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Lagrangian Approach to the Study of Level Sets: Application to a Free Boundary Problem in Climatology

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Abstract

We study the dynamics and regularity of level sets in solutions of the semilinear parabolic equation

$$u_t - \Delta u \in a\,\mathbb{H}(u-\mu) \quad \text{in} \quad Q = \Omega \times (0, T],$$

where $\Omega \subset \mathbb{R}^n$ is a ring-shaped domain, a and μ are given positive constants, $\mathbb{H}(\cdot)$ is the Heaviside maximal monotone graph: $\mathbb{H}(s) = 1$ if $s > 0$, $\mathbb{H}(0) = [0, 1]$, $\mathbb{H}(s) = 0$ if $s < 0$. Such equations arise in climatology (the so-called Budyko energy balance model), as well as in other contexts such as combustion. We show that under certain conditions on the initial data the level sets $\Gamma_\mu = \{(\mathbf{x}, t) : u(\mathbf{x}, t) = \mu\}$ are n-dimensional hypersurfaces in the $(\mathbf{x}, t)$-space and show that the dynamics of Γ_μ is governed by a differential equation which generalizes the classical Darcy law in filtration theory. This differential equation expresses the velocity of advancement of the level surface Γ_μ through spatial derivatives of the solution u. Our approach is based on the introduction of a local set of Lagrangian coordinates: the equation is formally considered as the mass balance law in the motion of a fluid and the passage to Lagrangian coordinates allows us to watch the trajectory of each of the fluid particles.

1. Introduction

1.1. Statement of the problem

Let $\Omega \in \mathbb{R}^n$, $n \geqq 1$, be a ring-shaped domain with exterior boundary $\partial_e \Omega$ and interior boundary $\partial_i \Omega$, $\partial_i \Omega \cap \partial_e \Omega = \emptyset$. Given $T > 0$, we denote by D_T a cylinder $D_T = \Omega \times (0, T]$ with the "lateral boundary"

$$S_T = \{\partial \Omega_e \times (0, T]\} \cup \{\partial \Omega_i \times (0, T]\}$$

and the "parabolic boundary" $P_D = S_T \cup (\overline{\Omega} \times \{0\})$. We consider the problem

$$\begin{cases} u_t - \Delta u \in a\,\mathbb{H}(u-\mu) & \text{in } D_T, \\ u = \phi & \text{on } S_T, \\ u(\mathbf{x},0) = u_0(\mathbf{x}) & \text{in } \Omega \end{cases} \tag{1}$$

where $\mu > 0$ and a are prescribed constants and $\mathbb{H}(\cdot)$ is the Heaviside maximal monotone graph in $\mathbb{R}^2$ given by

$$\mathbb{H}(s) = \begin{cases} 1 & \text{if } s > 0, \\ [0,1] & \text{if } s = 0, \\ 0 & \text{if } s < 0. \end{cases}$$

The solution of problem (1) is understood as follows:

Definition 1. A function $u : D_T \mapsto \mathbb{R}$ is said to be a continuous weak solution of problem (1) if

1. $u \in C^0(\overline{D}_T) \cap L^2(0,T; H^1(\Omega))$ and satisfies the initial and boundary conditions by continuity,
2. there exists a function $h_u \in L^\infty(D_T)$, such that

$$h_u : D_T \mapsto [0,1], \quad h_u(\mathbf{x},t) \in \mathbb{H}(u(\mathbf{x},t)-\mu) \text{ for almost every } (\mathbf{x},t) \in D_T, \tag{2}$$

3. for every test-function η satisfying the conditions $\eta \in L^2(0,T; H_0^1(\Omega))$, $\eta_t \in L^2(D_T)$, $\eta(\mathbf{x},T) = 0$, the following identity holds:

$$\int_{D_T} [\eta_t u - \nabla\eta \cdot \nabla u + a\,\eta\, h_u]\, \mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_\Omega u_0(\mathbf{x})\,\eta(\mathbf{x},0)\,\mathrm{d}\mathbf{x} = 0.$$

Problem (1) is thus reformulated as the problem of finding the functions u and h_u such that

$$\begin{cases} u_t - \Delta u = a\,h_u \in a\,\mathbb{H}(u-\mu) & \text{in } D_T, \\ u = \phi & \text{on } S_T, \\ u(\mathbf{x},0) = u_0(\mathbf{x}) & \text{in } \Omega. \end{cases} \tag{3}$$

It is assumed that the data of problem (3) are subject to the following assumptions:

$$\begin{cases} \partial\Omega_e \text{ and } \partial\Omega_i \text{ are Lipschitz-continuous}, \\ \phi \in C^0(S_T), \quad \phi(\mathbf{x},0) = u_0(\mathbf{x}) > 0 \quad \text{on} \quad \overline{S_T} \cap \{t=0\}, \\ \phi > \mu > 0 \text{ on } \partial\Omega_e \times [0,T], \quad \mu > \phi > 0 \quad \text{on} \quad \partial\Omega_i \times [0,T], \\ u_0 \in C^{2+\beta}(\Omega) \quad \text{for some} \quad \beta \in (0,1). \end{cases} \tag{4}$$

Let us introduce the notation

$$\Gamma_\mu = \{(\mathbf{x},t) \in D_T : u(\mathbf{x},t) = \mu\}, \quad \Gamma_\mu(t) = \Gamma_\mu \cap \{t = \text{const}\}.$$

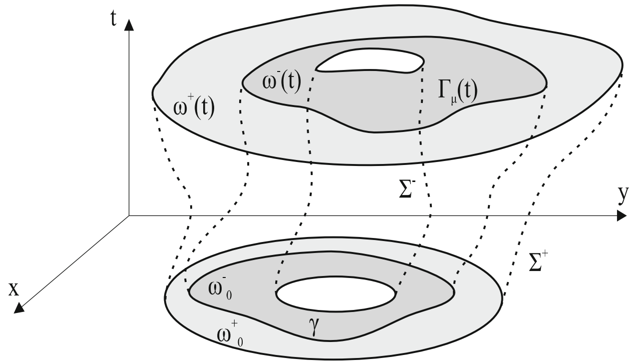

Fig. 1. Evolution of the moving domain $\omega(t) = \omega^+(t) \cup \omega^-(t)$

We will assume that

$$\left\{\begin{array}{l} \Omega \text{ can be represented in the form } \Omega = \Omega^+ \cup \Gamma_\mu(0) \cup \Omega^-, \text{ where} \\ \quad \Omega^+ = \Omega \cap \{\mathbf{x}: \ u_0(\mathbf{x}) > \mu\}, \quad \Omega^- = \Omega \cap \{\mathbf{x}: \ u_0(\mathbf{x}) < \mu\} \\ \text{are ring-shaped domains, } \Gamma_\mu(0) \text{ is simple-connected and} \\ \text{contains a } (n-1)\text{-dimensional hypersurface } \gamma \in C^{2+\beta} \text{ such that} \\ \Omega^- \text{ is contained in the domain bounded by } \gamma. \end{array}\right. \tag{5}$$

(See Fig. 1.)

Remark 1. The assumption $\mu > 0$ is not restrictive: since problem (1) is invariant with respect to the translation $u \mapsto u + \text{const}$, in the case $\mu \leqq 0$ we introduce the new unknown $v = u + |\mu| + 1$ which solves problem (3) with the data satisfying conditions (4).

The main goal of this paper is to describe the level set Γ_μ which separates the regions

$$D_T^+ = \{(\mathbf{x}, t) \in D_T : \ u(\mathbf{x}, t) > \mu\} \quad \text{and} \quad D_T^- = \{(\mathbf{x}, t) \in D_T : \ u(\mathbf{x}, t) < \mu\}.$$

We want to answer the following questions:

1. Given the initial function u_0, how does Γ_μ start moving at the time $t = 0$?
2. Is it possible to characterize the evolution of Γ_μ in terms of the solution u and its derivatives?

As a byproduct of our study, we shall also analyze some topological and regularity properties of the level set Γ_μ. To the best of our knowledge, these questions have been discussed so far only in papers [12,28] in the one-dimensional setting. It is shown in [12] that the level set Γ_μ is a C^∞-curve on every interval $[\tau, T]$ with $\tau > 0$. Unfortunately, the method of [12] would not work in the case of several space variables. The study of interface regularity in [28] is also performed by essentially one-dimensional methods.

1.2. Physical background

Our interest in problem (1) is motivated by its application in climatology. Equation (1) is a simplification of Budyko's energy balance model [5]. This is the heat equation for energy u which includes the term

$$R_a = QS(t, \mathbf{x})\mathbb{H}(u + 10),$$

that represents the amount of solar energy absorbed by the Earth. Here Q is the solar constant, the function $S(t, \mathbf{x})$ is an insolation function that gives the distribution of the solar radiation incident at the top of the atmosphere. When the averaging time is of the order of a year or more, it is assumed that there exists a constant $S_0 > 0$ such that $S(t, \mathbf{x}) \geqq S_0$ for all t and $\mathbf{x}$.

The balance equation includes also a diffusion term and a term R_e that represents the amount of energy radiated into space. BUDYKO suggested representing $R_e(u)$ performing a linear regression adjustment of the empirical data:

$$R_e(u) = Bu + C,$$

with given constants B and C.

Proceeding in a parallel way, SELLERS [22] proposed a similar energy balance model but with the term R_e given in accordance to the Stefan–Boltzman law: $R_e = \sigma u^4$, where σ is the emissivity constant and u is measured in degrees Kelvin. In both cases, the resulting equation has the form

$$u_t - \Delta u + \mathcal{G}(\mathbf{x}, u) \in QS(t, \mathbf{x})\mathbb{H}(u + 10) + f(t, \mathbf{x}) \tag{6}$$

where $\mathcal{G}(\mathbf{x}, u)$ is a strictly increasing function of u and $R_e(u) = \mathcal{G}(\mathbf{x}, u) - f(t, \mathbf{x})$, $f(t, \mathbf{x})$ is a prescribed forcing term, and $\mathbb{H}(\cdot)$ is the Heaviside maximal monotone graph.

