1}; \mu_-(\lambda_1)\big).$$

Then, for any y, $|y| \leqslant 1 - l(\lambda)$, $l(\lambda) := (\lambda_1/\lambda)^{(p-1-m)/(q-m)(p-1)}$, the function

$$r(x;y) = \begin{cases} (\frac{\alpha}{\lambda})^{1/(q-m)} u_1((x-y)L(\alpha,\lambda_1)), & \text{for } |x-y| \leqslant l(\lambda), \\ 0, & \text{for } |x-y| > l(\lambda), \end{cases}$$

is a solution of $\mathcal{P}_{\lambda,\alpha}$. In fact, if N is a positive integer and $\lambda \geqslant \lambda_1 N^{(q-m)/(p-1-m)}$, given a vector $\mathbf{y} = (y_1, y_2, \dots, y_N)$ with

$$-1 \leqslant y_i - l(\lambda),\ y_i + l(\lambda) \leqslant y_{i+1} - l(\lambda), \quad i = 1,\dots,N-1,\ y_N + l(\lambda) \leqslant 1$$

and the set of solutions of $\mathcal{S}_N(\lambda)$ is given by the functions of the form

$$r(x,\mathbf{y}) = \begin{cases} (\frac{\alpha}{\lambda})^{1/(q-m)} u_1((x-y_i)L(\alpha,\lambda_1)), & \text{for } |x-y_i| \leqslant l(\lambda), \\ 0, & \text{for } |x-y_i| > l(\lambda), \end{cases}$$

then the set of nontrivial and nonnegative solutions of $\mathcal{P}(\lambda)$ is formed by $\mathcal{S}(\lambda)$ jointly with $q(\cdot,\lambda)$ where $\mathcal{S}(\lambda)$ is the set defined by $\mathcal{S}(\lambda) = \bigcup_{j=1}^N \mathcal{S}_j(\lambda)$.

(f) *If we assume* (16) *and replace the condition* (35) *by*

$$-\frac{1}{p+1} > m > -1, \tag{37}$$

then the conclusions (d) *and* (e) *remain true and there exists a small* $\delta > 0$ *such that there are exactly two positive solutions if* $m \in (-\frac{1}{1+p} - \delta, -\frac{1}{1+p})$ *and* $\lambda \in (\lambda_0, \lambda_1)$ *but there is a unique positive solution if* $m \in (-1, -1+\delta)$ *and* $\lambda \in (\lambda_0, \lambda_1)$. *Moreover, in the special case of*

$$q = 1 + 2m$$

the exact number of positive solutions, for $\lambda \in (\lambda_0, \lambda_1)$, *is*

$$\begin{cases} 2 & \textit{if } p > \dfrac{2(1+m)}{(-m)}, \\ 1 & \textit{if } p = \dfrac{2(1+m)}{(-m)}, \\ 1 & \textit{if } p < \dfrac{2(1+m)}{(-m)}. \end{cases}$$

The proof will be obtained as a consequence of the study of the bifurcation diagram for the auxiliary problem

$$\mathcal{P}(L) \quad \begin{cases} -\left(|u'|^{p-2}u'\right)' = u^q - u^m & \text{in } (-L, L), \\ u(\pm L) = 0, \end{cases}$$

where we consider the equation on a general interval $(-L, L)$ and take L as variable parameter.

Proof of Theorem 2. Let $u_{\lambda,\alpha}$ be a solution of $\mathcal{P}_{\lambda,\alpha}$ Then the change of variables

$$u_{\lambda,\alpha}(x) = \left(\frac{\alpha}{\lambda}\right)^{1/(q-m)} u\left(\alpha^{-(p-1-q)/(p(q-m))}\lambda^{(p-1-m)/(p(q-m))}x\right) \tag{38}$$

transforms $u_{\lambda,\alpha}$ into a solution u of the problem $\mathcal{P}(L)$ with $L := \alpha^{-\frac{p-1-q}{p(q-m)}}\lambda^{\frac{p-1-m}{p(q-m)}}$ and, conversely, any solution u of problem $\mathcal{P}(L)$ into a solution of $\mathcal{P}_{\lambda,\alpha}$. Indeed, if we write formula (38) in the form

$$u_{\lambda,\alpha}(x) = au(bx) \tag{39}$$

then

$$\left(\left|u'_{\lambda,\alpha}\right|^{p-2}u'_{\lambda,\alpha}\right)'(x) = a^{p-1}b^p\left(|u'|^{p-2}u'\right)'(bx),$$

and u is a solution u of the equation $-(|u'|^{p-2}u')' = u^q - u^m$ if and only if

$$\lambda a^{q-(p-1)} = \alpha a^{m-(p-1)} = b^p. \tag{40}$$

The first identity leads to

$$a = \left(\frac{\alpha}{\lambda}\right)^{1/(q-m)} \tag{41}$$

and the substitution in the second identity of (40) gives

$$b = \alpha^{-(p-1-q)/(p(q-m))}\lambda^{m-p+1/(p(m-q))}. \tag{42}$$

By replacing (41) and (42) into (39) we get (38).

We define now $\lambda_0 := [\gamma(\mu_0)\alpha^{(p-1-q)/(p(q-m))}]^{(p(q-m))/(p-1-m)}$. All the conditions of Theorem 1 are satisfied: for instance assumption (14) holds with

$$\mu_1 = \left[\frac{(\frac{p}{1+m}-1)}{(\frac{p}{1+q}-1)}\right]^{1/(q-m)} \quad \left(> r_F = \left((1+q)/(1+m)\right)^{1/(q-m)}\right),$$

$$\mu_2 = \left[\frac{(1+m)(\frac{p}{1+m}-1)}{(1+q)(\frac{p}{1+q}-1)}\right]^{1/(q-m)} \quad (< \mu_1),$$

and

$$\mu_3 = \mu_2 \ (< \mu_1).$$

The equivalence between conditions (12) and (35), once we assume $f(w) = w^q - w^m$ is more delicate.

