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ON THE RETENTION OF THE INTERFACES IN SOME ELLIPTIC AND PARABOLIC NONLINEAR PROBLEMS

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Abstract. We study some *retention phenomena* on the free boundaries associated to some elliptic and parabolic problems of reaction-diffusion type. This is the case, for instance, of the *waiting time phenomenon* for solutions of suitable parabolic equations. We find sufficient conditions in order to have a discrete version of the waiting time property (the so called *nondiffusion of the support*) for solutions of the associated family of elliptic equations and prove how to pass to the limit in order to get this property for the solutions of the parabolic equation.

1. **Introduction.** We consider a general class of nonlinear elliptic and parabolic equations of the following form:

$$(EE) \qquad Au + \beta(x,u) = g(x)$$

and

$$(PE) \qquad \frac{\partial}{\partial t}\psi(u) + Au + \beta(x,u) = G(t,x)$$

where A represents an elliptic second order quasilinear operator (eventually degenerate), $\beta(.,u)$ and $\psi(u)$ are nondecreasing real functions (eventually discontinuous or multivalued). Instead to make explicit now the more general assumption that we can assume on A, β, and ψ, we shall mention some applied models in which (EE) and (PE) are of relevant interest:

i) *Reaction - diffusion equations* ([8]). In the study of a single irreversible reaction the density u of the reactant satisfies (EE) or (PE) in stationary or evolution regime. Some natural choices in that setting are $Au = Lu$ a linear second order elliptic operator (not necessarily in divergence form), $\beta(x,u) \geq \lambda u^q$ for some $q \geq 0$ (called the order of the reaction) and $\psi(u) = \alpha u$

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ii) *Non-newtonian fluid type equations* ([35]). In the one dimensional case, u represents a velocity and it is not necessarily nonnegative. Some special cases of interest are

$$Au = -\Delta_p u \equiv -div\left(|\nabla u|^{p-2}\nabla u\right), \qquad p > 1$$

(the p-Laplacian operator) and $\psi(u) = \alpha u$ (the presence of some magnetic field leads to the term $\beta(x,u) \geq \lambda |u|^{q-1} u$ for some $q \geq 0$).

iii) *Flows through porous media* ([11]). Due to the Darcy law, the more characteristic fact in that large class of problems is that in (PE) $\psi(u)$ is non-linear e.g. $\psi(u) = u^{1/m}$ with $m > 1$. The principal part of the elliptic operator can be taken as linear or not according the Reynolds number. Convection terms in A are also of interest in vertical infiltrations. A natural assumption on the absorption term is again $\beta(x,u) \geq \lambda u^q$.

It is clear that many other phenomena can be formulated in terms of some of the above settings. For instance, multivalued terms $\beta(x,u)$ and/or $\psi(u)$ appears in the study of variational inequalities, ice sheets, lubrication theory, etc. (see, e.g. [19]).

We point out that, in fact, the formulation (EE) and (PE) does not require any concrete boundary condition. Our results will be obtained for any local weak solution on subsets where they are bounded.

Some papers containing many references on the existence of solutions for the different boundary value problems (including possible dynamic boundary conditions) associated to the formulation (EE) and (PE) are [29], [13], [6] and [28], among many others.

A common fact of the above models is the occurrence of a free boundary (or interface) which usually is defined by the boundary of the support of the solution

$$\mathcal{F}(u) = \partial S(u) \cap \partial N(u)$$

where $S(u) \equiv \{Support\ of\ u\}$ and $N(u) = \{u = 0\} \equiv \{null\ set\ of\ u\}$. In the terminology of chemical kinetics $N(u)$ is called as the *dead core* and in infiltration theory $\mathcal{F}(u)$ is the *wetting front*. Some other related free boundaries are defined in other terms. For instance in the model of Non-Newtonian fluids the set where ∇u vanishes is called the *quasi-solid zone* and so its boundary $\mathcal{F}(u) = \partial\{\nabla u \neq 0\}$ defines a free boundary of the problem.

From a mathematical point of view, those free boundaries are formed when some degeneracy or singular terms arise at the equation. For instance, $0 \leq q < 1$ in i), $(p-1) > q$ in ii) and $m > 1$ in iii). Nevertheless the merely degeneracy of the equations is not enough for the formation of the free boundary $\mathcal{F}(u)$. Roughly speaking, the existence of $\mathcal{F}(u)$ depends of two different kind of conditions: 1) a balance between two of the terms of the equation that represent the particular characteristics of the phenomenon (diffusion, absorption, convection, evolution, etc.); and 2) a balance between the "*sizes*" of the domain and of the solution.

The main goal of this work is to study some *retention phenomena* in which the interface does not "*move*" with respect to the data of the problem. That rough affirmation needs to be stated with a more precision. So, in the parabolic problem it may means that $\mathcal{F}(u(t)) = \mathcal{F}(u(0))$ for any $t \in [0, t^*]$ and for some t^* (called the *waiting time* in the setting of porous media). In the elliptic case, the retention may be understood in the sense that $\mathcal{F}(u) = \mathcal{F}(g)$ (that properly seems to be noted and proved by the first time in [19] where it was called as the *nondiffusion of the support*).

The organization of the article is the following: in Section 2 we consider the elliptic problem. The nondiffusion of the support is proved under a great generality extending, in this way, the results of [2].

The parabolic problem is considered in Section 3. We study the initial behavior of the free boundary by means of the consideration of an implicit approximation scheme

$$\frac{\psi(u_n)-\psi(u_{n-1})}{\tau}+Au_n+\beta(x,u_n)=G_n(x). \tag{1}$$

In particular, we show that under suitable assumptions $\mathcal{F}(u_{n^*})=\mathcal{F}(u_0)$ for some $n^*\in N$. Then, thanks to an additional argument we pass to the limit and obtain the *waiting time* property for the solutions of (PE).

The special case of

$$\beta(x,u)=G(t,x)\equiv 0 \quad \text{ and } \quad Au=-\Delta_p u \tag{2}$$

leads to the *doubly nonlinear parabolic problems* and have been intensively studied in the literature. A usual reformulation arises if we assume that ψ is strictly increasing and we introduce the notation

$$v=\psi(u) \text{ and } \varphi:=\psi^{-1},$$

(if ψ is not strictly increasing $\varphi:=\psi^{-1}$ can be understood as a maximal monotone graph of $\mathbb{R}^2$). More precisely, we consider the free boundary associated to local weak solutions of the nonlinear diffusion equation on subsets ω where they are bounded.