Equations of type (6) were studied under different assumptions on the problem domain and the nonlinear terms involved in the equation. Among other results available in the literature, let us specially mention here papers [28] and [6], where the problem in the one-dimensional setting is studied, and [7], which discusses the possibility of formulating the problem on a compact Riemannian manifold without boundary. The manifold represents the Earth's surface in the two-dimensional case. It is easy to adapt this formulation to the case in which the spatial domain is a ring surrounding one of the ice caps.

The numerical approximation of the problem by the finite element method is performed in [2].

Remark 2. For the sake of presentation, throughout the paper we deal with the model problem (1). All our results extend to the case of the complete equation of type (6); an outline of these arguments is given in the final section.

Remark 3. Equations of type (1) arise in the mathematical modelling of other physical phenomena such as combustion. We refer to papers [1,6,7] for further information on this issue.

1.3. The change of independent variables: a mechanical approach

Let us notice first that the solution of problem (3) is locally smooth, as a solution of the heat equation, away from the level set Γ_μ where the forcing term $\mathbb{H}(u-\mu)$ is discontinuous. The dynamics of every smooth level surface Γ_ν can be described in a standard way. Calculating the total time derivative along the surface Γ_ν,

$$\left.\frac{\mathrm{d}u(\mathbf{x},t)}{\mathrm{d}t}\right|_{\Gamma_\nu} = u_t(\mathbf{x},t) + \nabla u(\mathbf{x},t)\cdot\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = 0,$$

and using the fact that the normal vector to Γ_ν has the form $\mathbf{n} = \dfrac{\nabla u}{|\nabla u|}$, we find that the advancement of the point of Γ_ν in the normal direction is defined by the differential equation

$$\mathbf{n}\cdot\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = -\left.\frac{u_t}{|\nabla u|}\right|_{\Gamma_\nu}. \tag{7}$$

Equation (7) need not be true on the level set Γ_μ, however. Indeed, (3) is a uniformly parabolic equation with the right-hand side in $L^\infty(D_T)$, and generic regularity results show that its solution belongs to the parabolic space $W^{2,1}_q(D_T)$, that is, the time derivative need not be continuous in D_T. This observation explains why questions of dynamics and regularity of the set Γ_μ require a special study.

Our study of the level set Γ_μ is based on the introduction of a system of Lagrangian coordinates frequently used in continuum mechanics. Every positive solution of problem (3) can be formally considered as a solution of the problem

$$\begin{cases} u_t = \operatorname{div}\,(u\nabla\,\ln u) + a\,h_u \quad \text{in} \quad D_T, \\ u = \phi \text{ on } S_T, \quad u(\mathbf{x},0) = u_0 \quad \text{in} \quad \Omega, \end{cases} \tag{8}$$

understood in the sense of Definition 1. The reason for this conversion from semilinear equation (3) to Eq. (8) with the nonlinear diffusion operator is that the latter can be understood as the mass balance law in the motion of a continuous medium,

$$\rho_t + \operatorname{div}\,(\rho\,\mathbf{v}) = f,$$

with density $\rho = u$ and velocity $\mathbf{v} = -\nabla\,\ln\,u$ in the presence of the source of mass $f = a\,h_\rho$ nonlinearly depending on the density ρ.

Let us assume that we already have a solution u of problem (8). By the continuity of u, we may choose a subdomain $\mathcal{C}\subset D_T$, bounded by two continuous surfaces $\Sigma^\pm$, such that

$$\begin{aligned} &\text{(a)} \qquad \Gamma_\mu\cap\Sigma^\pm = \emptyset, \qquad \Gamma_\mu\subset\mathcal{C}, \\ &\text{(b)} \qquad \forall\, t>0 \quad \int_{\omega(t)} u(\mathbf{x},t)\,\mathrm{d}\mathbf{x} = \text{const}, \\ &\qquad\qquad \text{where } \omega(t) = \mathcal{C}\cap\{t=\text{const}\}. \end{aligned} \tag{9}$$

It is worth noting here that there is a continuum of domains $\mathcal{C}$ satisfying these conditions but, as we shall see, this is not relevant for our further arguments. Let us introduce the auxiliary free-boundary problem: to find a positive function u, a function $h_u \in \mathbb{H}(u-\mu)$, and surfaces $\Sigma^{\pm}$ satisfying the the following conditions

$$\begin{cases} u_t = \operatorname{div}\,(u\nabla \ln u) + a\,h_u(\mathbf{x},t) \quad \text{in} \quad \mathcal{C} = \bigcup_{t\in(0,T]}\omega(t), \\ u(\mathbf{x},0) = u_0 \quad \text{in} \quad \omega_0, \\ \forall\, t \in [0,T] \qquad \displaystyle\int_{\omega(t)} u(\mathbf{x},t)\,\mathrm{d}\mathbf{x} = \int_{\omega_0} u_0(\mathbf{x})\,\mathrm{d}\mathbf{x}, \qquad \omega_0 \equiv \omega(0), \\ \Gamma_\mu \cap \Sigma^{\pm} = \emptyset. \end{cases} \tag{10}$$

Instead of posing any specific boundary condition on the surfaces $\Sigma^{\pm}$, we claim that the total mass of fluid in the domain $\omega(t)$ is constant at every moment $t \geqq 0$.

Definition 2. Surfaces $\Sigma^{\pm} \in C^{1+\alpha,(1+\alpha)/2}$, $\alpha \in (0,1)$, and a function $u \in C^0(\overline{\mathcal{C}})$ are said to be a weak continuous solution of problem (10) if $u > 0$ in $\overline{\mathcal{C}}$, ln $u \in L^2(0,T;H^1(\omega(t)))$, there exists a function h_u satisfying conditions (2), and for every test-function $\eta \in C^1(\mathcal{C}) \cap C^0(\overline{\mathcal{C}})$, vanishing as $t = T$, the following identity holds:

$$\int_{\mathcal{C}} [u\,\eta_t - u\,\nabla \ln u \cdot \nabla\,\eta + a\,h_u\,\eta]\,\mathrm{d}\mathbf{x}\,\mathrm{d}t + \int_{\omega_0} u_0(\mathbf{x})\eta(\mathbf{x},0)\,\mathrm{d}\mathbf{x} = 0. \tag{11}$$

Using mechanical terminology, we may interpret the auxiliary problem (10) as follows: to find the density $u(\mathbf{x},t)$ of a fluid in a domain $\omega(t)$ knowing that the mass source is defined by the function ah_u, that the total mass of the fluid contained in $\omega(t)$ is constant for every time $t \in [0,T]$, and that the velocity has the form $-\nabla$ ln u. So, equation (10) can then be viewed as the mass balance law in the Euler description where all the flux functions, such as density, velocity, pressure, etc., are considered as functions of time t and coordinates $\mathbf{x}$ (which is a coordinate system not connected with the medium).

An alternative description is due to J. Lagrange. In his approach, the characteristics of motion are functions of the initial positions of particles and the time t. If the fluid volume is constituted by the same particles at every time, and if the total mass of fluid is constant, the domain occupied by the fluid is time-independent in the Lagrangian description. This feature is of special convenience if the boundaries of the flow domain are a priori unknown. The passage to Lagrangian coordinates provides us with a non-local change of independent variables which renders the free boundaries $\Sigma^{\pm}$ stationary. Moreover, if we claim that the surface Γ_μ is also constituted by the same particles for all $t > 0$, the image of the set Γ_μ in the space of Lagrangian coordinates is also time-independent.

This is the method we shall use to study the free-boundary problem (10). Once the free-boundary problem formulated in Lagrangian coordinates is solved, we show that the constructed solution generates a solution of the same problem in the Euler formulation [alias problem (10)], and then extend the constructed solution from $\mathcal{C}$ to the rest of the cylinder D_T by solving Eq. (3).

We thus obtain a double description for function u: on one hand, it solves the initial and boundary-value problem (3), on the other hand, it represents the distribution of density in the flow of an “imaginary” fluid and the motion of each of the fluid particles is described in terms of its initial position and the time. In this mechanical framework, the level set $\Gamma_\mu \cap \{t = \text{const}\}$ of $u(\cdot, t)$ is understood as the set of points of Ω occupied by the particles whose density at the time t equals μ.

The idea of using Lagrangian coordinates in the study of the free-boundary problems for evolution equations was independently proposed in [3,15,20]. The review of results obtained with this method in the study of evolution equation in divergence form can be found in the monograph [21] or in paper [8]. Articles [9,25,26] contain the first applications of the method to the study of evolution equations with one spatial variable which are not in divergence form. In the present paper we follow the method of introducing a local system of Lagrangian coordinates proposed in [23,24]. These papers deal with the porous medium equation with continuous strong absorption terms. Let us point out that linearity of the diffusion operator coupled with discontinuity of the source term gives rise to new difficulties which are studied here, by means of introduction of Lagrangian coordinates, for the first time in the literature.

1.4. Main results

Let $\gamma \subset \Gamma_\mu(0)$ be the hypersurface from condition (5) and let ω_0^+ and ω_0^- be ring-shaped domains chosen as follows:

$$\begin{cases} u_0 \geqq \mu \text{ in } \overline{\omega}_0^+,\, u_0 \leqq \mu \text{ in } \overline{\omega}_0^-,\, \gamma = \overline{\omega}_0^+ \cap \overline{\omega}_0^-, \\ \overline{\omega}_0^+ \cap \partial\Omega_e = \emptyset,\, \overline{\omega}_0^- \cap \partial\Omega_i = \emptyset, \\ \text{the exterior boundary } \partial\omega_0^+ \text{ of } \omega_0^+ \text{ and} \\ \text{the interior boundary } \partial\omega_0^- \text{ of } \omega_0^- \text{ belong to } C^{2+\beta} \text{ with } \beta \in (0,1). \end{cases} \tag{12}$$

Theorem 1. *Let conditions* (4), (5), (12) *be fulfilled and* $a > 0$. *Then there exists* $T^* > 0$, *depending on n,* $|u_0|_\Omega^{(2+\beta)}$, *and the properties of the surfaces* γ *and* $\partial\omega_0^\pm$, *such that*

1. *Problem* (10) *has a solution* $(v,\ \Sigma^\pm)$ *in the sense of Definition* 2*;*
2. *Problem* (3) *has a solution* $u(x,t)$ *in the sense of Definition* 1*;*
3. *For every* $t \in (0, T^*]$ *the set* $\Gamma_\mu(t)$ *is a* $(n-1)$*-dimensional simple-connected hypersurface,*
4. $u \equiv v$ *in a neighborhood of the surface* Γ_μ.