Lemma 2. *Assume $f(w) = w^q - w^m$ with* (16). *Then* (12) *holds if and only if condition* (35) *is satisfied.*

Proof of Lemma 2. For $f(u) = u^q - u^m$, the function γ may be written as follows

$$\gamma(\mu) = (q+1)^{\frac{1}{p}} \int_0^1 l_1(\tau)\left[\frac{\mu^{p-q-1}}{l_2(\tau) - \frac{(q+1)\mu^{m-q}}{m+1}}\right]^{\frac{1}{p}} d\tau, \tag{43}$$

where l_1 and l_2 are defined by

$$l_1(\tau) = \frac{1}{(1-\tau^{m+1})^{1/p}} \quad \text{and} \quad l_2(\tau) = \frac{1-\tau^{q+1}}{1-\tau^{m+1}}. \tag{44}$$

Since γ is continuous, and there exists $\lim_{\mu\to r_F^+} \gamma'(\mu)$ it is satisfied that $\gamma'(r_F) = \lim_{\mu\to r_F^+} \gamma'(\mu)$ and then

$$\gamma'(r_F) = \frac{(q+1)^{\frac{1}{p}} r_F^{-(q+1)/p}}{p} \int_0^1 \frac{l_1(\tau)}{(l_2(\tau)-1)^{1/p}}\left[\frac{(l_2(\tau)-1)(p-q-1)+m-q}{l_2(\tau)-1}\right] d\tau. \tag{45}$$

For convenience, we rewrite (45) as

$$\gamma'(r_F) = \frac{(q+1)^{\frac{1}{p}} r_F^{-(q+1)/p}}{p} \int_0^1 \frac{1}{(\tau^{m+1})^{1+1/p}(1-\tau^{q-m})^{1/p}} l_3(\tau)\, d\tau, \tag{46}$$

where l_3 is given by

$$l_3(\tau) = \frac{\tau^{m+1}(1-\tau^{q-m})(p-q-1) + (m-q)(1-\tau^{m+1})}{1-\tau^{q-m}}, \tag{47}$$

which is continuous in [0, 1] once extended by continuity. Since l_3 is continuous in [0, 1] (46) will be divergent if and only if $(m+1)(1+1/p) \geqslant 1$ which is equivalent to (35). Taking into account that $\Lambda(r_F)$ is bounded, the identity (26) and (46), the condition (12) is satisfied if and only if the inequality (35) is satisfied. □

For the proof of Theorem 2 we shall need to carry out a careful study of the different subcases associated to the assumption (13). Moreover, we will use the following lemma.

Lemma 3. *Let $\alpha > 0$, $\beta > 0$ and let $g : [0,1] \to \mathbb{R}$ be the function defined by $g(\tau) = \frac{(1-\tau^\alpha)}{(1-\tau^\beta)}$ and $g(1) = \frac{\alpha}{\beta}$. If $\alpha/\beta > 1$* (*respectively $\alpha/\beta < 1$*)*, then g is monotone increasing* (*respectively decreasing*) *and attains its maximum* (*respectively minimum*) *at $\tau = 1$.*

Proof of Lemma 3. It is easily seen that g is continuous in $[0,1]$. The function g is such that $g'(\tau) = \frac{\tau^{\alpha-1}h(\tau)}{(1-\tau^\beta)^2}$, where $h(\tau) = -\alpha - (\beta-\alpha)\tau^\beta + \beta\tau^{(\beta-\alpha)}$. Since, for $\alpha > \beta$ (respectively $\alpha < \beta$), $h'(\tau) < 0$ (respectively > 0) for all $\tau \in (0,1)$, h is monotone, and taking into account that $h(1) = 0$ the sign of $h(\tau)$ is given by the sign of $\frac{\alpha}{\beta} - 1$ and the result follows. □

Proposition 4. *Assume* $f(w) = w^q - w^m$ *with* m *and* q *satisfying* (16).

(i) *Then* (13) *holds if and only if condition* (37) *is satisfied. Moreover there exists* $\delta > 0$ *small enough such that* $-\infty < \gamma'(r_F) < 0$ *if* $m \in (-\frac{1}{1+p} - \delta, -\frac{1}{1+p})$ *and* $0 < \gamma'(r_F) < +\infty$ *if* $m \in (-1, -1+\delta)$.

(ii) *If, in addition,* $q = 2m+1$ *then*

$$-\infty < \gamma'(r_F) < 0 \quad \text{if } p > \frac{2(1+m)}{(-m)},$$
$$\gamma'(r_F) = 0 \quad \text{if } p = \frac{2(1+m)}{(-m)},$$
$$0 < \gamma'(r_F) < +\infty \quad \text{if } p < \frac{2(1+m)}{(-m)}.$$

Proof of Proposition 4. In order to prove the first part of (i) we obtain the following estimate

$$l_3(\tau) \leqslant (p-q-1)\tau^{m+1} + (m-q)\max\left(1, \frac{m+1}{q-m}\right), \tag{48}$$

which follows from the continuity of l_3 and Lemma 3. According to (48) we get

$$\gamma'(r_F) \leqslant \frac{(q+1)^{\frac{1}{p}} r_F^{-(q+1)/p}}{p(q-m)} \left(\mathrm{B}\left(\frac{1}{q-m}\left(1 - \frac{m+1}{p}\right), 1 - \frac{1}{p}\right) + (m-q)\max\left(1, \frac{m+1}{q-m}\right) \mathrm{B}\left(\frac{1}{q-m}\left(1 - (m+1)\left(1+\frac{1}{p}\right)\right), 1 - \frac{1}{p}\right)\right), \tag{49}$$

where $\mathrm{B}(\alpha, \beta)$ is the Beta function, usually defined as

$$\mathrm{B}(\alpha, \beta) := \int_0^1 u^{\alpha-1}(1-u)^{\beta-1}\,du. \tag{50}$$

Taking into account that $m - q < 0$, (49) and $\lim_{\tau\to 0} \mathrm{B}(\tau, 1-1/p) = +\infty$, we obtain

$$\gamma'(r_F) < 0 \tag{51}$$

if $m \in ((-\frac{1}{1+p} - \delta, \frac{-1}{1+p}))$ for δ small enough.