$$\begin{cases} v_t=\Delta_p\varphi(v) & \text{in} \quad \omega\times(0,T), \\ v(x,0)=v_0(x) & \text{on} \quad \omega, \end{cases} \tag{3}$$

where φ is a continuous strictly increasing function, $\omega\subset\mathbb{R}^N$ is an open set (not necessarily bounded). We recall that the assumptions $p>2$ or $\varphi'(0)=0$ imply that the equation becomes degenerate (i.e. non uniformly parabolic) and that one of the many consequences of this fact is the *finite speed of propagation* property: if the support of v_0 is a compact set strictly contained in Ω then the same occurs for $v(.,t)$, at least for any $t>0$ small enough. A sharper property concerns the, so called, *waiting time property* typical of "flat" initial data near the boundary of its support. So, if, for instance, $\varphi(v)=|v|^{m-1}v$ with $m(p-1)>1$, it is well known (see references in the mentioned papers) that if the initial datum $v_0(x)$ satisfies

$$|v_0(x)|\leq C_0\,|x-x_0|^{\frac{p}{m(p-1)-1}} \quad \text{a.e. } x\in\omega \ \text{ with } |x-x_0|<\delta, \tag{4}$$

for some $x_0\in\partial\mathrm{supp}v_0$, and for some positive constants C_0 and δ, then there exists a *waiting time* $t^*>0$ such that $x_0\in\partial\mathrm{supp}v(.,t)$ for any $t\in[0,t^*]$. So, if, for instance, v is continuous, then

$$v(x_0,t)=0 \quad \mathit{for} \quad 0\leq t\leq t^*.$$

In this work we shall prove that the waiting time property also holds for the associated discrete solutions (and, in fact it can be proved by passing to the limit). As indicated before, and in a similar way to the existence theory (via *accretive operators*), we discretize in time equation (3) using an implicit scheme. Then we get a problem of the type (1) under the additional condition (2): more exactly, we lead to the problem

$$\begin{cases} -\varepsilon\Delta_p\varphi(v_{n,\varepsilon})+v_{n,\varepsilon}=v_{n-1,\varepsilon} & \text{in} \quad \omega, \\ v_{0,\varepsilon}(x)=v_0(x) & \text{in} \quad \omega, \end{cases} \tag{5}$$

where $1 \leq n \leq \left[\frac{t^*}{\varepsilon}\right]$. Notice that, again, we are not specifying any boundary condition on $\partial\omega$. We assume that $v_{n,\varepsilon}$ represent an approximation of the solution $v(x,t)$ at time $t_n = n\varepsilon$. The compactness of the support of the solutions of this type of problems, assumed $v_0(x)$ with compact support follows as in Section 2.

The main result of this Section (improving the previous results by the authors [3]) gives some sufficient conditions for the waiting time property.

We shall indicate also the obvious modifications to extend the above arguments to the case of equation (PE) relaxing the assumption (2) to the condition types assumed in Section 2.

We point out that the method of proofs in this Section 3 is of relevance for the numerical analysis since we show that the retention of the free boundary holds for local solutions of the semidiscrete elliptic iterative scheme (1).

2. **Nondiffusion of the support in elliptic problems.** In this Section we shall study the retention of the free boundary for elliptic equations of the form (EE). Our result will have a local character and so they will apply to bounded weak solutions u satisfying (EE) on an open bounded set ω of $\mathbb{R}^N$. Global consequences will be derived for solutions of (EE) on an open set Ω where the boundary conditions on $\partial\Omega$ and the structure assumptions allow to have estimates on $\|u\|_{L^\infty(\omega)}$ for a suitable subset ω of Ω.

The main assumption on the absorption term $\beta(x,u)$ will be the following

$$\beta(x,r) \geq f(r) \tag{6}$$

for some continuous nondecreasing function $f(.)$ such that $f(0) = 0$. The relevant condition on the elliptic operator A will be its degree of homogeneity. We can take as A a linear operator

$$A = -L \qquad Lu = \sum_{i,j=1}^{N} a_{ij} \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^{N} b_i \frac{\partial u}{\partial x_i} \tag{7}$$

with $a_{ij} \in L^\infty(\Omega)$, $b_i \in L^\infty(\Omega)$ such that there exists $\Lambda, \lambda \in L^\infty(\Omega)$, $\lambda \geq 0$ for which

$$\Lambda(x) \left|\xi\right|^2 \geq \sum a_{ij}(x)\xi_i\xi_j \geq \lambda(x) \left|\xi\right|^2 \qquad \forall\, \xi \in \mathbb{R}^N. \tag{8}$$

Another possible choice of A is the (p-Laplacian operator

$$A(u) = -\Delta_p u \equiv -div\left(\left|\nabla u\right|^{p-2} \nabla u\right), \qquad p > 1 \tag{9}$$

We note that the degree of homogeneity of both choices of A are 1 and $(p-1)$ respectively. In order to study the existence and behavior of the free boundary $\mathcal{F}(u)$ we shall use suitable barrier functions of the form $\eta(|x - x_0|)$ defined on balls $B_{R_0}(x_0)$, $x_0 \in \omega$. So it is useful to note that making $r = |x - x_0|$ and $z_i = x_i - x_{0,i}$ then

$$\begin{aligned} L(\eta(r)) =& \eta''(r) \sum_{i,j} a_{ij}(x) \frac{z_i z_j}{r^2} \\ &+ \frac{\eta'(r)}{r} \left[\sum_i a_{ii}(x) - \sum_{i,j} a_{ij} \frac{z_i z_j}{r^2} + \sum_i b_i(x) z_i\right]. \end{aligned} \tag{10}$$

In particular if $\eta', \eta'' \geq 0$ we have that

$$-\Lambda_R(x_0)\eta''(r) - \frac{B_{1,R}(x_0)}{r}\eta'(r) \leq -L(\eta(r)) \leq -\lambda_R(x_0)\eta''(r) + \frac{B_{2,R}(x_0)}{r}\eta'(r) \quad (11)$$

where

$$\Lambda_R(x_0) = \sup_{B_R(x_0)} \Lambda(x) \qquad \lambda_R(x_0) = \inf_{B_R(x_0)} \lambda(x), \quad (12)$$

$$B_{1,R}(x_0) = \sup_{B_R(x_0)} \left\{ \left| \sum_i a_{ii}(x) - \sum_{i,j} a_{ij}\frac{z_i z_j}{r^2} + \sum_i b_i(x) z_i \right| \right\}, \quad (13)$$

$$B_{2,R}(x_0) = \inf_{B_R(x_0)} \left\{ \left| \sum_i a_{ii}(x) - \sum_{i,j} a_{ij}\frac{z_i z_j}{r^2} + \sum_i b_i(x) z_i \right| \right\}$$

(note that $B_{1,R}(x_0)$ and $B_{2,R}(x_0)$ are real numbers due to the ellipticity assumption). On the other hand if A is the quasilinear operator given by (9) then

$$\Delta_p(\eta(r)) = \left(|\eta'|^{p-2}\eta'\right)' + \frac{N-1}{r}|\eta'|^{p-2}\eta'. \quad (14)$$

As we shall see, the existence of the free boundary $\mathcal{F}(u)$ is related to the existence of local supersolutions $\eta(|x-x_0|)$ where $\eta(r)$ satisfies $\eta'(0) = \eta(0) = 0$. So we shall introduce the general operator given by

$$\mathcal{L}(u) := -\left(|u'|^{p-2}u'\right)' - \frac{C_1}{r}|u'|^{p-2}u' + f(u) \quad (15)$$

for $p > 1$, $C_i > 0$, and we shall study the homogeneous Cauchy problem

$$\mathcal{L}(u) = 0, \qquad u(0) = u'(0) = 0. \quad (16)$$

Notice that, obviously, $u \equiv 0$ is always a solution of (16) and so we shall need some additional assumptions of f in order to have nontrivial solutions. In the autonomous case ($C_1 \equiv 0$) it is not difficult to check (see e.g. [19]) that the existence of nontrivial solutions of (16) is equivalent to the condition

$$\int_0 \frac{ds}{F(s)^{1/p}} < \infty, \qquad F(t) = \int_0^t f(s)ds \quad (17)$$

(in fact if we are only interested in nonnegative solutions of (16) it is enough to assume (17) replacing 0 by 0^+, (respectively 0^- for nonpositive solutions). The treatment of the general case $C_1 \neq 0$ is more delicate and was performed in [19] (see [15]) for a related result .