The existence result stated in Theorem 1 is local in time and the restriction on T^* is of topological character: this is the moment when the surface $\Gamma_\mu(t)$ may split into several simply-connected components. In our conditions on the data, the initial function u_0 is allowed to have a "flat zone" $\{\mathbf{x} \in \Omega :\ u_0(\mathbf{x}) = \mu\} \subset \Gamma_\mu(0)$ of nonzero measure. Nonetheless, Theorem 1 guarantees the existence of at least one solution such that its level set $\Gamma_\mu(t)$ is a $(n-1)$-dimensional hypersurface.

Let us introduce several notations. Throughout the rest of the paper we denote by $\mathbf{H}(V)$ the Hessian matrix with the entries $D^2_{ij}V$, $\mathbf{I}$ always stands for the identity matrix. Set $Q_T^\pm = \omega_0^\pm \times (0, T]$ and denote

$$W_q^k(Q_T^\pm) = \big\{u(\mathbf{y},t) :\ D_{\mathbf{y}}^\gamma u \in L^q(Q_T^\pm),\ 0 \leqq |\gamma| \leqq k\big\}, \quad q \in (1,\infty).$$

The sets $W_q^k(Q_T^{\pm})$ equipped with the norms

$$\|u\|^{(k)}_{q,Q_T^{\pm}} = \sum_{0\leqq|\gamma|\leqq k} \|D^{\gamma}_{\mathbf{y}} u\|_{L^q(Q_T^{\pm})}$$

are Banach spaces. By $C^{\alpha}(\overline{Q_T^{\pm}})$ with $\alpha \in (0,1)$ we denote the space of functions obtained as the closure of $C^{\infty}(\overline{Q_T^{\pm}})$ with respect to the norms

$$|u|^{(\alpha)}_{\overline{Q_T^{\pm}}} = \sup_{\overline{Q_T^{\pm}}} |u| + \langle u\rangle^{(\alpha)}_{\overline{Q_T^{\pm}}},$$

where $\langle\,\cdot\,\rangle^{(\alpha)}_{\overline{Q_T^{\pm}}}$ is the Hölder quotient

$$\langle u\rangle^{(\alpha)}_{\overline{Q_T^{\pm}}} = \sup_{(\mathbf{x},t),(\mathbf{y},\tau)\in\overline{Q_T^{\pm}},\,(\mathbf{x},t)\neq(\mathbf{y},\tau)} \frac{|u(\mathbf{x},t)-u(\mathbf{y},\tau)|}{|\mathbf{x}-\mathbf{y}|^{\alpha}+|t-\tau|^{\alpha/2}}.$$

The notation $W_q^{2,1}(Q_T^{\pm})$, $q\in(1,\infty)$, is used for parabolic spaces with the norms

$$\|u\|_{W_q^{2,1}(Q_T^{\pm})} = \sum_{0\leqq|\beta|+2j\leqq 2} \|D_t^j D^{\beta}_{\mathbf{x}} u\|_{L^q(Q_T^{\pm})}.$$

Theorem 2. *Let the conditions of Theorem 1 be fulfilled.*

1. *There exists a function $U(\mathbf{y},t)$ such that*

$$\begin{aligned} &U,\ tU_t \in W_q^4(Q_{T^*}^{\pm}),\quad U_t,\ tU_{tt}\in W_q^2(Q_{T^*}^{\pm}) \quad \textit{with some}\quad q>n+2,\\ &U(\mathbf{y},0)=0,\quad U=0\quad \textit{on}\quad \partial\omega_0^{\pm}\times[0,T^*], \end{aligned} \tag{13}$$

   Det $[\mathbf{I}+\mathbf{H}(U)]$ *is separated away from zero and infinity in $\overline{Q_{T^*}^{\pm}}$, and the solution to problem (10) $(u,\ \mathcal{C})$ is defined by the formulas*

$$u(\mathbf{x},t) = \frac{u_0(\mathbf{y})}{\operatorname{Det}[\mathbf{I}+\mathbf{H}(U)]},\quad \mathbf{x}=\mathbf{y}+\nabla U(\mathbf{y},t)\quad \textit{with}\quad \mathbf{y}\in\overline{\omega}_0^{\pm}. \tag{14}$$

2. *The level surface Γ_μ and the boundaries $\Sigma^{\pm}$ of the set $\mathcal{C}$ are parametrized by means of the bijective mapping*

$$\begin{aligned} &\text{(a)}\quad \gamma\ni\mathbf{y}\mapsto\mathbf{x}=\mathbf{y}+\nabla U(\mathbf{y},t)\in\Gamma_\mu(t),\\ &\text{(b)}\quad \partial\omega_0^{\pm}\ni\mathbf{y}\mapsto\mathbf{x}=\mathbf{y}+\nabla U(\mathbf{y},t)\in\partial\omega^{\pm}(t). \end{aligned} \tag{15}$$

The function U in the conditions of the above theorem is the solution of problem (10) formulated in local Lagrangian coordinates; this formulation is discussed in detail in Section 2.

Theorem 3. *Let the conditions of Theorem 2 be fulfilled and let* $(u, \Sigma^{\pm})$ *be the solution of problem* (10) *defined by formulas* (14)–(15). *The solution* $(u, \Sigma^{\pm})$ *then describes the motion of a fluid with density* $u(\mathbf{x}, t)$ *and (potential) velocity*

$$\mathbf{v}(\mathbf{x}, t) \equiv \nabla_{\mathbf{x}} [p - \ln u], \tag{16}$$

where the function $p(\cdot, t)$*, depending on* t *as a parameter, is a weak solution of the linear elliptic problem*

$$\operatorname{div}(u \nabla p) + a\, \chi_{\omega^+(t)} = 0 \quad in \quad \omega^{\pm}(t), \quad p = 0 \quad on \quad \partial\omega^{\pm}(t), \tag{17}$$

and $\chi_{\omega^+(t)}$ *denotes the characteristic function of the set* $\omega^+(t)$*. For* $t \in (0, T^*]$*, the level set* $\Gamma_\mu(t)$*, associated with the constructed solution, moves with velocity* $\nabla_{\mathbf{x}}[p - \ln u]|_{\Gamma_\mu(t)}$

Theorem 4. *Under the foregoing conditions:*

1. *for every* $\tau \in (0, T^*]$

$$\Gamma_\mu \in C^{\sigma}\left([0, T^*];\ C^{2+\sigma}(\gamma)\right) \cap C^{1+\sigma}\left([\tau, T^*]; C^{2+\sigma}(\gamma)\right)$$

   with $\sigma \in (0, 1)$*;*
2. $\mathbf{v}(\mathbf{x}, t) \in C^0([0, T^*];\ C^{\alpha}(\omega(t)))$ *with some* $\alpha \in (0, 1)$*,*

$$|\mathbf{v}(\mathbf{x}(\mathbf{y}, t), t) - \mathbf{v}_0(\mathbf{y})|^{(\alpha)}_{\omega_0^{\pm}} \to 0 \quad as \quad t \to 0,$$

   where $\mathbf{x}(\mathbf{y}, t)$ *is defined by formula* (15) (a)*,* $\mathbf{v}_0(\mathbf{y}) = -\nabla(\ln u_0 - p_0)$ *and* p_0 *is the solution of the problem*

$$\operatorname{div}(u_0 \nabla p_0) + a\, \chi_{\omega_0^+} = 0 \quad in\ \omega(0), \quad p_0 = 0 \quad on\ \partial\omega_0^{\pm};$$

3. *if* $\Gamma_\mu(0)$ *is not an* $(n-1)$*-dimensional hypersurface and the surface* γ *is not uniquely defined, then the solution of problem* (3) *is not unique.*
4. *if* $\Gamma_\mu(0) \equiv \gamma$ *and if the continuous weak solution of problem* (3) *is unique, then the velocity* $\mathbf{v}(\mathbf{x}, t)|_{\Gamma_\mu(t)}$ *does not depend on the choice of the sets* $\omega_0^{\pm}$*.*

The questions of uniqueness and nonuniqueness of solutions to problem (3) were discussed in papers [6,7,12] (see also [10,11] for the discussion of uniqueness of a solution for a one-dimensional model in combustion theory). It is shown in [6] that in the case $n = 1$ the solution of Eq. (8) with the co-normal derivative boundary condition is unique if the initial function satisfies the *nondegeneracy condition*: $|u_0'| \geqq \epsilon > 0$ in $\Omega = (0, 1)$. Conversely, if $|u_0'| = 0$ at the level $u_0 = \mu$, then problem (3) may have infinitely many solutions—see [6] for rigorous proofs and examples of nonuniqueness. In the case when Ω is a compact two-dimensional Riemannian manifold without boundary, a similar criterion for uniqueness and nonuniqueness of solutions to problem (3) is established in [7]. The examples of nonuniqueness in [6,7] are given for a class of initial data which is different from the class of nonuniqueness considered in the present work [see Theorem 4 (3)]. In [12] the existence of a unique solution of problem (3) in the case $n = 1$ is also stated under the assumption of nondegeneracy of the initial function u_0.

It is worth noting here that the assertions of the above theorems cease to be true in the limit case $a = 0$ (see Remark 4 below.)

1.5. Organization of the paper

In Section 2 we introduce a local system of Lagrangian coordinates and then formulate problem (10) in the Lagrangian setting for a potential flow. By using a De Rham type orthogonal decomposition we reformulate the problem, first giving a definition of the weak solution and then showing that, under suitable conditions on the initial data, every solution of the problem in Lagrangian formulation generates a solution of the original problem (3).