In order to prove the last part of (i) we now rewrite (46) as

$$\gamma'(r_F) = \frac{(q+1)^{\frac{1}{p}} r_F^{-(q+1)/p}}{p} \left[\int_0^1 \frac{p-q-1}{(\tau^{m+1})^{1+1/p}(1-\tau^{q-m})^{1/p}}\,d\tau + \int_0^1 l_4(\tau) l_1(\tau) \frac{1}{(\tau^{m+1})^{1+1/p}}\,d\tau \right] \tag{52}$$

where l_4 is given by

$$l_4(\tau) := (\tau^{m+1} - 1)\left[p - q - 1 + \frac{(q-m)}{(1-\tau^{q-m})}\right]. \tag{53}$$

By virtue of Lemma 3, if $m+1 < q-m$ and $c > 1$, l_4 is such that (notice that the integrand below is nonnegative and bounded above by 1)

$$\begin{aligned}\int_0^1 \left|l_4(\tau)\right|^c d\tau &\leqslant \left(2\max\left((p-q-1), q-m\right)\right)^c \int_0^1 \left(\frac{1-\tau^{m+1}}{2} + \frac{1-\tau^{m+1}}{2(1-\tau^{q-m})}\right)^c d\tau \\ &\leqslant \left(2\max\left((p-q-1), q-m\right)\right)^c \\ &\quad \times \left(\int_0^1 \frac{1-\tau^{m+1}}{2}\,d\tau + \int_0^{\frac{1}{2}} \frac{1-\tau^{m+1}}{2(1-(\frac{1}{2})^{q-m})}\,d\tau + \int_{\frac{1}{2}}^1 \frac{1-\frac{1}{2}^{m+1}}{2(1-(\frac{1}{2})^{q-m})}\,d\tau \right)\end{aligned}$$

$$= \frac{(2\max((p-q-1), q-m))^c}{2}\left(\frac{m+1}{m+2} + \frac{m+2-(1/2)^{m+1}}{2(m+2)(1-(1/2)^{q-m})} + \frac{1-(1/2)^{m+1}}{2(1-(1/2)^{q-m})}\right). \tag{54}$$

Upon application of Hölder inequality to the second integral of (52) with $c = p(1+p)/2$ and $c' = \frac{p(1+p)}{p(1+p)-2}$ we obtain

$$\int_0^1 \left| l_4(\tau) l_1(\tau) \frac{1}{(\tau^{m+1})^{1+1/p}} \right| d\tau \leqslant \left(\int_0^1 \left| l_4(\tau) \right|^c \right)^{1/c} \left(\int_0^1 \left| \frac{l_1(\tau)}{(\tau^{m+1})^{1+1/p}} \right|^{c'} \right)^{1/c'}. \tag{55}$$

The inequalities (54) and (55) imply that the second integral in (52) goes to zero as $m > -1$ goes to -1. Since the first integral in (52) is bounded from below by a positive constant as $m > -1$ goes to -1 for a fixed value of q that satisfies (16), the result follows and ends the proof of (i).

In order to proof (ii) just notice that for $q = 2m+1$ the function l_3 takes the form

$$l_3(\tau) = \tau^{m+1}(p-q-1) + (m-q). \tag{56}$$

Then $\gamma'(r_F)$ may be written in closed form in terms of the Euler Beta function as

$$\begin{aligned} \gamma'(r_F) = {} & \frac{(q+1)^{\frac{1}{p}} r_F^{-(q+1)/p}}{p(m+1)} \left((p-q-1)\mathrm{B}\left(\frac{1}{m+1} - \frac{1}{p}, 1 - \frac{1}{p} \right) \right. \\ & \left. + (m-q)\mathrm{B}\left(\frac{1}{m+1} - \frac{1}{p} - 1, 1 - \frac{1}{p} \right) \right). \end{aligned} \tag{57}$$

By means of the identities $\mathrm{B}(\alpha,\beta) = \Gamma(\alpha)\Gamma(\beta)/\Gamma(\alpha+\beta)$, $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$, and taking into account that now $q = 2m+1$ we obtain

$$\gamma'(r_F) = \frac{(q+1)^{\frac{1}{p}} r_F^{-(q+1)/p} \mathrm{B}(\frac{1}{m+1} - \frac{1}{p} - 1, 1 - \frac{1}{p})}{p(m+1)} \left(\big(p - 2(m+1)\big) \frac{p-(m+1)(p+1)}{p-2(m+1)} - (m+1) \right), \tag{58}$$

where $\Gamma(\alpha) = \int_0^{+\infty} t^{\alpha-1}\exp(-t)\,dt$ is the Gamma function. From (58), (ii) immediately follows and the proof is complete. □