Lemma 2.1. *Assume that (17) holds. For $\mu > 0$ and $\tau \in \mathbb{R}^+$ define*

$$\psi_\mu(\tau) = \left(\frac{p-1}{p\mu}\right)^{\frac{1}{p}} \int_0^\tau \frac{ds}{F(s)^{1/p}}, \qquad F(t) = \int_0^t f(s)ds \quad (18)$$

and

$$\eta(r,\mu) = \psi_\mu^{-1}(r) \qquad for\ r \in [0, \psi_\mu(+\infty)). \quad (19)$$

Then $\eta(0,\mu) = \eta'(0,\mu) = 0$ $\forall\mu$. Moreover we have : i) If $0 < \mu < \mu_0$, $\mu_0 = C_2/(C_1+1)$ then $\mathcal{L}(\eta(r,\mu)) > (C_2 - \mu(C_1+1)f(\eta(r,\mu)) \geq 0$. ii) For every $\tau > 0$ the function $\eta_\tau(r,\mu) = \eta([r-\tau]^+,\mu)$ satisfies $\mathcal{L}(\eta_\tau(r,\mu)) > 0$ if $\mu > \mu_0$ for $r > \mu$. Where $\mathcal{L}$ is defined in 15

In the special case of $A = -\Delta_p$ and $f(r) = \lambda|r|^{q-1}r$ with $q \geq 0$, condition (17) holds if and only if $q < (p-1)$. In this case Lemma 2.1 can be improved. It is not difficult to see [19], that for $C > 0$ the function

$$u(r) = Cr^{\frac{p}{p-1-q}} \tag{20}$$

satisfies

$$\mathcal{L}(u(r)) = \left[\lambda C^q - C^{p-1}\frac{p^{(p-1)}(pq + N(p-1-q))}{(p-1-q)^p}\right] r^{\frac{pq}{p-1-q}}. \tag{21}$$

In particular, if we define

$$K_{N,\lambda} = \left[\frac{\lambda(p-1-q)^p}{p^{(p-1)}(qp + N(p-1-q))}\right]^{\frac{1}{p-1-q}} \tag{22}$$

then $\mathcal{L}(u) \equiv 0$ if $C = K_{N,\lambda}$ and $\mathcal{L}(u) > 0$ (resp. $\mathcal{L}(u) < 0$) if $C < K_{N,\lambda}$ (resp. $C > K_{N,\lambda}$).

Now, we return to the consideration of the elliptic problem (EE). The existence of the free boundary under the assumption (17) is an easy task with the help of Lemma 2.1 (see Theorem 1.9 of [19]). The next result shows the nondiffusion of the support $S(u)$ (with respect to $S(g)$). We shall state it (by simplicity) for nonnegative solutions.

Theorem 2.2. *Let A given by (7) or (9). Assume that β satisfies (6) for some f such that (17) holds, where $p-1$ is the degree of homogeneity of A. Assume $g \in L^\infty(\omega)$ such that there exist μ and K small enough and there exists*

$$\Gamma \subset \partial S(g) \subset \overline{\omega} \tag{23}$$

for which

$$0 \leq g(x) \leq Kf(\eta(d(x), \mu)) \qquad a.e. \quad x \in \omega \text{ such that } d(x) := d(x, \Gamma) \leq R \tag{24}$$

for some $R > 0$ ($\eta(d(x), \mu)$ is defined 19). Let $u \in L^\infty(\omega)$ be a nonnegative weak solution of (EE) such that $\|u\|_{L^\infty(\omega)} \leq M$. Then if

$$R \geq \psi_\mu(M) \tag{25}$$

we conclude that $u(x) = 0$ for $x \in \Gamma$ such that $d(x, S(u|_{\partial\omega})) \geq R$. Moreover, if $\Gamma = \partial S(g)$ and $u|_{\partial\omega} = 0$, then $\mathcal{F}(u) = \partial S(g)$.

Proof. Let $x_0 \in \Gamma$ and $\tilde{\omega} = \omega \cap B_R(x_0)$ with $R = d(x, S(u|_{\partial\omega}))$. Consider the function $\bar{u}(x) = \eta(|x - x_0|, \mu)$. Then, by Lemma 2.1 , if $\mu < \mu_0$ we conclude that

$$A\bar{u} + f(\bar{u}) \geq \Lambda_R(x_0)\mathcal{L}(\eta(|x - x_0|, \mu) \geq (C_2 - \mu(C_1 + 1))f(\bar{u}).$$

So, by (24), if $K \leq C_2 - \mu(C_1 + 1)$ we have that

$$A\bar{u} + f(\bar{u}) \geq g(x) = Au + \beta(x, u) \geq Au + f(u)$$

a.e. $x \in \tilde{\omega}$ (notice that $d(x) \leq |x - x_0|$ because $x_0 \in \Gamma$). In order to show that $u(x_0) = 0$ it is enough to show that $\bar{u}(x) \geq u(x)$ on $\partial\tilde{\omega}$ because it implies (by the comparison principle) that $0 \leq u(x) \leq \bar{u}(x)$ on $\tilde{\omega}$. On the set $\partial\tilde{\omega}$ we know (by the choice of R) that $0 = u \leq \bar{u}$. Moreover if $x \in \partial\tilde{\omega} \cap \partial B_R(x_0)$ then

$$\bar{u} = \eta(R, \mu) \geq M \geq u(x)$$

which holds by (25). Finally, if $u|_{\partial\omega} = 0$ we first show that

$$N(u) \supset \{x \in N(g)\ d(x, \partial S(g)) \geq \psi_\mu(M)\}$$

(that is shown by means of the same supersolution $\eta(|x-x_0|,\mu)$ but now applied to $x_0 \in N(g)$ and $\tilde{\omega} = \omega \cap B_R(x_0)$ with $R = d(x_0, \partial S(g))$. In order to obtain that $\mathcal{F}(u) = \partial S(g)$ or equivalently $N(u) = N(g)$ we only need to observe that on the set $\tilde{\omega}^* = \{x \in N(g)\ d(x, \partial S(g)) \geq \psi_\mu(M)\}$ u satisfies $0 = Au + \beta(x,u) \geq Au + f(u)$ and $u = 0$ on $\partial\tilde{\omega}^*$. So $u = 0$ in $\tilde{\omega}^*$ which proves the Theorem. □

The operator $A = -\Delta_p u$ degenerates near the set $\{\nabla u = 0\}$ if $p > 2$ and so a free boundary $\mathcal{F}(u)$ may appears as the given by $\partial\{\nabla u = 0\}$ (see Theorem 1.14 of [19], [30]). Notice that u is constant on each connected component of the set $\{\nabla u = 0\}$ and that its exact value can be obtained from the equation (EE). Again a retention phenomenon may occur.