Sections 3–4 contain the analysis of the linearized problem in Lagrangian variables. The complete nonlinear problem is solved in Section 5. To this end, we use an abstract version of the modified Newton method which allows us to reduce the nonlinear problem to a sequence of linear problems. The proofs of the main theorems are given in Section 6

In Section 7 we extend our arguments and conclusions to the case of the general equation (6).

2. Local system of Lagrangian coordinates

2.1. The Euler and Lagrangian descriptions of an evolving fluid volume

Let us consider the motion of a fluid volume occupying a moving region $\omega(t) \subset \mathbb{R}^n$ which contains a surface Γ_μ on which the density of the fluid equals a prescribed positive constant μ. As is usual in continuum mechanics (see, for example, [14]), we assume that

(a) the mass of the fluid contained in $\omega(t)$ is constant,
(b) the fluid volume is constituted by the same particles for all $t \geqq 0$,
(c) the moving boundaries $\partial\omega(t)$ of the domain $\omega(t)$ and the surface $\Gamma_\mu(t)$ are constituted by the same particles for all $t > 0$.

We also assume that the velocity field $\mathbf{v}(\mathbf{x}, t)$ is a given vector-valued function of the particle position $\mathbf{x} \in \mathbb{R}^n$ and time t, that at the time $t = 0$ the fluid occupies the region ω_0, and that the initial density $\rho_0(\mathbf{x})$ is prescribed. Denote by $\rho(\mathbf{x}, t)$ the density of the fluid considered as a function of the variables $(\mathbf{x}, t)$. Given the initial distribution of the density $\rho_0(\mathbf{x}) = u(\mathbf{x}, 0)$, the Euler description of this flow has the form

$$\begin{cases} \rho_t + \operatorname{div}(\rho\, \mathbf{v}) = 0 \quad \text{in} \quad \mathcal{C} = \bigcup_{t\in(0,T]} \omega(t), \\ \rho(\mathbf{x}, 0) = u(\mathbf{x}, 0) \quad \text{in} \quad \omega_0, \quad \rho = \mu \quad \text{on} \quad \Gamma_\mu \subset \mathcal{C}, \\ \forall\, t \in [0, T] \quad \displaystyle\int_{\omega(t)} \rho(\mathbf{x}, t)\, \mathrm{d}\mathbf{x} = \int_{\omega_0} \rho(\mathbf{x}, 0)\, \mathrm{d}\mathbf{x}, \end{cases} \tag{18}$$

Definition 3. By a weak continuous solution of problem (18) we mean a couple $(\rho, \mathcal{C})$ such that $\rho \in C^0(\overline{\mathcal{C}})$ and for every test-function $\phi \in C^1(\mathcal{C}) \cap C^0(\overline{\mathcal{C}})$, $\phi(\mathbf{x}, T) = 0$,

$$\int_{\mathcal{C}} [\rho\phi_t + \rho\, \mathbf{v} \cdot \nabla\phi]\, \mathrm{d}\mathbf{x}\mathrm{d}t + \int_{\omega_0} \rho(\mathbf{x}, 0)\phi(\mathbf{x}, 0)\, \mathrm{d}\mathbf{x} = 0. \tag{19}$$

Let us give the alternative Lagrangian description of this flow. Assumption (b) allows us to label each of the particles by its initial position $\mathbf{y} \in \omega_0$. Denoting by $\mathbf{X}(\mathbf{y}, t)$ the position, at time t, of the particle initially located at the point $\mathbf{y} \in \omega_0$, we obtain (for fixed t) the map from ω_0 on $\omega(t)$:

$$\mathbf{y} \mapsto \mathbf{x} = \mathbf{X}(\mathbf{y}, t).$$

The function $\mathbf{X}(\mathbf{y}, t)$ satisfies the equation of trajectories

$$\begin{cases} \dfrac{\mathrm{d}\mathbf{X}}{\mathrm{d}t}(\mathbf{y}, t) = \mathbf{v}[\mathbf{X}(\mathbf{y}, t), t], \quad t > 0, \\ \mathbf{X}(\mathbf{y}, 0) = \mathbf{y} \in \overline{\omega}_0. \end{cases} \tag{20}$$

(In what follows, when dealing with the Lagrangian description of an Eulerian function we shall always use square brackets). Equation (20) is a system of ordinary differential equations for the components of the vector $\mathbf{X} = (X_1, \ldots, X_n)$. The solution of this system is understood in a weak sense (see Definition 4 below). We claim that system (20) describes the motion of the particles which belong to the closed set $\overline{\omega}_0$ at time $t = 0$. Since the "free" boundaries of the moving volume $\omega(t)$ and the surface $\Gamma_\mu(t)$ are assumed to be constituted by the same particles at every time t, the solution of system (20) with $\mathbf{y} \in \partial\omega_0$ or $\mathbf{y} \in \gamma$ describes the motion of the free boundaries $\partial\omega(t)$ and $\Gamma_\mu(t)$. This condition is usually termed *the dynamical boundary condition*.

Assumption (a) means that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho\left[\mathbf{X}(\mathbf{y}, t), t\right] \operatorname{Det}[\mathbf{J}]\right) = 0 \quad \forall\,(\mathbf{y}, t) \in \omega_0 \times [0, T], \tag{21}$$

where $\mathbf{J}$ is the Jacobi matrix of the map $\mathbf{y} \mapsto \mathbf{X}(\mathbf{y}, t)$ (notice the abuse of notation: in fact $\mathbf{J} = \mathbf{J}(\mathbf{y}, t)$). Indeed: take an arbitrary fluid volume $\sigma(t)$ whose mass is preserved in time, that is

$$\int_{\sigma(t)} \rho(\mathbf{x}, t)\,\mathrm{d}\mathbf{x} = \text{const};$$

if $\dfrac{\mathrm{d}}{\mathrm{d}t}(\rho \operatorname{Det}[\mathbf{J}]) \in L^1(\sigma(0))$, we may formally pass to the coordinates $\mathbf{y}$, which gives

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\sigma(t)} \rho(\mathbf{x}, t)\,\mathrm{d}\mathbf{x}\right) \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\sigma(0)} \rho[\mathbf{X}(\mathbf{y}, t), t] \operatorname{Det}[\mathbf{J}]\;\mathrm{d}\mathbf{y}\right) \\ &= \int_{\sigma(0)} \frac{\mathrm{d}}{\mathrm{d}t}\left(\rho[\mathbf{X}(\mathbf{y}, t), t] \operatorname{Det}[\mathbf{J}]\right)\,\mathrm{d}\mathbf{y}. \end{aligned}$$

The conclusion follows since the domain $\sigma(t)$ is arbitrary.

According to assumption (c), the velocity of the boundaries $\partial\omega(t)$ and $\Gamma_\mu(t)$ coincides with the velocity of the particles constituting these surfaces, that is, the motion of $\partial\omega(t)$ and $\Gamma_\mu(t)$ is described by Eq. (20). Any function $f(\mathbf{x}, t)$ defined on $\bigcup_{t\in[0,T]}\omega(t)$ can now be considered as the function of the variables $(\mathbf{y}, t)$ defined in the cylinder $(\mathbf{y}, t) \in Q_T := \omega_0 \times [0, T]$:

$$F(\mathbf{y}, t) \equiv f[\mathbf{X}(\mathbf{y}, t), t] = f(\mathbf{x}, t), \qquad (\mathbf{x}, t) \in \bigcup_{t\in[0,T]} \omega(t), \quad (\mathbf{y}, t) \in Q_T.$$

In the rest of the paper we shall follow this rule and use capital letters for the Lagrangian counterparts of the functions defined as functions of the variables $(\mathbf{x}, t)$. For instance, we shall write

$$\rho(\mathbf{x}, t) = \rho[\mathbf{X}(\mathbf{y}, t), t] = R(\mathbf{y}, t),\ p(\mathbf{x}, t) = p[\mathbf{X}(\mathbf{y}, t), t] = P(\mathbf{y}, t),$$

etc. Thus, the mass balance law (21) is written in the form

$$R(\mathbf{y}, t)\,\mathrm{Det}\,[\mathbf{J}] = \rho_0(\mathbf{y}) \quad \forall\,(\mathbf{y}, t) \in Q_T. \tag{22}$$

The initial and boundary conditions for the functions defining the flow in the plane of Lagrangian coordinates are

$$R(\mathbf{y}, 0) = \rho_0(\mathbf{y}) \text{ in } \omega_0, \quad \mathrm{Det}[\mathbf{J}] = 1 \text{ on } \gamma \times [0, T]. \tag{23}$$

2.2. *Potential flows*

Let us consider the special class of flows for which the position of the particle is given in the form

$$\mathbf{X}(\mathbf{y}, t) = \mathbf{y} + \nabla U(\mathbf{y}, t)$$

for some function $U(\mathbf{y}, t)$ defined in the cylinder Q_T. We also assume that $U = 0$ on the parabolic boundary of Q_T. Notice that the mechanical meaning of the vector ∇U is nothing other than the displacement $\mathbf{X}(\mathbf{y}, t) - \mathbf{y}$. System (20), (22), (23) takes the form

$$\begin{cases} \nabla U_t = \mathbf{v}[\mathbf{y} + \nabla U, t), t], & \text{in } \overline{Q}_T, \\ U = 0 \quad \text{on the parabolic boundary of } Q_T, \\ R(\mathbf{y}, 0) = \rho_0 \quad \text{in} \quad \omega_0, \\ \mathrm{Det}[\mathbf{I} + \mathbf{H}(U)] = 1 \quad \text{on} \quad \gamma \times [0, T], \end{cases} \tag{24}$$

where $\mathbf{H}(U)$ denotes the Hessian matrix of the function U, $H_{ij}(U) = D^2_{ij}U$, and $\mathbf{I}$ is the identity matrix. The function R is then defined by the equation

$$R = \frac{\rho_0}{\mathrm{Det}[\mathbf{I} + \mathbf{H}(U)]} \quad \text{in} \quad Q_T.$$