Then, from Theorem 1 we get that the bifurcation equation $L = \gamma(\mu)$ for the solutions of $\mathcal{P}(L)$ leads to the equivalent bifurcation equation

$$\alpha^{-(p-1-q)/(p(q-m))} \lambda^{(p-1-m)/(p(q-m))} = \gamma\left(\|u_{\lambda,\alpha}\|_\infty \left(\frac{\lambda}{\alpha}\right)^{1/(q-m)} \right) \tag{59}$$

for the solutions of $\mathcal{P}_{\lambda,\alpha}$. Since $\gamma(\mu_0)$ is the minimum value of γ we deduce that if $\lambda < \lambda_0$, Eq. (59) has no solution and for $\lambda = \lambda_0$ there is only one solution. This proves (a) and (b). Since the range of the branch γ_+ is $[\gamma(\mu_0), +\infty)$ we deduce that for any $\lambda > \lambda_0$ Eq. (59) has, at least, a solution, which implies the existence of a solution of $\mathcal{P}_{\lambda,\alpha}$, $q(\cdot,\lambda)$. If $\lambda \in (\lambda_0, \lambda_1]$, from the continuity of the branch γ_-, we deduce the existence of a second solution of Eq. (59) which implies the existence of the solution $p(\cdot,\lambda)$. Both roots correspond to the two solutions $\mu_- < \mu_+$ of the equation $L = \gamma(\mu)$ and then

$$\mu_- = \big\|P(\cdot,\lambda)\big\|_\infty \left(\frac{\lambda}{\alpha}\right)^{1/(q-m)}, \qquad \mu_+ = \big\|Q(\cdot,\lambda)\big\|_\infty \left(\frac{\lambda}{\alpha}\right)^{1/(q-m)}$$

for a suitable λ, which proves that $\|p(\cdot,\lambda)\|_\infty < \|q(\cdot,\lambda)\|_\infty$. Moreover, since

$$p(\cdot,\lambda) = \left(\frac{\alpha}{\lambda}\right)^{1/(q-m)} P\big(\alpha^{-(p-1-q)/(p(q-m))} \lambda^{(p-1-m)/(p(q-m))} x, L\big)$$

and

$$q(\cdot,\lambda) = \left(\frac{\alpha}{\lambda}\right)^{1/(q-m)} Q\big(\alpha^{-(p-1-q)/(p(q-m))} \lambda^{(p-1-m)/(p(q-m))} x, L\big),$$

using (iii) of Theorem 1, we get (c). Part (d) is proved in a similar way. The information (33) on function $\psi(\tau)$ leads to estimate (36).

The conclusion (f), ending the proof of Theorem 2, is a consequence of Proposition 2. □

Remark 9. Estimate (36) implies that if $m \in (-1, 0)$ in fact $f(u) \in L^r(-1, 1)$ for any $r \in [1, \frac{p-(1+m)}{p(-m)})$.

3.2. The case $0 = m < q < p - 1$

We consider the problem

$$\mathcal{P}^H_{\lambda,\alpha} \quad \begin{cases} -\big(|u'|^{p-2}u'\big)' + \alpha H(u) \ni \lambda u^q & \text{in } (-1, 1), \\ u(\pm 1) = 0, \end{cases} \tag{60}$$

which "formally" coincides with problem $\mathcal{P}_{\lambda,\alpha}$ with $m = 0$. Here

$$0 < q < p - 1, \tag{61}$$

and $H(u)$ denotes the maximal monotone graph of $\mathbb{R}^2$ associated to the Heaviside function

$$H(r) = \begin{cases} 1 & \text{if } r > 0, \\ [0, 1] & \text{if } r = 0, \\ 0 & \text{if } r < 0. \end{cases}$$

Notice that if u is a positive solution of $\mathcal{P}^H_{\lambda,\alpha}$ then $H(u(x)) = 1$ for any $x \in (-1, 1)$ and so u necessarily satisfy the easier formulation

$$\widehat{\mathcal{P}}^H_{\lambda,\alpha} \quad \begin{cases} -\big(|u'|^{p-2}u'\big)' = \lambda u^q - \alpha & \text{in } (-1, 1), \\ u(\pm 1) = 0. \end{cases} \tag{62}$$

Moreover, since $0 < q < p - 1$, the required condition $u \in W^{1,p}_0(-L, L)$ implies that $u^q \in L^1(-1, 1)$ and so the notion of solution derived from Definition 1 is clarified.

For nonnegative solutions the meaning of the expression $H(u(x))$ in Definition 2 needs to be made more precise:

Definition 3. We say that $u \in W^{1,p}_0(-1,1)$ is a *nonnegative strong solution* of problem $\mathcal{P}^H_{\lambda,\alpha}$ if $u \geqslant 0$ on $(-1, 1)$, $u^q \in L^1(-1, 1)$, $(|u'|^{p-2}u')' \in L^1(-1, 1)$, and there exists $h \in L^\infty(-1, 1)$ such that $h(x) \in H(u(x))$ for a.e. $x \in (-1, 1)$ and $-(|u'|^{p-2}u')'(x) + \alpha h(x) = \lambda u(x)^q$ for a.e. $x \in (-1, 1)$.

Remark 10. There is some analogy between this situation and the different kinds of solutions (of types I and II) introduced by Stuart [52] when dealing with semilinear elliptic problems with discontinuous nonlinearities. On the other hand, the case of 2π-periodic solutions on the real line was studied in [9] for a related equation. Finally, for a different problem having both free-boundary and (under some additional assumption) positive solutions, see [3] and [36,37].