Corollary 1. *Let $A = -\Delta_p$ and $\beta(x,u) = f(u)$ with f a continuous increasing real function such that $f(0) = 0$. Let $k > 0$ and assume $0 \leq g(x) \leq k$ such that the set*

$$\{g = k\} := N(k-g) \tag{26}$$

has a positive measure. Let $a > 0$ such that $f(a) = k$, and assume that

$$\int_{0^+} \frac{ds}{F_a(s)^{1/p}} < \infty \tag{27}$$

where

$$F_a(s) = \int_0^s f_a(t)dt \qquad f_a(t) = f(a) - f(a-t) \text{ if } 0 \leq t \leq a.$$

We also suppose that there exist μ and K small enough and $\Gamma \subset \partial S(k-g)$ for which

$$k - g(x) \leq K f_a(\eta_a(d(x),\mu)) \qquad a.e.\ x \in \omega \text{ such that } d(x) = d(x,\Gamma) \leq R \tag{28}$$

for some $R > 0$, η_a defined as in (19) but replacing F by F_a. Let u be a weak solution of (EE) with $0 \leq u \leq a$ on ω. Then if R is large enough we conclude that

$$u(x) \equiv a \quad if\ x \in N(k-g) \text{ and } d(x, S(a - u|_{\partial\omega})) \geq R.$$

Proof. The function $w = a - u$ satisfies that $0 \leq w \leq a$ and

$$-\Delta_p w + f_a(w) = \tilde{g}$$

with $\tilde{g} = k - g$ and $f_a(t) = f(a) - f(a-t)$ (notice that f_a is increasing and $f_a(0) = 0$). Then it suffices to apply Theorem 2.2 to w □

In the case of

$$A = -\Delta_p \quad \text{and} \quad f(r) = \lambda|r|^{q-1}r \quad \text{with} \quad 0 \leq q < (p-1), \tag{29}$$

assumption (24) becomes more explicit.

Corollary 2. *Assume (29) and let $g \in L^\infty(\omega)$ such that there exists $K > 0$ (small enough) and $\Gamma \subset \partial S(g)$ for which*

$$0 \leq g(x) \leq K d(x,\Gamma)^{\frac{pq}{p-1-q}} \qquad a.e.\ x \in \omega \text{ with } d(x,\Gamma) \leq R \tag{30}$$

for some $R > 0$. Let $u \in L^\infty(\omega)$ be a nonnegative weak solution of (EE) such that $\|u\|_{L^\infty(\omega)} \leq M$. Then if

$$R \geq \left(\frac{M}{K_{N,\lambda}}\right)^{\frac{p-1-q}{p}}, \tag{31}$$

$K_{N,\lambda}$ given in (22), we conclude that $u(x) = 0$ for $x \in \Gamma$ such that $d(x, S(u|_{\partial\omega})) \geq R$. Moreover, if $\Gamma = \partial S(g)$ and $u|_{\partial\omega} = 0$, then $\mathcal{F}(u) = \partial S(g)$.

Now we shall study the optimality of assumptions (30) and (31). Our first result state the strict local diffusion of the support when the external perturbation $g(x)$ is greater than the critical growing :

Theorem 2.3. *Let $g \in L^1_{loc}(\omega)$, $g \geq 0$, $x_0 \in \partial S(g) \cap \omega$ and $u \geq 0$ such that*

$$-\Delta_p u + \lambda u^q \geq g \qquad \text{in } \omega, \tag{32}$$

for some $\lambda > 0$ and $0 < q < (p-1)$. Let $1 < \delta < 1 + (\alpha q + 1)/(N-1)$, (with $\alpha = p/(p-1-q)$), then there exist $C, K_1, K_2, K_3 > 0$ such that if $\varepsilon > 0$, $x_1 \in \omega$ satisfy $\delta\varepsilon > |x_1 - x_0| \geq ((\delta+1)/2)\varepsilon$, $B_\varepsilon(x_1) \subset \omega$ and

$$g(x) \geq C|x - x_0|^{\alpha q} \qquad \text{a.e. } x \in B_\varepsilon(x_1). \tag{33}$$

Then

$$u(x) \geq \begin{cases} K_1\varepsilon^\alpha - K_2|x-x_1|^\alpha & if \quad 0 \leq |x-x_1| \leq \varepsilon, \\ K_3(\delta\varepsilon - |x-x_1|)^\alpha & if \quad \varepsilon \leq |x-x_1| \leq \delta\varepsilon, \end{cases}$$

In particular, $u > 0$ in $B_{(\delta\varepsilon - |x_1-x_0|)}(x_0)$.

Proof. We shall construct a positive subsolution $\underline{u}(x)$ on the ball $B_{\varepsilon\delta}(x_1)$ and so, by the comparison principle $0 < \underline{u}(x) \leq u(x)$ *a.e.* $x \in B_{\varepsilon\delta}(x_1)$. We define $\underline{u}(x) = \theta(|x-x_1|)$ in the following way :

$$\theta(r) = \begin{cases} \theta_1(r) = K_1\varepsilon^\alpha - K_2 r^\alpha & if \quad 0 \leq r \leq \varepsilon, \\ \theta_2(r) = K_3(\delta\varepsilon - r)^\alpha & if \quad \varepsilon \leq r \leq \delta\varepsilon, \end{cases}$$

where $r = |x - x_1|$, $\alpha = p/(p-1-q)$, $1 < \delta < 1 + (\alpha q+1)/(N-1)$, and

$$\begin{aligned} K_3 &= \left(\frac{\lambda\alpha^{1-p}}{\alpha q + 1 - (\delta-1)(N-1)}\right)^{\frac{1}{p-1-q}} \\ K_2 &= K_3(\delta-1)^{\alpha-1} \\ K_1 &= K_3(\delta-1)^{\alpha-1}\delta \end{aligned}$$

Thanks to the choice of the constants K_1, K_2 and K_3 we have that $\theta \in C^1([0, \delta\varepsilon])$. On the other hand

$$-\Delta_p\underline{u} + \lambda\underline{u}^q = -(|\theta'|^{p-2}\theta')' - \frac{N-1}{r}|\theta'|^{p-2}\theta' + \lambda\theta^q := \mathcal{L}(\theta)$$

on the region $r \in (\varepsilon, \delta\varepsilon)$ we write $r = \delta\varepsilon - t\varepsilon(\delta-1)$ with $0 < t < 1$. Then

$$\begin{aligned} \mathcal{L}(\theta_2) = &-(p-1)\alpha^{p-1}(\alpha-1)K_3^{p-1}[t\varepsilon(\delta-1)]^{(\alpha-1)(p-1)-1} + \\ &\frac{N-1}{r}\alpha^{p-1}K_3^{p-1}[t\varepsilon(\delta-1)]^{(\alpha-1)(p-1)} + \lambda K_3^q[t\varepsilon(\delta-1)]^{\alpha q}. \end{aligned}$$

But $(\alpha-1)(p-1)-1 = \alpha q$ so from the choice of δ and K_3 we conclude that

$$\mathcal{L}(\theta_2) \leq (\alpha K_3)^{p-1}[t\varepsilon(\delta-1)]^{\alpha q}\left[(N-1) + \lambda K_3^{q-p+1}\alpha^{1-p} - (p-1)(\alpha-1)\right] \leq 0.$$