Definition 4. A function U defined on $\overline{Q}_T$, $U = 0$ on the parabolic boundary of Q_T, is said to be a solution of problem (24) if

1. the mapping $\mathbf{y} \mapsto \mathbf{x} = \mathbf{y} + \nabla U(\mathbf{y}, t)$ is a bijection from $\overline{\omega}_0$ to $\overline{\omega}(t)$, and

$$\lambda_1 < \mathrm{Det}[\mathbf{I} + \mathbf{H}(U)] < \lambda_2$$

in $\overline{Q}_T$ for some positive constants λ_1, λ_2,

2. $\rho_0(\mathbf{I}+\mathbf{H}(U))^{-1}\mathbf{v}(\mathbf{y}+\nabla U,t) \in \left(W^1_q(Q_T)\right)^n,$
$\rho_0(\mathbf{I}+\mathbf{H}(U))^{-1}\nabla U_t \in \left(W^1_q(Q_T)\right)^n$ with some $q \in (1,\infty)$,
3. $\mathrm{Det}[\mathbf{I}+\mathbf{H}(U)] = 1$ on $\gamma \times [0,T]$,
4. for every test-function $\eta \in W^{1,q'}(Q_T)$, $q' = q/(q-1)$,

$$\int_{Q_T} \nabla\eta \cdot \left(\rho_0\,(\mathbf{I}+\mathbf{H}(U))^{-1}\,(\nabla U_t - \mathbf{v}[\mathbf{y}+\nabla U,t])\right)\,\mathrm{d}\mathbf{y}\,\mathrm{d}t = 0. \tag{25}$$

We are now in a position to establish the correspondence between the Euler and Lagrangian descriptions of the motion of a fluid volume.

Theorem 5. *Let* $\mathbf{v}(\mathbf{x},t)$ *be a given continuous vector-field. If function* U *is a solution of problem* (24) *in the sense of Definition* 4*, then the function* $\rho(\mathbf{x},t)$ *defined by the formulas*

$$\begin{cases} \rho(\mathbf{x},t) = R(\mathbf{y},t) = \dfrac{\rho_0(\mathbf{y})}{\mathrm{Det}\,[\mathbf{I}+\mathbf{H}(U)]}, \\ \mathbf{x} = \mathbf{X}(\mathbf{y},t) \equiv \mathbf{y}+\nabla U(\mathbf{y},t), \quad (\mathbf{y},t) \in Q_T, \end{cases}$$

is a solution of problem (18) *in the sense of Definition* 3.

Proof. It is easy to see that the domain

$$\mathcal{C} = \{(\mathbf{x},t):\ \mathbf{x} = \mathbf{y}+\nabla U(\mathbf{y},t),\ (\mathbf{y},t) \in Q_T\},$$

satisfies all the conditions of Definition 3. It suffices to verify that identity (19) is fulfilled for the function $\rho(\mathbf{x},t)$. Let ϕ be a suitable test-function. According to Definition 4

$$\begin{aligned}
I_0 &= -\int_{\omega(0)} \phi(\mathbf{y},0)\rho_0(\mathbf{y})\,\mathrm{d}\mathbf{y} = \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\omega(t)} \phi(\mathbf{x},t)\,\rho(\mathbf{x},t)\,\mathrm{d}\mathbf{x}\right)\mathrm{d}t \\
&= \int_0^T \frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\omega_0} \phi[\mathbf{X}(\mathbf{y},t),t]\,\rho[\mathbf{X}(\mathbf{y},t),t]\,\mathrm{Det}[\mathbf{J}]\,\mathrm{d}\mathbf{y}\right)\mathrm{d}t \\
&= \int_0^T \left(\int_{\omega_0} \left(\frac{\mathrm{d}}{\mathrm{d}t}\phi[\mathbf{X}(\mathbf{y},t),t]\right)\rho[\mathbf{X}(\mathbf{y},t),t]\,\mathrm{Det}[\mathbf{J}]\right. \\
&\qquad \left. +\,\phi[\mathbf{X}(\mathbf{y},t),t]\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho[\mathbf{X}(\mathbf{y},t),t]\,\mathrm{Det}[\mathbf{J}]\right)\,\mathrm{d}\mathbf{y}\right)\mathrm{d}t \\
&= \int_{Q_T} \rho_0(\mathbf{y})\,\phi_t[\mathbf{X}(\mathbf{y},t),t]\mathrm{d}\mathbf{y}\,\mathrm{d}t + \int_{Q_T} \rho_0(\mathbf{y})\nabla_x\phi[\mathbf{X}(\mathbf{y},t),t]\cdot\mathbf{X}_t(\mathbf{y},t)\,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\
&\qquad + \int_{Q_T} \phi[\mathbf{X}(\mathbf{y},t),t]\frac{\mathrm{d}}{\mathrm{d}t}\left(R(\mathbf{y},t)\mathrm{Det}[\mathbf{J}]\right)\,\mathrm{d}\mathbf{y}\mathrm{d}t \\
&= I_1 + I_2 + I_3.
\end{aligned}$$

By the definition of solution of problem (20), (22) the term I_3 equals zero. The term I_2 can be transformed in the following way:

$$\begin{aligned}
I_2 &= \int_{Q_T} \rho_0\, \nabla_x \phi[\mathbf{X}(\mathbf{y},t),t] \cdot \mathbf{X}_t(\mathbf{y},t)\,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\
&= \int_{Q_T} \nabla_{\mathbf{y}} \phi[\mathbf{X}(\mathbf{y},t),t] \cdot \Big(\rho_0\, \mathbf{J}^{-1}(\mathbf{X}_t(\mathbf{y},t) - \mathbf{v}[\mathbf{X}(\mathbf{y},t),t])\Big)\,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\
&\quad + \int_{Q_T} \nabla_{\mathbf{y}} \phi[\mathbf{X}(\mathbf{y},t),t] \cdot \Big(\rho_0 \mathbf{J}^{-1} \cdot \mathbf{v}[\mathbf{X}(\mathbf{y},t),t]\Big)\,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\
&= \int_{Q_T} R(\mathbf{y},t)\, \nabla_x \phi[\mathbf{X}(\mathbf{y},t),t] \cdot \mathbf{v}[\mathbf{X}(\mathbf{y},t),t]\, \mathrm{Det}[\mathbf{J}]\,\mathrm{d}\mathbf{y}\,\mathrm{d}t \\
&= \int_{\mathcal{C}} \rho(\mathbf{x},t)\, \nabla_x \phi(\mathbf{x},t) \cdot \mathbf{v}(\mathbf{x},t)\,\mathrm{d}\mathbf{x}\,\mathrm{d}t.
\end{aligned}$$

Gathering these relations, we obtain (19). □

2.3. Orthogonal decomposition of spaces of vector-valued functions

Let us recall several known facts concerning the De Rham type decompositions of the spaces of vector-valued functions with components in $L^2(\omega_0)$. Although the following results are quite close to the well-known results in the literature (see, for example, [18,27]), for the sake of completeness we outline here the main idea of the proofs. Let us introduce the Hilbert spaces

$$\begin{aligned}
&L^2_n(\omega_0^\pm) = \{\mathbf{w} = (w_1, \dots, w_n) :\ w_i \in L^2(\omega_0^\pm)\}, \\
&(\mathbf{u}, \mathbf{w})_{L^2_n(\omega_0^\pm)} = \sum_i \int_{\omega_0^\pm} u_i v_i\,\mathrm{d}\mathbf{x}.
\end{aligned}$$

Set

$$\begin{aligned}
\mathbb{G}_\pm &= \{\nabla\phi :\ \phi \in H^1(\omega_0^\pm)\}, \\
\mathbb{J}_\pm &= \Big\{\mathbf{u} :\ \forall\, \mathbf{w} \in \mathbb{G}_\pm\ (\mathbf{u}, \mathbf{w})_{L^2_n(\omega_0^\pm)} = 0,\ \mathbf{u}\cdot\mathbf{n} = 0 \text{ on } \gamma \text{ and on } \partial\omega_0^\pm\Big\}.
\end{aligned}$$

Lemma 1. *Every vector*

$$\mathbf{f} \in L^2_n(\omega_0^+), \quad \mathbf{f}\cdot\mathbf{n}|_{\partial\omega_0^+} \in L^2(\partial\omega_0^+), \quad \mathbf{f}\cdot\mathbf{n}|_\gamma \in L^2(\gamma)$$

can be represented in the form $\mathbf{f} = \nabla\phi + \mathbf{g}$, *with* $\nabla\phi \in \mathbb{G}_+$ *and* $\mathbf{g} \in \mathbb{J}_+$.

Proof. Let ϕ be the weak solution of the Neumann problem for the Poisson equation

$$\begin{cases} \mathrm{div}\,(\nabla\phi - \mathbf{f}) = 0 & \text{in} \quad \omega_0^+, \\ (\nabla\phi - \mathbf{f})\cdot\mathbf{n} = 0 & \text{on} \quad \gamma \cup \partial\omega_0^+. \end{cases} \tag{26}$$

This problem has a solution $\phi \in H^1(\omega_0^+)$ understood as follows:

$$\forall\, \eta \in H^1(\omega_0^+) \qquad \int_{\omega_0^+} \nabla\eta \cdot (\nabla\phi - \mathbf{f})\,\mathrm{d}\mathbf{y} = 0. \tag{27}$$

Set $\mathbf{g} = \mathbf{f} - \nabla \phi$. Then $\mathbf{g} \in \mathbb{J}_+$: for every $\mathbf{s} \in \mathbb{G}_+$ there exists $\eta \in H^1(\omega_0^+)$ such that $\mathbf{s} = \nabla \eta$, and by virtue of the definition of $\mathbf{g}$

$$(\mathbf{g},\, \mathbf{s})_{L^2_n(\omega_0^+)} = \int_{\omega_0^+} \nabla \eta \cdot (\mathbf{f} - \nabla \phi)\, \mathrm{d}\mathbf{y} = 0.$$

□

Lemma 2. *Let A be a strictly positive defined symmetric matrix with bounded entries, and let $\mathbf{v}$ satisfy the conditions*

$$\mathbf{v} \in L^2_n(\omega_0^+), \quad \mathbf{v}\cdot\mathbf{n}|_\gamma \in L^2(\gamma), \qquad \mathbf{v}\cdot\mathbf{n}|_{\partial\omega_0^+} \in L^2(\partial\omega_0^+).$$

Then there exists a function $f \in H^1(\omega_0^+)$ such that $\mathbf{v} - A\,\nabla f \in \mathbb{J}_+$.