To state our main result for problem $\mathcal{P}^H_{\lambda,\alpha}$, as in the precedent subsection, it is useful to introduce the notation $f(w) = w^q - H(w)$ and

$$F(r) := \int_0^r f(s)\,ds = \frac{1}{1+q} r^{1+q} - r_+,$$

where

$$r_+ = \max(0, r), \quad \text{for any } r \in \mathbb{R}.$$

Notice that although f is multivalued at $w = 0$ (with $f(0+) = -1$ but $f(0-) = 0$) its primitive is a continuous function. Moreover the conditions of Section 2 are satisfied. For instance, $r_f = 1$ and $r_F = ((1 + q))^{1/q}$ (the unique zero of $F(r)$). Condition (7) holds and since

$$r \geqslant r - \frac{1}{1+q} r^{1+q} \geqslant \frac{q}{(1+q)} r \quad \text{for } r \in (0,1),$$

we get that there exist two positive constants $\underline{C} < \overline{C}$ such that

$$\underline{C}\tau^{(1-\frac{1}{p})} \leqslant \psi(\tau) = \frac{1}{[p/(p-1)]^{1/p}} \int_0^\tau \frac{dr}{(-F(r))^{1/p}} \leqslant \overline{C}\tau^{(1-\frac{1}{p})} \tag{63}$$

for any $\tau \in (0,1)$.

Let $\lambda_1 > 0$ be given by

$$\lambda_1 := \alpha^{(p-1-q)/(p-1)} \left(\frac{(p-1)^{1/p}}{p^{1/p}} \int_0^{r_F} \frac{dr}{(-F(r))^{1/p}} \right)^{(p-1)q/(p-1)}. \tag{64}$$

Notice that $\lambda_1 < \infty$ thanks to the assumption (61). We have

Theorem 3. *There exists $\lambda_0 \in (0, \lambda_1)$ such that*:

(a) *If $\lambda \in (0, \lambda_0)$, there is no positive solution.*
(b) *If $\lambda = \lambda_0$, there is a unique positive solution $u(\cdot, \mu_+(\lambda_0))$ $(\mu_+(\lambda_0) := \|u\|_\infty)$ and $u'(\pm 1, \mu_+(\lambda_0)) \lesseqgtr 0$.*
(c) *If $\lambda \in (\lambda_0, \lambda_1]$, there are two positive solutions $p(\cdot, \lambda) = u(\cdot, \mu_-(\lambda))$ and $q(\cdot, \lambda) = u(\cdot, \mu_+(\lambda))$ $(\mu_+(\lambda) := \|q\|_\infty$, $\mu_-(\lambda) := \|p\|_\infty)$. Moreover, $p(\cdot, \lambda) < q(\cdot, \lambda)$ on $(-1, 1)$, $u'(\pm 1, \mu_+(\lambda_0)) \lesseqgtr 0$, and $u'(\pm 1, \mu_-(\lambda_1)) = 0$ with*

$$\underline{k}|\pm 1 - x|^{\frac{p}{p-1}} \leqslant u(x, \mu_-(\lambda_1)) \leqslant \bar{k}|\pm 1 - x|^{\frac{p}{p-1}} \tag{65}$$

for some positive constants $\underline{k} < \bar{k}$ and for $|\pm 1 - x|$ small enough.
(d) *If $\lambda \geqslant \lambda_1$, there is one positive solution $q(\cdot, \lambda) = u(\cdot, \mu_+(\lambda))$ and $u'(\pm 1, \mu_+(\lambda)) \lesseqgtr 0$.*

Finally,

(e) *if $\lambda > \lambda_1$, there is a family of nonnegative solutions which are generated by $u(\cdot, \mu_-(\lambda_1))$ and for $\lambda > \lambda_1$ we have $\mu_-(\lambda) = (\frac{\alpha}{\lambda})^{1/q} C$ with $C = \|u(\cdot, \mu_-(\lambda_1))\|_\infty$. More precisely, let $u_1$be the function defined on $|z| \leqslant L(\alpha, \lambda_1)$, $L(\alpha, \lambda_1) := \alpha^{-(p-1-q)/(pq)} \lambda_1^{(p-1)/(pq)}$ by the identity*

$$u_1(z) = \left(\frac{\lambda_1}{\alpha}\right)^{1/q} u\big(zL(\alpha, \lambda_1)^{-1}; \mu_-(\lambda_1)\big).$$

Then, for any y, $|y| \leqslant 1 - l(\lambda)$, $l(\lambda) := (\lambda_1/\lambda)^{(p-1)/q(p-1)}$, the function

$$r(x; y) = \begin{cases} (\frac{\alpha}{\lambda})^{1/q} u_1((x-y)L(\alpha, \lambda_1)), & \text{for } |x-y| \leqslant l(\lambda), \\ 0, & \text{for } |x-y| > l(\lambda), \end{cases}$$

is a solution of $\mathcal{P}^H_{\lambda,\alpha}$. In fact, if N is a positive integer and $\lambda \geqslant \lambda_1 N^{q/(p-1)}$, given a vector $\mathbf{y} = (y_1, y_2, \ldots, y_N)$ with

$$-1 \leqslant y_i - l(\lambda),\ y_i + l(\lambda) \leqslant y_{i+1} - l(\lambda), \quad i = 1, \ldots, N-1,\ y_N + l(\lambda) \leqslant 1$$

and if we define the set of solutions of $\mathcal{S}_N(\lambda)$ is given by functions

$$r(x, y) = \begin{cases} (\frac{\alpha}{\lambda})^{1/q} u_1((x-y_i)L(\alpha, \lambda_1)), & \text{for } |x-y_i| \leqslant l(\lambda), \\ 0, & \text{for } |x-y_i| > l(\lambda), \end{cases}$$

and then the set of nontrivial and nonnegative solutions of $\mathcal{P}^H_{\lambda,\alpha}$ is formed by $\mathcal{S}(\lambda)$ jointly with $q(\cdot, \lambda)$ where $\mathcal{S}(\lambda)$ is the set defined by $\mathcal{S}(\lambda) = \bigcup_{j=1}^N \mathcal{S}_j(\lambda)$.