On the other hand, if $r = \varepsilon t$ with $0 < t < 1$ we have

$$\mathcal{L}(\theta_1) = \left[(\alpha q + N)(\alpha K_2)^{p-1}t^{\alpha q} + \lambda(K_1 - K_2 t^\alpha)^q\right]\varepsilon^{\alpha q} \leq K_4\varepsilon^{\alpha q}$$

where

$$K_4 = (\alpha q + N)(\alpha K_2)^{p-1} + \lambda {K_1}^q.$$

We observe that if $x \in B_\varepsilon(x_1)$

$$|x - x_0| \geq d(x, B_\varepsilon(x_1)) \geq \frac{\delta - 1}{2}\varepsilon,$$

therefore if we take

$$C = \frac{K_4 2^{\alpha q}}{(\delta - 1)^{\alpha q}}$$

from (33) we obtain for $x \in B_\varepsilon(x_1)$

$$g(x) \geq C|x - x_0|^{\alpha q} \geq \frac{K_4 2^{\alpha q}}{(\delta - 1)^{\alpha q}} \left(\frac{\delta - 1}{2}\varepsilon\right)^{\alpha q} = K_4 \varepsilon^{\alpha q} \geq \mathcal{L}(\theta_1.)$$

Finally, on the domain $\tilde{\omega} = B_{\varepsilon\delta}(x_1)$ we have $0 = \underline{\mathrm{u}}(x) \leq u(x)$ on $\partial\tilde{\omega}$ and

$$-\Delta_p \underline{\mathrm{u}} + \lambda_1 \underline{\mathrm{u}}^q \leq g(x)$$

then $\underline{\mathrm{u}}(x)$ is a subsolution and so the inequality $\underline{\mathrm{u}} \leq u$ holds on $B_{\varepsilon\delta}(x_1)$ □

From Theorem 2.3, we can deduce the optimality of the growth criterion (33).

Corollary 3. *Let $1 < \delta < 1 + (\alpha q + 1)/(N - 1)$ and assume that for $x_0 \in \partial S(g) \cap \omega$ there exists a cone H defined by*

$$H := \left\{ x \in \mathbb{R} \;\; < x - x_0, n > \geq |x - x_0| \sqrt{1 - \frac{4}{(\delta + 1)^2}} \right\},$$

such that

$$Lim_{\varepsilon \to 0} Inf \left\{ \frac{g(x)}{|x - x_0|^{\alpha q}} \;\; x \in H, \;\; 0 < |x - x_0| < \varepsilon \right\} > C, \tag{34}$$

C defined as in Theorem 2.3, $< .,. >$ is the usual scalar product in $\mathbb{R}^N$ and n is an unit vector (i.e. $< n, n >= 1$),then $u(x_0) > 0$..

Proof. Condition (34) implies that there exists $\varepsilon_0 > 0$ such that for any $x \in H \cap B_{\varepsilon_0}(x_0)$

$$g(x) \geq C|x - x_0|^{\alpha q}$$

let $\varepsilon_1 = \frac{2\varepsilon_0}{\delta + 3}$ and $x_1 = x_0 + \varepsilon_1 \frac{\delta+1}{2} n$. First we show that $B_{\varepsilon_1}(x_1) \subset B_{\varepsilon_0}(x_0)$. If $x \in B_{\varepsilon_1}(x_1)$

$$|x - x_0| \leq |x - x_1| + |x_1 - x_0| \leq \varepsilon_1 + \varepsilon_1 \frac{\delta + 1}{2} = \varepsilon_1 \frac{\delta + 3}{2} = \varepsilon_0$$

next we show that $B_{\varepsilon_1} \subset H$ if $x \in B_{\varepsilon_1}(x_1)$, $x = x_1 + \varepsilon_1 y$ where $< y, y > \leq 1$, then

$$\frac{< x - x_0, n >}{|x - x_0|} = \frac{< \varepsilon_1 \frac{\delta+1}{2} n + \varepsilon_1 y, n >}{\left|\varepsilon_1 \frac{(\delta+1)}{2} n + \varepsilon_1 y\right|} = \frac{\frac{\delta+1}{2} + < y, n >}{\sqrt{\left(\frac{\delta+1}{2}\right)^2 + (\delta + 1) < n, y > + < y, y >}}$$

a straightforward computation leads to the minimum with respect to $< y, n >$ of the above expression is attained when $< y, y >= 1$ and $< n, y >= -2/(\delta + 1)$. Therefore

$$\frac{< x - x_0, n >}{|x - x_0|} \geq \frac{\frac{\delta+1}{2} - \frac{2}{\delta+1}}{\sqrt{\left(\frac{\delta+1}{2}\right)^2 - 1}} = \sqrt{1 - \frac{4}{(\delta + 1)^2}}$$

and we conclude that $B_{\varepsilon_1}(x_1) \subset H$. So we have that

$$g(x) \geq C|x - x_0|^{\alpha q} \qquad a.e. \; x \in B_{\varepsilon_1}(x_1)$$

and $\delta\varepsilon_1 > |x_1 - x_0| = ((\delta + 1)/2)\varepsilon_1$, then, by applying Theorem 2.3 with $\varepsilon = \varepsilon_1$ we obtain that $u(x_0) > 0$ □

The assumption (31) is also necessary in order to conclude the nondiffusion of the support. We shall illustrate it by means of the following
Counterexample. Let $N = 1$ and $p = 2$. On the set $\omega = (-R, \varepsilon R)$ consider the function

$$u(x) = \begin{cases} u_1(x) := C(\varepsilon R - x)^\alpha & if \quad 0 \leq x < \varepsilon R, \\ u_2(x) := C\left(1 - R^{-\alpha}|x|^\alpha\right)(\varepsilon R - x)^\alpha & if \quad -R < x \leq 0, \end{cases}$$

where $\alpha = 2/(1-q)$ and

$$C = \left[\frac{\lambda(1-q)^p}{2(1+q)}\right]^{\frac{1}{1-q}}.$$

Then

$$\begin{array}{r} -u'' + \lambda u^q = g(x) \quad on \quad \omega \\ u(-R) = u(\varepsilon R) = 0 \end{array}$$

with $g \equiv 0$ in $[0, \varepsilon R)$ and

$$g(x) = C\alpha(\alpha-1)R^{-\alpha}|x|^{\alpha q}(R\varepsilon - x)^\alpha + C\alpha^2 R^{-\alpha}|x|^{\alpha-1} + \\ (R\varepsilon - x)^{\alpha q} C^q[(1 - R^{-\alpha}|x|^\alpha)^q - 1 + R^{-\alpha}|x|^\alpha].$$

on $(-R, 0]$. Then, it is easy to check that

$$Lim_{x\to 0^-}\frac{g(x)}{|x|^{\alpha q}} = C(\alpha(\alpha-1))\varepsilon^\alpha.$$

This shows how condition (30) may be fulfilled near $\partial S(g)$ but $u > 0$ on ω.