Proof. Let $f \in H^1(\omega_0^+)$ be a weak solution of the problem with the co-normal derivative

$$\begin{cases} \operatorname{div}(\mathbf{A}\,\nabla f - \mathbf{v}) = 0 & \text{in} \quad \omega_0^+, \\ (\mathbf{A}\,\nabla f - \mathbf{v})\cdot\mathbf{n} = 0 & \text{on} \quad \gamma \cup \partial\omega_0^+. \end{cases}$$

The conclusion follows by setting $\mathbf{g} = \mathbf{v} - \mathbf{A}\,\nabla f$. □

Using the above lemmas we may reduce problem (24) to a single scalar equation for the potential U. It follows from Lemma 2 and the proof of Theorem 5 that instead of solving the complete system (24) it is sufficient to claim that the trajectory equations are fulfilled in the following sense:

$$\mathbf{J}\,(\nabla U_t - \mathbf{v}[\mathbf{y} + \nabla U, t]) \in \mathbb{J}_\pm.$$

Lemma 3. *Let $\mathbf{J}\nabla U_t,\ \mathbf{J}\mathbf{v}[\mathbf{y} + \nabla U, t] \in (W^1_2(Q^\pm_T))^n$ and*

$$\begin{cases} \operatorname{div}(\mathbf{J}\ (\nabla U_t - \mathbf{v}[\mathbf{y} + \nabla U, t), t])) = 0 \ \ \textit{for almost every } \mathbf{y} \in \omega_0^\pm,\ t > 0, \\ \\ U = 0 \textit{ on } \partial\omega_0^\pm. \end{cases} \tag{28}$$

Then $\mathbf{w} = \mathbf{J}\,(\nabla U_t - \mathbf{v}[\mathbf{y} + \nabla U, t]) \in \mathbb{J}_\pm$.

Proof. Let us multiply Eq. (28) by an arbitrary test-function $\eta \in H^1(\omega_0^+)$ and integrate over ω_0^+. Then

$$0 = \int_{\omega_0^+} \eta \operatorname{div}\mathbf{w}\, \mathrm{d}\mathbf{y} = \int_{\partial\omega_0^+} \eta\, \mathbf{w}\cdot\mathbf{n}\, \mathrm{d}S + \int_\gamma \eta\, \mathbf{w}\cdot\mathbf{n}\, \mathrm{d}S - \int_{\omega_0^+} \nabla\eta\cdot\mathbf{w}\, \mathrm{d}\mathbf{y}$$

and the assertion follows since the test-function is arbitrary. □

Now we may specify the assertion of Theorem 5 in the following way.

Theorem 6. *Let $\mathbf{v}(x, t)$ be a given vector-field and let $U(\mathbf{y}, t)$ be a function satisfying the conditions*

1. *the mapping* $\mathbf{y} \mapsto \mathbf{x} = \mathbf{y} + \nabla U(\mathbf{y}, t)$ *is a bijection from* $\overline{\omega_0}$ *to* $\overline{\omega(t)}$,

$$\lambda_1 < \text{Det}\,[\mathbf{I} + \mathbf{H}(U)] < \lambda_2 \quad in \quad Q_T$$

for some positive constants λ_1, λ_2,

2. $U = 0$ *on the parabolic boundary of* Q_T, $|D^2_{ij} U|$ *are bounded in* Q_T,

$$\mathbf{v}(\mathbf{y} + \nabla U, t),\ \nabla U_t \in \left(W^1_q(Q^{\pm}_T)\right)^n \quad with\ some \quad q \in [2, \infty),$$

3. U *satisfies* (28), $\text{Det}[\mathbf{I} + \mathbf{H}(U)] = 1$ *on* $\gamma \times [0, T]$.

Then the function $\rho(\mathbf{x}, t)$ *and the set* $\mathcal{C}$, *defined by the formulas*

$$\begin{cases} \rho(x, t) = \dfrac{\rho_0}{\text{Det}[\mathbf{I} + \mathbf{H}(U)]} \quad in \quad \mathcal{C}, \\ \mathcal{C} = \{(\mathbf{x}, t) :\ \mathbf{x} \equiv \mathbf{y} + \nabla U(\mathbf{y}, t)\ for\ (\mathbf{y}, t) \in Q_T\}, \end{cases} \tag{29}$$

give a continuous weak solution of problem (18) *in the sense of Definition* 3.

2.4. Lagrangian coordinates associated with a solution of a parabolic equation not in divergent form

In the previous subsection we formulated the Euler and Lagrange descriptions of motion of a fluid volume. We derived conditions on the solution of the problem in the Lagrangian setting which are sufficient to generate a solution of the problem in the Euler formulation. In so doing, we assumed that the velocity field $\mathbf{v}$ was prescribed. Let us now choose the velocity field $\mathbf{v}$ in such a way that the solution of problem (24) generates a solution of the free boundary problem (10). It follows from the proof of Theorem 5 that it suffices to set

$$u(\mathbf{x}, t) = R(\mathbf{y}, t), \quad \mathbf{x} = \mathbf{X}(\mathbf{y}, t) \equiv \mathbf{y} + \nabla U(\mathbf{y}, t), \quad (\mathbf{y}, t) \in Q_T,$$

and to choose $\mathbf{v}(\mathbf{x}, t)$ from the condition

$$\int_{\mathcal{C}} u\mathbf{v} \cdot \nabla\phi \, \mathrm{d}\mathbf{x}\, \mathrm{d}t = \int_{\mathcal{C}} [-u \nabla \ln\, u \cdot \nabla\phi + a\, h_u \phi\,] \, \mathrm{d}\mathbf{x}\, \mathrm{d}t,$$

where ϕ is an arbitrary test-function satisfying the conditions of Definition 3. This condition suggests introducing a new unknown function, $p(\mathbf{x}, t)$, which will play the role of an *artificial pressure*, and assuming that the velocity of the fluid obeys the following generalization of the Darcy law:

$$\mathbf{v} = -\nabla \ln\, u + \nabla\, p.$$

The function $p(\mathbf{x}, t)$ is thus defined as follows:

$$\forall\, \phi \in H^1(\omega(t)) \quad \int_{\omega(t)} [u\, \nabla p \cdot \nabla\, \phi - a\, h_u \phi]\, \mathrm{d}\mathbf{x} = 0,$$

where h_u is the function associated with u by conditions (2). The function p depends on t as a parameter. Since we claim that the velocity $\mathbf{v}$ is continuous across the surface Γ_μ, the artificial pressure p has to be a solution of the linear elliptic problem

$$\begin{cases} \operatorname{div}\,(u(\cdot,t)\nabla p(\cdot,t)) = -a\,\chi_{\{u>\mu\}} & \text{in } \omega^+(t)\cup\omega^-(t), \\ [\nabla\, p\cdot\mathbf{n}]_{\Gamma_\mu(t)} = [\nabla\,\ln\, u\cdot\mathbf{n}]_{\Gamma_\mu(t)}, \end{cases} \tag{30}$$

where $\chi_{\{u>\mu\}}$ is the characteristic function of the set $\{u>\mu\}$ and the symbol $[\,\cdot\,]_{\Gamma_\mu(t)}$ denotes the jump across $\Gamma_\mu(t)$. Problem (30) is completed by posing boundary conditions for p on $\partial\omega^\pm(t)$. These conditions follow from the dynamical boundary conditions: we claim that the trajectory Eq. (20) is fulfilled on the boundaries $\partial\omega^\pm(t)$. This means that

$$\nabla p\cdot\mathbf{n} = (\nabla\ln u + \mathbf{v})\cdot\mathbf{n} \quad \text{on} \quad \partial\omega^\pm(t).$$

Formally passing to the Lagrangian variables $(\mathbf{y},t)$ and excluding from the appearing system the function $R(\mathbf{y},t)$, we arrive at the following problem: to find the functions $U(\mathbf{y},t)$ (the potential) and $P(\mathbf{y},t) = p(\mathbf{x},t)$ (the artificial pressure) which satisfy the system of nonlinear equations

$$\begin{cases} \operatorname{div}\,\left(\mathbf{J}\nabla U_t + \nabla\left(\ln\,(\rho_0\,\mathrm{Det}[\mathbf{J}^{-1}]) - P\right)\right) = 0 \quad \text{in} \quad Q_T^\pm, \\ \operatorname{div}\,\left(\rho_0\,(\mathbf{J}^{-1})^2\nabla P\right) = -a\,\mathrm{Det}[\mathbf{J}]\,\chi_{\omega_0^+} \end{cases} \tag{31}$$

and the initial and boundary conditions

$$\begin{cases} U=0 \quad \text{on the parabolic boundary of } Q_T, \\ \nabla P\cdot\mathbf{n} = \left(\mathbf{J}\cdot\nabla U_t + \nabla\left(\rho_0\mathrm{Det}[\mathbf{J}^{-1}]\right)\right)\cdot\mathbf{n} \quad \text{on} \quad \partial\omega_0^\pm\times(0,T], \\ \left[\mathbf{J}^{-1}\nabla(P - \ln(\rho_0\,\mathrm{Det}[\mathbf{J}^{-1}])\cdot\mathbf{n}\right]\Big|_{\gamma\times[0,T]} = 0, \\ \mathrm{Det}[\mathbf{J}] = 1 \quad \text{on} \quad \gamma\times[0,T]. \end{cases} \tag{32}$$

In what follows, problem (31), (32) will be termed *Problem* (PL). The solution of this problem is understood in the sense of Definition 4. The following assertion is an immediate byproduct of Theorem 6.