Proof of Theorem 3. As in Theorem 2, the proof will be obtained as a consequence of the study of the bifurcation diagram for the auxiliary problem

$$\mathcal{P}^H(L) \quad \begin{cases} -\big(|u'|^{p-2}u'\big)' + H(u) \ni u^q & \text{in } (-L, L), \\ u(\pm L) = 0, \end{cases}$$

in which we consider the equation on a general interval $(-L, L)$ and take L as variable parameter. The only important modification introduced in the proof of Theorem 2 is that now

$$j(u)'(x) \in \partial j\big(u(x)\big)u'(x),$$

where $\partial j(\cdot)$ means the multi-valuated subdifferential associated to a given convex lower semi continuous real function j. As before, let $u_{\lambda,\alpha}$ be a solution of $\mathcal{P}^H_{\lambda,\alpha}$ Then the change of variables

$$u_{\lambda,\alpha}(x) = \left(\frac{\alpha}{\lambda}\right)^{1/q} u\big(\alpha^{-(p-1-q)/(pq)}\lambda^{(p-1)/(pq)}x\big) \tag{66}$$

transforms $u_{\lambda,\alpha}$ into a solution u (defined in an analogous way to Definition 3) of the problem $\mathcal{P}^H(L)$ with $L := \alpha^{-\frac{p-1-q}{pq}}\lambda^{\frac{p-1}{pq}}$ and, conversely, any solution u of problem $\mathcal{P}^H(L)$ into a solution of $\mathcal{P}^H_{\lambda,\alpha}$.

We define $\lambda_0 := [\gamma(\mu_0)\alpha^{(p-1-q)/(pq)}]^{(pq)/(p-1)}$. All the conditions of Theorem 1 are satisfied: for instance assumption (14) holds with

$$\mu_1 = \left[\frac{(p-1)}{(\frac{p}{1+q}-1)}\right]^{1/q} \big(> r_F = (1+q)^{1/q}\big),$$

$$\mu_2 = \left[\frac{(p-1)}{(1+q)(\frac{p}{1+q}-1)}\right]^{1/q} \quad (= \mu_1),$$

and

$$\mu_3 = \mu_2 \ (= \mu_1).$$

We also point out that, by Lemma 2, we know that condition (12) holds since $q \in (0, p-1)$. Then, from Theorem 1 we get that the bifurcation equation $L = \gamma(\mu)$ for the solutions of $\mathcal{P}(L)$ leads to the equivalent bifurcation equation

$$\alpha^{-(p-1-q)/(pq)}\lambda^{(p-1)/(pq)} = \gamma\left(\|u_{\lambda,\alpha}\|_\infty\left(\frac{\lambda}{\alpha}\right)^{1/q}\right) \tag{67}$$

for the solutions of $\mathcal{P}_{\lambda,\alpha}$. Since $\gamma(\mu_0)$ is the minimum value of γ we deduce that if $\lambda < \lambda_0$ Eq. (67) has no solution and for $\lambda = \lambda_0$ there is only one solution. The rest of the statement is proved in a similar way. The information (63) on function $\psi(\tau)$ leads to estimate (65) and the proof ends. □

Remark 11. Notice that, a posteriori, we can prove that function $h \in L^\infty(-1, 1)$, in Definition 3, satisfying that $h(x) \in H(u(x))$ in fact must verify that

$$h(x) = 0 \quad \text{on the interior set of } \big\{x \in (-1, 1)\colon\ u(x) = 0\big\}$$

because $(|u'|^{p-2}u')' \in L^1(-1, 1)$ and then the result follows from a well-known property of Sobolev spaces (see, e.g., Kinderlehrer and Stampacchia [43]).

Remark 12. The study of bifurcation diagrams associated to nonlinear eigenvalue problems formulated in terms of variational inequalities (of a quite similar nature to problem $\mathcal{P}^H_{\lambda,\alpha}$) has been considered in the literature by many authors (see, e.g. [13,26]).

3.3. The case $-1 < m < q = 0$

Finally, we consider the problem

$$\mathcal{P}_{\lambda,\alpha,H} \quad \begin{cases} -\left(|u'|^{p-2}u'\right)' + \alpha g_m(u) \in \lambda H(u) & \text{in } (-1,1), \\ u(\pm 1) = 0, \end{cases} \tag{68}$$

with

$$g_m(r) = \begin{cases} r^m & \text{if } r > 0, \\ 0 & \text{if } r = 0, \end{cases}$$

which "formally" coincides with problem $\mathcal{P}_{\lambda,\alpha}$ with $q = 0$. Here

$$-1 < m < 0 < p - 1, \tag{69}$$

and, again, $H(u)$ denotes the maximal monotone graph of $\mathbb{R}^2$ associated to the Heaviside function. For positive solutions u of $\mathcal{P}_{\lambda,\alpha,H}$ we have that $H(u(x)) = 1$ for any $x \in (-1,1)$ and so u necessarily satisfy the easier problem

$$\widehat{\mathcal{P}}_{\lambda,\alpha,H} \quad \begin{cases} -\left(|u'|^{p-2}u'\right)' + \alpha u^m = \lambda & \text{in } (-1,1), \\ u(\pm 1) = 0. \end{cases}$$

Then we arrive to the natural adaptation of positive solution given in Definition 1 (we replace $(-1,1)$ by $(-L,L)$):

Definition 4. We say that $u \in W_0^{1,p}(-L,L)$ is a *positive strong solution* of problem $\mathcal{P}_{\lambda,\alpha,H}$ if $u > 0$ on $(-L,L)$, $u^m \in L^1(-L,L)$, $(|u'|^{p-2}u')' \in L^1(-L,L)$ and $-(|u'|^{p-2}u')'(x) + \alpha u(x)^m = \lambda$ for a.e. $x \in (-L,L)$.