We shall end this Section with several remarks:

Remark 1. The optimality of the diffusion-absorption balance condition (17) was shown in [36]. Many other properties of the free boundary $\mathcal{F}(u)$ was given in the monograph [19] where abundant bibliographic comments are made. On the other hand, the study of the free boundary $\mathcal{F}(u)$ was carried out in [21], [34] for the case of the general quasilinear operators $Au = div(Q(|\nabla u|)\nabla u)$ including the minimal surface operator. We also point out that the degeneracy of the $p-$Laplacian operator when $p > 2$ (leading for instance to Theorem 1) is the reason of the peculiar structure of solutions of some nonmontone reaction-diffusion equations of the type $-\Delta_p u = u(u-a)(1-u)$ (see, e.g., [25] and its references).

Remark 2. In some special cases it is interesting to study the influence of the convection term $\sum b_i \frac{\partial u}{\partial x_i}$ on the formation and size of the null set $N(u)$. It is easy to see from the constants $B_{1,R}$ and $B_{2,R}$ and Lemma 2.1 that the size of the null set is bigger for outward pointing drift vector field ($\sum b_i(x)x_i \geq 0$) and smaller for inward pointing drift vector field ($\sum b_i(x)x_i \leq 0$). We also remark that the study of the formation and diffusion or not of the free boundary $\mathcal{F}(u)$ may be made under a suitable diffusion - convection balance (see for instance the study of the equation $-\Delta u + |\nabla u|^q = g(x)$, for $0 \leq q < 1$, made in [10] and [34]).

Remark 3. Notice that the proofs of Theorem 2 and Corollary 3 do not require the boundedness of the weak solution u. Moreover both results can be easily extended to the case of $A = -L$ (see (7)) by using the inequalities (11). We also point out that the estimate (2.3) allows to conclude the convergence of the approximation of the free boundary (see [1], [5], [33]).

Remark 4. Using the same type of arguments as in [19] (Section 2.2), we can extend the above results to the case of $\beta(u)$ a maximal monotone graph multivalued at $u = 0$. The case of a linear operator $A = -L$ (see (7)) but degenerate (i.e. such that $\lambda(x) = 0$ for some $x \in \omega$) was considered in [20] when studying an stochastic control problem arising in the economics of the environment.

3. **Waiting time phenomenon.** In this Section we consider the parabolic problem

$$(PE) \qquad \frac{\partial}{\partial t}\psi(u) + Au + \beta(x,u) = G(t,x)$$

where A is given by (7) or (9), β satisfies (6) and ψ is (for simplicity) a continuous nondecreasing function. It is well known that under suitable assumptions on ψ, A and β, a free boundary (or moving free boundary)

$$\mathcal{F} = \bigcup_{t\geq 0} \mathcal{F}(t) \qquad \mathcal{F}(t) = \partial S[u(t,.)] \cap \partial N[u(t,.)]$$

may occur. The possibility that for some $x_0 \in \partial S[u(0,.)]$ the free boundary let static for some period $t \in [0, t^*(x_0)]$ was first noted by D.G. Aronson in [9] (for the one-dimensional porous media equation) and has received an important attention since then. The time $t^*(x_0)$ was called the waiting time for the point x_0.

The main goal of this Section is to show how the results of the precedent Section can be used in order to give an approximation scheme for the discretization in time of that equation in such a way that the retention phenomenon is conserved. As by product we shall obtain new results on the waiting time for the original problem.

We start by the case where β, A and G satisfy (2):

Theorem 3.1. *Let v be a solution of (3) such that $\|v\|_{L^\infty((0,T)\times\omega)} \leq M^*$, for some $M^* > 0$. Assume that $\varphi \in C^0[0, M^*]$ is an increasing function satisfying (17) with $f = \varphi^{-1}(s)$, i.e.*

$$\theta(r) =: \int_0^{\varphi(r)} \frac{ds}{\left(\int_0^s \varphi^{-1}(t)dt\right)^{\frac{1}{p}}} < \infty. \tag{35}$$

Moreover, we assume that

$$\theta(kr) \geq k^q \theta(r) \tag{36}$$

for some $q > 0$ and any $0 < r < M^*$, $\frac{pq}{pq+1} \leq k \leq 1$. Let $x_0 \in \omega$ and $R > 0$ be such that $B_R(x_0) \subset \omega$. Let $C_0 > 0$ and $h(r)$ be defined by the relation

$$\theta(h(r)) = C_0 r \qquad 0 \leq r \leq M^*. \tag{37}$$

Finally, assume $v_0(x)$ be such that

$$|v_0(x)| \leq h(|x - x_0|) \qquad \text{in } B_R(x_0). \tag{38}$$

Then there exists $t^* > 0$ such that $v(x_0, t) = 0$ for any $0 \leq t \leq t^*$. Moreover

$$t^* \geq \min\left\{\frac{R^p}{\theta(M^*)^p(p-1)Nq}, \frac{1}{C_0^p(p-1)Nq}\right\}. \tag{39}$$

For the proof we shall need an useful auxiliary result obtained in ([3]):

Lemma 2. *Let* $\phi(s) = s - s^m$ *for* $s \in (0, \left(\frac{1}{m}\right)^{\frac{1}{m-1}}]$ *and let* $\phi^n(s) = \overbrace{\phi \circ \phi \circ \ldots\ldots \circ \phi}^{n}(s)$. *Then*

$$\lim_{n\to\infty} \phi^n(s) n^{\frac{1}{m-1}} = \left(\frac{1}{m-1}\right)^{\frac{1}{m-1}}.$$

Remark 5. In the terminology of discrete dynamical systems (see, e.g. [18]), the above Lemma indicates that although the fixed point $s = 0$ is not hyperbolic (since $|\phi'(0)| = 1$) it is attractive for the logistic type system $s_{n+1} = \phi(s_n)$. It also indicate the rate of convergence for any $s \in (0, (\frac{1}{m})^{\frac{1}{m-1}}]$.

Proof. **Proof of Theorem 3.** We will show, that if t^* is equal to the right part of inequality (39), then the function $\bar{v}(x,t)$ defined by the relation

$$\theta(\bar{v}(x,t)) = \frac{|x - x_0|}{((t^* - t)(p-1)Nq)^{1/p}} \qquad 0 \le t < t^*,\ x \in B_R(x_0) \tag{40}$$

satisfies

$$|v(x,t)| \le \bar{v}(x,t) \qquad 0 \le t < t^*,\ x \in B_R(x_0). \tag{41}$$

We notice that the conclusion of the Theorem follows from this inequality taking into account that $\bar{v}(x_0, t) = 0$ for $0 \le t < t^*$.