Theorem 7. *Let* (U,P) *be a solution of Problem* (PL) *such that the conditions of Theorem* 6 *are fulfilled with*

$$\mathbf{v}[\mathbf{y}+\nabla U,t] = \mathbf{J}\nabla\left(\ln\,(\rho_0\,\mathrm{Det}[\mathbf{J}^{-1}]) - P\right). \tag{33}$$

Then the pair $(u,\mathcal{C})$ *defined by formulas* (29) *is a continuous weak solution of problem* (10).

3. Auxiliary nonlinear problem

In this section we consider the auxiliary problem of finding a function U under the assumption that the second unknown, P, is given. This problem splits into two similar problems posed on the cylinders Q_T^+ and Q_T^-. We limit ourselves by considering the problem in Q_T^+, the problem in Q_T^- is studied in the same way.

3.1. Formulation of the problem

Let us fix a function P and consider the auxiliary problem of defining the function U from the conditions

$$\begin{cases} \mathcal{H}_1(U) \equiv \operatorname{div}\left[\mathbf{J}\,\nabla U_t + \nabla\left(\ln\left(\rho_0 \operatorname{Det}\left[\mathbf{J}^{-1}\right]\right) - P\right)\right] = 0 \quad \text{in} \quad Q_T^+, \\ \mathcal{H}_2(U) \equiv \operatorname{Det}[\mathbf{J}] - 1 = 0 \quad \text{on} \quad \gamma \times [0, T], \\ U = 0 \text{ on the lateral boundary of } Q_T^+, \quad U(\mathbf{y}, 0) = 0. \end{cases} \tag{34}$$

This problem can be formulated as the functional equation

$$\mathcal{H}(U) \equiv \{\mathcal{H}_1(U), \mathcal{H}_2(U)\} = \mathbf{0}.$$

The existence of a unique solution of problem (34) will be proved by means of an abstract version of the modified Newton method [16, Chapter XVIII].

Theorem 8. *Let $\mathcal{X}$, $\mathcal{Y}$ be Banach spaces and assume that the following conditions hold:*

1. *the operator $\mathcal{H} : \mathcal{X} \mapsto \mathcal{Y}$ admits a strong (Frechét) differential $\mathcal{H}'(\cdot)$ in a ball $B_r(0) \subset \mathcal{X}$ of radius $r > 0$,*
2. *the operator $\mathcal{H}'(V) : \mathcal{X} \mapsto \mathcal{Y}$ is Lipschitz-continuous in $B_r(0)$,*

$$\|\mathcal{H}'(U_1) - \mathcal{H}'(U_2)\| \leqq L\,\|U_1 - U_2\|, \quad L = \text{const},$$

3. *there exists the inverse operator $\left[\mathcal{H}'(0)\right]^{-1}$ and*

$$\left\|\left[\mathcal{H}'(0)\right]^{-1}\right\| = M, \quad \left\|\left[\mathcal{H}'(0)\right]^{-1}\langle\mathcal{H}(0)\rangle\right\| = \Lambda.$$

Then, if $\lambda = M\Lambda L < 1/4$, the equation $\mathcal{H}(U) = 0$ has a unique solution U^* in the ball $B_{\Lambda t_0}(0)$ where t_0 is the least root of the equation $\lambda t^2 - t + 1 = 0$. Moreover, the solution U^* is obtained as the limit of the sequence

$$U_{n+1} = U_n - [\mathcal{H}'(0)]^{-1}\langle\mathcal{H}(U_n)\rangle, \quad U_0 = 0. \tag{35}$$

It is known that if an operator is weakly differentiable (in the sense of Gateaux), and its Gateaux differential is Lipschitz-continuous, then the operator is strongly differentiable and its weak and strong differentials coincide [16, Chapter XVIII]. Due to condition (2) of Theorem 8, we may take for $\mathcal{H}(0)$ the weak differential of $\mathcal{H}$ at the initial state $U_0 = 0$, which is easy to obtain by means of formal linearization. The proof of existence of a solution to the nonlinear problem (34) reduces then to the detailed study of the linear problem $\mathcal{H}'(0)\langle U\rangle = (F, \Phi)$.

Let us fix $q > n + 2$ and introduce the Banach spaces

$$\begin{aligned} \mathcal{Z}^+ &= \left\{U : \begin{array}{l} U, tU_t \in W_q^4(Q_T^+),\ U_t,\ (tU_t)_t \in W_q^2(Q_T^+), \\ U = 0 \text{ on } \partial\omega_0^+ \times (0, T],\ U(\mathbf{y}, 0) = 0 \text{ in } \omega_0^+ \end{array}\right\}, \\ \mathcal{Y}^+ &= \left\{f : \ f,\ tf_t \in W_q^2(Q_T^+)\right\}, \\ \mathcal{X}^+ &= \{\phi : \ \phi,\ t\phi_t \in W_q^{2,1}(Q_T^+),\ \phi(\mathbf{y}, 0) = 0 \text{ in } \omega_0^+\} \end{aligned}$$

with the norms

$$\|U\|_{\mathcal{Z}^+} = \|U\|^{(4)}_{q,Q_T^+} + \|U_t\|^{(2)}_{q,Q_T^+} + \|tU_t\|^{(4)}_{q,Q_T^+} + \|tU_{tt}\|^{(2)}_{q,Q_T^+},$$
$$\|f\|_{\mathcal{Y}^+} = \|f\|^{(2)}_{q,Q_T^+} + \|tf_t\|^{(2)}_{q,Q_T^+},$$
$$\|\phi\|_{\mathcal{X}^+} = \|\phi\|_{W_q^{2,1}(Q_T^+)} + \|t\phi_t\|_{W_q^{2,1}(Q_T^+)}.$$

3.2. The linear problem

To calculate the Gateaux derivative of $\mathcal{H}$ we use its definition as

$$\mathcal{H}_i'(0)\langle U\rangle = \left.\frac{\mathrm{d}\mathcal{H}_i(\epsilon U)}{\mathrm{d}\epsilon}\right|_{\epsilon=0},$$

where ϵ is a small parameter and

$$\mathcal{H}_1(\epsilon U) = \operatorname{div}\Big(\epsilon\,(\mathbf{I}+\epsilon\,\mathbf{H}(U))\nabla U_t + \nabla\Big(\ln\,(\rho_0\,|\mathbf{I}+\epsilon\mathbf{H}(U)|^{-1})\rangle - P\Big)\Big),$$
$$\mathcal{H}_2(\epsilon U) = \mathrm{Det}[\mathbf{I}+\epsilon\,\mathbf{H}(U)] - 1.$$

Obviously,

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\operatorname{div}\left[\epsilon\,(\mathbf{I}+\epsilon\,\mathbf{H}(U))\nabla U_t]\rangle\right|_{\epsilon=0} = \Delta U_t.$$

According to Newton's formulas, for every matrix $\mathbf{B}$ and $\mu = \mathrm{const}$

$$\mathrm{Det}\,[\mu\,\mathbf{I}-\mathbf{B}] = \sum_{k=0}^{n}(-1)^k\alpha_k\,\mu^{n-k}, \tag{36}$$

where $\alpha_0 = 1$, $k\,\alpha_k = \sum_{i=1}^{k}\alpha_{k-i}\,\mathrm{trace}\,\big(\mathbf{B}^i\big)$ for $1 \leqq k \leqq n$. Then $\mathcal{H}_2'(0)\langle U\rangle = \Delta\,U$ and

$$\ln\,(\rho_0\,\mathrm{Det}[\mathbf{I}+\epsilon\mathbf{H}(U)]^{-1}) = \ln\,\rho_0 + \ln\Big(1-\epsilon\,\Delta U + \mathcal{O}\left(\epsilon^2\right)\Big).$$

Gathering these formulas, we find that

$$\mathcal{H}_1'(0)\langle U\rangle = \Delta(U_t - \Delta U), \qquad \mathcal{H}_2'(0)\langle U\rangle = \Delta U.$$

The linear problem $\mathcal{H}'(0)\langle U\rangle = (\Delta f, \phi)$ now reads as follows: it is requested to find a function $U \in \mathcal{Z}^+$ such that

$$\begin{cases} \Delta(U_t - \Delta U) = \Delta f & \text{in} \quad Q_T^+, \\ (\Delta\,U - \phi)|_{\gamma\times[0,T]} = 0. \end{cases} \tag{37}$$

Theorem 9. *For every $f \in \mathcal{Y}^+$, $\phi \in \mathcal{X}^+$ problem (37) has at least one solution $U \in \mathcal{Z}^+$ satisfying the estimate*

$$\|U\|_{\mathcal{Z}^+} \leqq C\,\left(\|f\|_{\mathcal{Y}^+} + \|\phi\|_{\mathcal{X}^+}\right), \quad C \equiv C\,(n,q). \tag{38}$$

Proof. Set $W = \Delta U$ and choose W according to the conditions

$$\begin{cases} W_t - \Delta W = \Delta f \in L^q(Q_T^+), \\ W - \phi = 0 \quad \text{on} \quad \gamma \times [0,T], \quad W = 0 \quad \text{on} \quad \partial\omega_0^+ \times [0,T], \\ W(\mathbf{y},0) = 0 \quad \text{in} \quad \omega_0^+. \end{cases} \tag{39}$$

The function $\phi \in W_q^{2,1}(Q_T^+)$ with $q > n+2$ is Hölder-continuous in $\overline{Q}_T^+$ and satisfies the zero-order compatibility condition on the hypersurface γ as $t = 0$. Since $\partial\omega_0^+, \gamma \in C^2$, then for every $\Delta f \in L^q(Q_T^+)$ problem (39) has a unique solution $W \in W_q^{2,1}(Q_T^+)$ which satisfies the estimate

$$\|\Delta U\|_{W_q^{2,1}(Q_T^+)} = \|W\|_{W_q^{2,1}(Q_T^+)} \leqq C\left(\|\phi\|_{W_q^{2,1}(Q_T^+)} + \|f\|_{q,Q_T^+}^{(2)}\right) \tag{40}$$