For nonnegative solutions the meaning of the expression $H(u(x))$ in Definition 2 needs to be made more precise:

Definition 5. We say that $u \in W_0^{1,p}(-L,L)$ is a *nonnegative strong solution* of problem $\mathcal{P}_{\lambda,\alpha,H}$ if $u \geqslant 0$ on $(-L,L)$, $g_m(u) \in L^1(-L,L)$, $(|u'|^{p-2}u')' \in L^1(-L,L)$, and there exists $h \in L^\infty(-L,L)$ such that $h(x) \in H(u(x))$ for a.e. $x \in (-1,1)$ and $-(|u'|^{p-2}u')'(x) + \alpha u(x)^m = \lambda h(x)$ for a.e. $x \in (-L,L)$ for a.e. $x \in (-L,L)$ such that $u(x) > 0$ and the set of points $x \in (-L,L)$ such that $u(x) = 0$ is not empty.

To state our main result for problem $(\mathcal{P}_{\lambda,\alpha,H})$, again, we introduce the notation $f(w) = H(w) - w^m$ and

$$F(r) := \int_0^r f(s)\,ds = r_+ - \frac{1}{1+m} r^{1+m}.$$

Notice that the assumption (69) implies that $f(0+) = -\infty$ if $m < 0$ but $f(0-) = 0$. Moreover, $r_f = 1$ and $r_F = (1/(1+m))^{1/(-m)}$ (the unique zero of $F(r)$). Condition (7) holds and since

$$\frac{1}{1+m} r^{1+m} \geqslant \frac{1}{1+m} r^{1+m} - r \geqslant \frac{(-m)}{(1+m)} r^{1+m} \quad \text{for } r \in (0,1),$$

we get that there exists two positive constants $\underline{C} < \overline{C}$ such that

$$\underline{C}\tau^{(1-\frac{1+m}{p})} \leqslant \psi(\tau) = \frac{1}{[p/(p-1)]^{1/p}} \int_0^\tau \frac{dr}{(-F(r))^{1/p}} \leqslant \overline{C}\tau^{(1-\frac{1+m}{p})} \tag{70}$$

for any $\tau \in (0,1)$.

Let $\lambda_1 > 0$ be given by

$$\lambda_1 := \alpha^{(p-1)/(p-1-m)} \left(\frac{(p-1)^{1/p}}{p^{1/p}} \int_0^{r_F} \frac{dr}{(-F(r))^{1/p}} \right)^{(p-1)(-m)/(p-1-m)}. \tag{71}$$

Notice that $\lambda_1 < \infty$ thanks to the assumption (16). We have

Theorem 4. *Let us assume that* (69) *and* (35) *are satisfied. There exists* $\lambda_0 \in (0, \lambda_1)$ *such that*:

(a) *If* $\lambda \in (0, \lambda_0)$ *there is no positive solution.*
(b) *If* $\lambda = \lambda_0$*, there is a unique positive solution* $u(\cdot, \mu_+(\lambda_0))$ $(\mu_+(\lambda_0) := \|u\|_\infty)$ *and* $u'(\pm 1, \mu_+(\lambda_0)) \lesseqgtr 0$.
(c) *If* $\lambda \in (\lambda_0, \lambda_1]$*, there are two positive solutions* $p(\cdot, \lambda) = u(\cdot, \mu_-(\lambda))$ *and* $q(\cdot, \lambda) = u(\cdot, \mu_+(\lambda))$ $(\mu_+(\lambda) := \|q\|_\infty$, $\mu_-(\lambda) := \|p\|_\infty)$. *Moreover,* $p(\cdot, \lambda) < q(\cdot, \lambda)$ *on* $(-1, 1)$, $u'(\pm 1, \mu_+(\lambda_0)) \lesseqgtr 0$, *and* $u'(\pm 1, \mu_-(\lambda_1)) = 0$ *with*

$$\underline{k}|\pm 1 - x|^{\frac{p}{p-(1+m)}} \leqslant u\big(x, \mu_-(\lambda_1)\big) \leqslant \bar{k}|\pm 1 - x|^{\frac{p}{p-(1+m)}} \tag{72}$$

for some positive constants $\underline{k} < \bar{k}$ *and for* $|\pm 1 - x|$ *small enough.*
(d) *If* $\lambda \geqslant \lambda_1$ *there is one positive solution* $q(\cdot, \lambda) = u(\cdot, \mu_+(\lambda))$ *and* $u'(\pm 1, \mu_+(\lambda)) \lesseqgtr 0$.

Finally,

(e) *If* $\lambda > \lambda_1$*, there is a family of nonnegative solutions which are generated by* $u(\cdot, \mu_-(\lambda_1))$ *and for* $\lambda > \lambda_1$ *we have* $\mu_-(\lambda) = (\frac{\alpha}{\lambda})^{1/(q-m)} C$ *with* $C = \|u(\cdot, \mu_-(\lambda_1))\|_\infty$. *More precisely, let* u_1 *be the function defined on* $|z| \leqslant L(\alpha, \lambda_1)$, $L(\alpha, \lambda_1) := \alpha^{-(p-1)/(-pm)} \lambda_1^{(p-1-m)/(-pm)}$ *by the identity*

$$u_1(z) = \left(\frac{\lambda_1}{\alpha}\right)^{1/(-m)} u\big(zL(\alpha, \lambda_1)^{-1}; \mu_-(\lambda_1)\big).$$