To simplify the exposition, in what follows, we will assume without loss of generality that $\theta(\infty) = \infty$ and that hypothesis (36) is satisfied by any $r > 0$. We define $\{C_{n,\varepsilon}\}_{0 \le n \le [\frac{t^*}{\varepsilon}]}$ in the following way

$$C_{[\frac{t^*}{\varepsilon}],\varepsilon} = \left(\frac{p}{\varepsilon(p-1)N}\right)^{\frac{1}{p}} \left(\frac{1}{2\,(pq+1)}\right)^{\frac{1}{p}}$$

$$C_{n-1,\varepsilon} = \left(1 - \varepsilon\frac{p-1}{p} N C_{n,\varepsilon}^p\right)^q C_{n,\varepsilon} \qquad 0 \le n \le \left[\frac{t^*}{\varepsilon}\right]. \tag{42}$$

We will show that the radial functions $x \to \bar{v}_{n,\varepsilon}(|x - x_0|)$ defined by the relation

$$\theta(\bar{v}_{n,\varepsilon}(r)) = C_{n,\varepsilon} r \tag{43}$$

satisfy

$$-\varepsilon \Delta_p \varphi(\bar{v}_{n,\varepsilon}) + \bar{v}_{n,\varepsilon} \ge \bar{v}_{n-1,\varepsilon} \qquad \textit{in } B_R(x_0). \tag{44}$$

Indeed , if we differentiate (43) with respect to r and taking into account (35), we obtain

$$\left(\varphi(\bar{v}_{n,\varepsilon})\right)'(r) = C_{n,\varepsilon} \left(\int_0^{\varphi(\bar{v}_{n,\varepsilon}(r))} \varphi^{-1}(t)dt\right)^{\frac{1}{p}} \tag{45}$$

and therefore

$$\left(\left((\varphi(\bar{v}_{n,\varepsilon}))'\right)^{p-1}\right)'(r) = C_{n,\varepsilon}^p \frac{p}{p-1} \bar{v}_{n,\varepsilon}(r). \tag{46}$$

From (45) we deduce that $\left(\varphi(\bar{v}_{n,\varepsilon})\right)'(0) = 0$ and since $\bar{v}_{n,\varepsilon}(r)$ is an increasing function, by integration in the above equality, we obtain

$$\left((\varphi(\bar{v}_{n,\varepsilon}))'\right)^{p-1}(r) \le C_{n,\varepsilon}^p \frac{p}{p-1} \bar{v}_{n,\varepsilon}(r) r. \tag{47}$$

On the other hand, the differential operator of equation (44) applied to a radially symmetric function can be expressed as

$$\mathbb{L}(v)(r) = \left[-\left(|\varphi(v)'|^{p-2}\varphi(v)'\right)' - \frac{N-1}{r}|\varphi(v)'|^{p-2}\varphi(v)'\right]\varepsilon + v.$$

Therefore, using the above relations we obtain

$$\mathbb{L}(\bar{v}_{n,\varepsilon})(r) \geq \left(1 - \varepsilon\frac{p-1}{p}NC^p_{n,\varepsilon}\right)\bar{v}_{n,\varepsilon}(r). \tag{48}$$

From (36), (43) and (48), since $\left(1-\varepsilon\frac{p-1}{p}NC^p_{n,\varepsilon}\right) \geq \frac{pq}{pq+1}$, we get

$$\theta\left(\left(1-\varepsilon\frac{p-1}{p}NC^p_{n,\varepsilon}\right)\bar{v}_{n,\varepsilon}\right) \geq \left(1-\varepsilon\frac{p-1}{p}NC^p_{n,\varepsilon}\right)^q C_{n,\varepsilon}r = \theta\left(\bar{v}_{n-1,\varepsilon}\right).$$

Then (44) is satisfied, that is, $\{\bar{v}_{n,\varepsilon}(x)\}_{0\leq n\leq\left[\frac{t^*}{\varepsilon}\right]}$ is a family of supersolutions for (5).

In order to study the behavior of $\{\bar{v}_{n,\varepsilon}(x)\}_{0\leq n\leq\left[\frac{t^*}{\varepsilon}\right]}$ when ε goes to 0, we first notice that if we define $f(s) = s - s^{1+pq}$ we can easily show (by induction) that

$$C^{\frac{1}{q}}_{\left[\frac{t^*}{\varepsilon}\right]-n,\varepsilon} = \left(\frac{p}{\varepsilon(p-1)N}\right)^{\frac{1}{pq}} f^n\left(\left(\frac{1}{2\,(pq+1)}\right)^{\frac{1}{pq}}\right),$$

where by $f^n(s)$ we mean the function composition $f\circ f\circ.....\circ f(s)$ (n times). Let $t<t^*$, using the above expression we have

$$C^{\frac{1}{q}}_{\left[\frac{t^*}{\varepsilon}\right]-\left[\frac{t^*-t}{\varepsilon}\right],\varepsilon} = \left(\frac{p}{\varepsilon(p-1)N}\right)^{\frac{1}{pq}} f^{\left[\frac{t^*-t}{\varepsilon}\right]}\left(\left(\frac{1}{2\,(pq+1)}\right)^{\frac{1}{pq}}\right). \tag{49}$$

Applying the above Lemma 2 we get that

$$Lim_{n\to\infty} f^n\left(\left(\frac{1}{2\,(pq+1)}\right)^{\frac{1}{pq}}\right) n^{\frac{1}{pq}} = \left(\frac{1}{pq}\right)^{\frac{1}{pq}}.$$

So, passing to limit in expression (49), when ε goes to 0, we obtain

$$\begin{aligned} Lim_{\varepsilon\to 0}C_{\left[\frac{t}{\varepsilon}\right],\varepsilon} &= Lim_{\varepsilon\to 0}\left(\frac{p}{\varepsilon(p-1)N}\right)^{\frac{1}{p}}\left(\frac{\varepsilon}{t^*-t}\right)^{\frac{1}{p}}\left(\frac{1}{pq}\right)^{\frac{1}{p}} \\ &= \left(\frac{1}{(t^*-t)(p-1)Nq}\right)^{\frac{1}{p}}. \end{aligned}$$

Now we choose $t^* > 0$ in order to obtain

$$\begin{aligned} &\bar{v}_{n,\varepsilon}(x) \geq M^* \quad in\ \partial B_R(x_0),\ 0\leq n\leq\left[\frac{t^*}{\varepsilon}\right], \\ &\bar{v}_{0,\varepsilon}(x) \geq v_0(x) \quad in\ B_R(x_0). \end{aligned} \tag{50}$$

Since $\theta(.)$ is an increasing function and $\bar{v}_{n,\varepsilon}(x)$ is increasing with respect to n, the above conditions are equivalent (when ε goes to 0) to

$$Lim_{\varepsilon\to 0}\theta\left(\bar{v}_{0,\varepsilon}(R)\right) = C_{0,\varepsilon}R \geq \theta(M^*) \text{ and } Lim_{\varepsilon\to 0}C_{0,\varepsilon} \geq C_0.$$

Therefore if t^* satisfies

$$t^* < \min\left\{\frac{R^p}{\theta(M^*)^P(p-1)Nq}, \frac{1}{C_0^p(p-1)Nq}\right\},$$

then, for ε small enough, by the comparison principle we obtain

$$\bar{v}_{n,\varepsilon}(x) \geq v_{n,\varepsilon}(x) \quad in\ B_R(x_0),\ 0\leq n\leq\left[\frac{t^*}{\varepsilon}\right].$$