Then, for any y, $|y| \leqslant 1 - l(\lambda)$, $l(\lambda) := (\lambda_1/\lambda)^{(p-1-m)/(q-m)(p-1)}$, *the function*

$$r(x, y) = \begin{cases} (\frac{\alpha}{\lambda})^{1/(-m)} u_1((x-y)L(\alpha, \lambda_1)), & \text{for } |x-y| \leqslant l(\lambda), \\ 0, & \text{for } |x-y| > l(\lambda), \end{cases}$$

is a solution of $\mathcal{P}_{\lambda,\alpha,H}$. *In fact, if* N *is a positive integer and* $\lambda \geqslant \lambda_1 N^{(-m)/(p-1-m)}$*, given a vector* $\mathbf{y} = (y_1, y_2, \ldots, y_N)$ *with*

$$-1 \leqslant y_i - l(\lambda),\ y_i + l(\lambda) \leqslant y_{i+1} - l(\lambda), \quad i = 1, \ldots, N-1,\ y_N + l(\lambda) \leqslant 1$$

and if we define the set of solutions of $\mathcal{S}_N(\lambda)$ *is given by the set of functions of the form*

$$r(x, y) = \begin{cases} (\frac{\alpha}{\lambda})^{1/(-m)} u_1((x-y_i)L(\alpha, \lambda_1)), & \text{for } |x-y_i| \leqslant l(\lambda), \\ 0, & \text{for } |x-y_i| > l(\lambda), \end{cases}$$

and then the set of nontrivial and nonnegative solutions of $\mathcal{P}(\lambda)$ *is formed by* $\mathcal{S}(\lambda)$ *jointly with* $q(\cdot, \lambda)$ *where* $\mathcal{S}(\lambda)$ *is the set defined by* $\mathcal{S}(\lambda) = \bigcup_{j=1}^N \mathcal{S}_j(\lambda)$.
(f) *If we assume* (16) *and replace the condition* (35) *by*

$$-\frac{1}{p+1} > m > -1, \tag{73}$$

then the conclusions (d) *and* (e) *remain true and there exists a small* $\delta > 0$ *such that there are exactly two positive solutions if* $m \in (-\frac{1}{1+p} - \delta, -\frac{1}{1+p})$ *and* $\lambda \in (\lambda_0, \lambda_1)$ *but there is a unique positive solution if* $m \in (-1, -1+\delta)$ *and* $\lambda \in (\lambda_0, \lambda_1)$. *Moreover, in the special case of*

$$m = -\frac{1}{2}$$

the exact number of positive solutions, for $\lambda \in (\lambda_0, \lambda_1)$*, is*

$$\begin{cases} 2 & \text{if } p > 2, \\ 1 & \text{if } p = 2, \\ 1 & \text{if } p < 2. \end{cases}$$

Proof of Theorem 4. The proof, as in the previous cases, will be obtained as a consequence of the study of the bifurcation diagram for the auxiliary problem

$$\mathcal{P}_H(L) \quad \begin{cases} -\left(|u'|^{p-2}u'\right)' + u^m = H(u) & \text{in } (-L, L), \\ u(\pm L) = 0, \end{cases}$$

in which we consider the equation on a general interval $(-L, L)$ and take L as variable parameter. Let $u_{\lambda,\alpha}$ be a solution of $\mathcal{P}_{\lambda,\alpha,H}$. Then the change of variables

$$u_{\lambda,\alpha}(x) = \left(\frac{\alpha}{\lambda}\right)^{1/(-m)} u\left(x\alpha^{-(p-1)/(-pm)}\lambda^{(p-1-m)/(-pm)}\right) \tag{74}$$

transforms $u_{\lambda,\alpha}$ into a solution u of the problem $\mathcal{P}_H(L)$ with $L := \alpha^{-\frac{p-1}{(-pm)}}\lambda^{\frac{p-1-m}{(-pm)}}$ and conversely.

We define $\lambda_0 := [\gamma(\mu_0)\alpha^{(p-1)/(-pm)}]^{(-pm)/(p-1-m)}$. All the conditions of Theorem 1 are satisfied: for instance assumption (14) holds with

$$\mu_1 = \left[\frac{(\frac{p}{1+m} - 1)}{(p-1)}\right]^{1/(-m)} \left(> r_F = \left(1/(1+m)\right)^{1/(-m)}\right),$$
$$\mu_2 = 1 \ (< \mu_1),$$

and

$$\mu_3 = \mu_2 \ (< \mu_1).$$

Then the result follows from Theorem 1. The information (63) on function $\psi(\tau)$ leads to estimate (72) and the proof ends by applying, again, Proposition 2. □

Remark 13. When $p = 2$ then (35) becomes

$$-\frac{1}{3} \leqslant m,$$

and, as we mentioned at the Introduction, a careful study of positive solutions was initiated by Choi, Lazer and McKenna [12] and then improved by Horváth and Simon [38]. It was shown in this last paper that, if $p = 2$, then the critical value of m (in the case (f)), i.e. for $-\frac{1}{3} > m > -1$) is $m = -\frac{1}{2}$. Part (f) of the above theorem gives some answers of a similar nature for different values of p.

Remark 14. In the case of part (f) of Theorem 4 with a unique solution, i.e., under the assumptions (73), $\mu_0 = r_F$ and $\lambda = \lambda_1$, a difference appearing between the cases associated to $\gamma'(r_F) = 0$ and $0 < \gamma'(r_F) < +\infty$ seems to be the one concerning the stability of the associate solutions: we conjecture that in the second case the solution is stable (as it is any of the positive solutions in the increasing part of the bifurcation diagram) but this cannot be ensured in the first case. The linearized stability in the second case was proved in Dávila and Montenegro [16] and in [37].

Remark 15. The regularity $g_m(u) \in L^r(-1, 1)$ for any $r \in [1, \frac{p-(1+m)}{p(-m)})$ holds for any $m \in (-1, 0)$ as indicated in Remark 2 (notice that this is independent of the value of q).

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