Therefore, passing to the limit when ε goes to 0, we conclude

$$\bar{v}(x,t) \geq v(x,t) \quad in\ B_R(x_0) \times [0,t^*].$$

Using the same argument, we can show that the functions $\{-\bar{v}_{n,\varepsilon}(x)\}_{0\leq n\leq [\frac{t^*}{\varepsilon}]}$ are subsolutions of the problem (5) and by the comparison principle (and passing to the limit when ε goes to 0) we obtain

$$-\bar{v}(x,t) \leq v(x,t) \quad in\ B_R(x_0) \times [0,t^*],$$

and finally

$$|v(x,t)| \leq \bar{v}(x,t) \quad in\ B_R(x_0) \times [0,t^*]$$

which concludes the proof of the Theorem □

Remark 6. We notice that the assumption (36) is much more general that to assume that θ is a power function. For instance if $\theta(.)$ is a nonincreasing function then for any q, the function

$$\theta(s) = \Theta(s)s^q$$

satisfies assumption (36). Indeed if $k < 1$

$$\theta(kr) = \Theta(kr)(kr)^q \geq k^q\Theta(r)r^q = k^q\theta(r).$$

We can prove the optimality of the growth assumption (38) by showing a recurrence lower estimate quite similar to Theorem 2. We shall illustrate it for the special (and global) formulation of problem (3) with $\varphi(s) = s$. More concretely, we consider the Dirichlet problem

$$(DP)\left\{\begin{array}{lll} v_t = \Delta_p v & \text{in} & \Omega \times (0,T), \\ v = 0 & \text{on} & \partial\Omega \times (0,T), \\ v(x,0) = v_0(x) & \text{on} & \Omega, \end{array}\right. \tag{51}$$

Theorem 3.2. *Let $v_0 \in W_0^{1,\infty}(\Omega)$, with $\Delta_p v \in L^\infty(\Omega)$ and $v_0 \geq 0$ a.e. on Ω. Let $x_0 \in \partial S(v_0)$ and $1 < \delta < 1+(\alpha+1)/(N-1)$, (with $\alpha = p/(p-2)$). Then there exist $C, K_1, K_2, K_3 > 0$ such that if $\varepsilon > 0$, $x_1 \in \omega$ satisfy $\delta\varepsilon > |x_1 - x_0| \geq ((\delta+1)/2)\varepsilon$, $B_\varepsilon(x_1) \subset \Omega$ and*

$$v_0(x) \geq C|x-x_0|^\alpha \qquad a.e.\ x \in B_\varepsilon(x_1). \tag{52}$$

the solution v of (51) verifies that $v > 0$ in $B_{(\delta\varepsilon - |x_1-x_0|)}(x_0) \times (0,T)$.

Proof. Since the associate operator A is m-T-accretive in $L^\infty(\Omega)$ (see,e. g. [19]) we know the convergence in $L^\infty(\Omega)$ of the implicit iteration scheme

$$\begin{array}{lll} v_{n+1} - v_n - \Delta t \Delta_p v_{n+1} = 0 & \text{in} & \Omega, \\ v_{n+1} = 0 & \text{on} & \partial\Omega. \end{array}$$

In addition, since $v_0 \in D(A)$ we have the estimate ([17], [32])

$$\|v(t,.) - v_n(t,.)\|_{L^\infty(\Omega)} \leq \frac{t}{\sqrt{n}} \|Av_0\|_{L^\infty(\Omega)}.$$

A careful reading of the proof of Theorem 2 allow to see that it is possible to control the constants appearing in the proof of this theorem (with $g = v_0$) as to have that

$$v_1(t,.)(x) \geq \left\{\begin{array}{lll} K_1\varepsilon^\alpha - K_2|x-x_1|^\alpha & if & 0 \leq |x-x_1| \leq \varepsilon, \\ K_3(\delta\varepsilon - |x-x_1|)^\alpha & if & \varepsilon \leq |x-x_1| \leq \delta\varepsilon. \end{array}\right.$$

and so (by using that once that the solution is strictly positive in a point x^* and a time t_0 it remains positive at point x^* and for any $t \geq t_0$: see, e. g. [29]) we get that

$$v(t,x_0) \geq v_n(t,x_0) - \mu > 0$$

for μ small enough, n sufficiently large and t near zero, which conclude the proof. $\square$

Remark 7. Different techniques for the study of the absence of waiting time were used in [4].

Coming back to the general formulation (i.e. without assuming (2)), we can easily adapt the previous result by means of the implicit discretization in time scheme given by

$$\psi(u_{n+1}) - \psi(u_n) + \Delta t(Au_{n+1} + \beta(x, u_{n+1})) = G(t_n, x)$$

where $G(t_n, .)$ is an approximation of $G(t, .)$ in $L^1(0, T : L^1(\omega))$.

Remark 8. For the case $A = L$ and $\beta = G = 0$ the Theorem 3 remains true (changing p by 2) since the modifications in the proof are obvious (use the inequalities (11)).

For the case $\beta \neq 0$ we can get easily a result similar to Theorem 3 if we assume that $G \equiv 0$. Here is a particular, but indicative, statement:

Theorem 3.3. *Let A given by (7) or (9). Assume that β satisfies (6) for some f such that for any $\mu \in R$ there exists $C, K_\mu > 0$ such that*

$$\psi(r) + \mu f(r) \geq K_\mu |r|^{q-1} r \qquad for\ any\ r \in R$$

and

$$C|r|^{q-1} r \geq \psi(r)$$

with $0 \leq q < p - 1$, where p is the degree of homogeneity of A. Assume that $\|u_0\|_{L^\infty(\omega)} \leq M$ and there exists $C_0 > 0$ and $R > 0$ for which

$$0 \leq \psi(u_0(x)) \leq C_0 |x - x_0|^{\frac{pq}{p-1-q}} \quad a.e.\ x \in \omega \cap B_R(x_0).$$

Then, if μ is small enough there exists $n^*(x_0) \in N$ such that

$$\psi(u_n(x)) \leq C_n |x - x_0|^{\frac{pq}{p-1-q}} \quad a.e.\ x \in \omega \cap B_R(x_0)$$

for some $C_n > 0$ and for any $n = 1, 2, ..., n^*(x_0)$.

Proof. The main idea is to adapt the proof of Theorem 3 to this context. So, it is enough to consider the family of barrier functions given by

$$\psi(\bar{v}_n(x)) = C_n |x - x_0|^{\frac{p}{p-1-q}}$$

and to use that

$$-\mu \Delta_p \bar{v}_{n+1} + C_\mu (\bar{v}_{n+1})^q \geq \psi(\bar{v}_n)$$

for $\mu = \Delta t$. $\square$

Remark 9. The study of the waiting time when $G \neq 0$ seems to be unexplored before in the literature, nevertheless, our technique of proof allows to get easily some results in the spirit of Theorem 4.

Remark 10. For the connection between other qualitative properties of solutions and their version for the solutions of some discretized algorithms see [27].

Remark 11. Many of the results of this paper also apply to *Stefan type problems* for which $\psi(u)$ is a multivalued maximal monotone graph of the form

$$\begin{aligned} \psi(u) &= \alpha_1(u) \qquad if\ u < 0 \\ \psi(0) &= [0, k] \\ \psi(u) &= \alpha_2(u) \qquad if\ u > 0 \end{aligned}$$

with α_i be nondecreasing functions, $\alpha_1(0) = 0 < k = \alpha_2(0)$. See Díaz [23] for an approach via an energy method. In the case of the Stefan problem it is clear that the interfaces are defined by the separation of each of the phases. Nevertheless it is well known (see, e.g. [31]) that the set $M(u) = \{(t, x) : u(t, x) = 0\}$ can be of positive measure (*the mushy region*) and so its boundary defines another free boundary.

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