(see [17, Chapter 4, Section 9]). The function U solves the problem

$$\begin{cases} U_t - \Delta U = f + G \quad \text{in} \quad Q_T^+, \\ \Delta U - \phi = 0 \quad \text{on} \quad \gamma \times [0,T], \quad U(\mathbf{y},0) = 0, \\ \Delta U = 0 \quad \text{on} \quad \partial\omega_0^+ \times [0,T], \end{cases} \tag{41}$$

where $G(\cdot,t)$ is an arbitrary harmonic in the ω_0^+ function, depending on t as a parameter. Let us take for G the solution of the problem

$$\begin{cases} \Delta G(\cdot,t) = 0 \quad \text{in} \quad \omega_0^+, \\ G(\cdot,t) + f(\cdot,t) = 0 \quad \text{on} \quad \partial\omega_0^+, \\ G(\cdot,t) + f(\cdot,t) + \phi(\cdot,t) = 0 \quad \text{on} \quad \gamma. \end{cases}$$

For almost every $t \in [0,T]$ this problem has a unique solution $G(\cdot,t)$ such that

$$\sum_{0 \leqq |\gamma| \leqq 2} \|D_y^\gamma G(\cdot,t)\|_{q,\omega_0^+} \leqq C \sum_{0 \leqq |\gamma| \leqq 2} \left(\|D_y^\gamma \phi(\cdot,t)\|_{q,\omega_0^+} + \|D_y^\gamma f(\cdot,t)\|_{q,\omega_0^+}\right).$$

The functions $f + G$ and $W = \Delta U$ have zero traces on $\partial\omega_0^+ \times [0,T]$, $W = \phi$ and $f + G = \phi$ on $\gamma \times [0,T]$. By virtue of the equation for U and the boundary conditions for G on γ we have that $U_t = 0$ on the lateral boundaries of Q_T^+. It follows that $U = U(\mathbf{y},0) = 0$ on the lateral boundaries of Q_T^+, which means that the solution W of problem (39) defines a solution $U(\mathbf{y},t)$ of problem (37) up to an arbitrary harmonic function $h(\mathbf{y})$ taken for the initial datum. Letting $h \equiv 0$, we obtain a function U satisfying (37).

Since $f \in L^q(Q_T^+)$, then $tf \to 0$ as $t \to 0$. By virtue of (37), the function $V = t\Delta U_t$ satisfies the conditions

$$\begin{cases} \Delta(V_t - \Delta V) = \Delta(f + tf_t + \Delta U) \in L^q(Q_T^+) \quad \text{in} \quad Q_T^+, \\ (\Delta V - t\phi_t)|_{\gamma \times [0,T]} = 0, \quad V = 0 \quad \text{on} \quad \partial\omega_0^+ \times [0,T], \\ V(\mathbf{y},0) = 0. \end{cases}$$

Repeating the above arguments, we find that this problem has at least one solution $V \in W_q^{2,1}(Q_T^+)$ satisfying the estimate

$$\|t\Delta U_t\|_{W_q^{2,1}(Q_T^+)} = \|V\|_{W_q^{2,1}(Q_T^+)} \leqq C\left(\|\phi\|_{\mathcal{X}^+} + \|f\|_{\mathcal{Y}^+}\right). \tag{42}$$

Estimate (38) follows now from (40) and (42). □

Corollary 1. *$M = \|\mathcal{H}^{-1}(0)\| \leqq C$ with the constant C from* (38).

Corollary 2.

$$\Lambda = \|\mathcal{H}^{-1}(0)\langle\mathcal{H}(0)\rangle\| \leqq C \left(T^{1/q} \|\Delta \ln \rho_0\|_{L^q(\omega_0^+)} + \|P\|_{\mathcal{Y}^+} \right)$$

with the constant C from (38).

Proof. The estimate follows from (38) with $\Delta f = \mathcal{H}_1(0) = \Delta(\ln \rho_0 - P)$, $\phi = 0$.
□

3.3. Existence of solution

To apply Theorem 8 we have to check Lipschitz continuity of the Gateaux derivative of the operator $\mathcal{H}$ defined by

$$\mathcal{H}'(V)\langle U\rangle = \frac{d}{d\epsilon}\mathcal{H}(V + \epsilon U)\Big|_{\epsilon=0},$$

and the relations

$$\forall U \in \mathcal{Z}^+ \quad \mathcal{H}_1(U) \in \mathcal{Y}^+, \quad \mathcal{H}_2(U) \in \mathcal{X}^+. \tag{43}$$

By the definition

$$\begin{aligned}\mathcal{H}_1(V + \epsilon U) = \operatorname{div}\left((\mathbf{I} + \mathbf{H}(V) + \epsilon\mathbf{H}(U))\nabla(V_t + \epsilon U_t)\right) \\ +\Delta\left[\ln\left(\rho_0 \operatorname{Det}[(\mathbf{I} + \mathbf{H}(V) + \epsilon\mathbf{H}(U))^{-1}]\right) - P\right].\end{aligned}$$

Using the easily verified formula

$$\begin{aligned}(\mathbf{I} + \mathbf{H}(V) + \epsilon\mathbf{H}(U))^{-1} &= (\mathbf{I} + \epsilon(\mathbf{I} + \mathbf{H}(V))^{-1}\mathbf{H}(U))^{-1}(\mathbf{I} + \mathbf{H}(V))^{-1} \\ &= (\mathbf{I} - \epsilon(\mathbf{I} + \mathbf{H}(V))^{-1}\mathbf{H}(U) + \mathcal{O}(\epsilon^2))(\mathbf{I} + \mathbf{H}(V))^{-1},\end{aligned}$$

we find that

$$\operatorname{Det}[(\mathbf{I} + \mathbf{H}(V) + \epsilon\mathbf{H}(U))^{-1}] = 1 - \epsilon \operatorname{trace}\left[(\mathbf{I} + \mathbf{H}(V))^{-1}\mathbf{H}(U)\right] + \mathcal{O}\left(\epsilon^2\right).$$

Then

$$\begin{aligned}\mathcal{H}_1'(V)\langle U\rangle &= \operatorname{div}\left(\mathbf{H}(U)\,\nabla V_t + (\mathbf{I} + \mathbf{H}(V))\,\nabla U_t\right. \\ &\quad \left. - \nabla\left(\operatorname{trace}\left[(\mathbf{I} + \mathbf{H}(V))^{-1}\mathbf{H}(U)\right]\right)\right), \\ \mathcal{H}_2'(V)\langle U\rangle &= \operatorname{trace}\left[(\mathbf{I} + \mathbf{H}(V))^{-1}\mathbf{H}(U)\right].\end{aligned}$$

The elements of the inverse matrix can be expressed through its algebraic adjoints and the determinant, which are polynomials of nth order. Further, the embedding theorems yield that since $D_{\mathbf{y}}^2 U \in W_q^{2,1}(Q_T^+)$, then for $q > n + 2$

$$\forall U \in \mathcal{Z}^+ \quad \sum_{|\gamma|=2,3} \left(\langle D_{\mathbf{y}}^\gamma U\rangle_{Q_T^+}^{(\alpha)} + \langle t D_{\mathbf{y}}^\gamma U_t\rangle_{Q_T^+}^{(\alpha)} \right) \leqq C\|U\|_{\mathcal{Z}^+} \tag{44}$$

with some $\alpha \in (0,1)$ (see, for example [17, Chapter 2, Lemma 3.3]). Since $U(\mathbf{y},0) \equiv 0$, it follows that

$$\sum_{|\gamma|=2,3} \left(\sup_{Q_T^+} |D_{\mathbf{y}}^{\gamma} U| + \sup_{Q_T^+} |t D_{\mathbf{y}}^{\gamma} U_t| \right) \leqq C T^{\alpha/2} \|U\|_{\mathcal{Z}^+}. \tag{45}$$

It is now straightforward to check that for every $V_1, V_2 \in \mathcal{Z}^+$ with $\|V_i\|_{\mathcal{Z}^+} \leqq 1$

$$\left\| \left(\mathcal{H}_i'(V_1) - \mathcal{H}_i'(V_2)\right) \langle U \rangle \right\| \leqq L \|V_1 - V_2\|_{\mathcal{Z}^+} \|U\|_{\mathcal{Z}^+} \tag{46}$$

with $L = L(n, \omega_0^+, n, T) \to 0$ as $T \to 0$. Relations (43) follow by the same arguments.

The next theorem is an immediate byproduct of Theorem 8.

Theorem 10. *Let P, $tP_t \in W_q^2(Q_T^+)$ with $q > n+2$. Then one may choose $T_* \equiv T_*(n, q, \omega_0) \in (0,1)$ so small that $\lambda = M\, L\, \Lambda < 1/4$ with the constants Λ, M and L from Corollaries 1, 2 and estimate (46), and problem (34) has a unique solution*

$$U \in \mathcal{B}_r(0) = \{W : \|W\|_{\mathcal{Z}^+} < r\}, \quad r = \frac{\Lambda}{2\lambda}\left(1 - \sqrt{1-4\lambda}\right) < 2\,\Lambda. \tag{47}$$

The same assertion is true for problem (34) in the cylinder Q_T^-.

4. Auxiliary linear elliptic problem

In this section we consider the problem of finding a function P satisfying the following conditions: for every $t \in [0,T]$

$$\begin{cases} \mathcal{M}P := \operatorname{div}\left(\rho_0 (\mathbf{J}^{-1})^2\, \nabla P\right) = -a \operatorname{Det}[\mathbf{J}]\, \chi_{\omega_0^+} \quad \text{in} \quad \omega_0^{\pm}, \\ \left[(\rho_0 (\mathbf{J}^{-1})^2 \nabla P - \Psi) \cdot \mathbf{n} \right]\Big|_{\gamma} = 0, \\ P = 0 \quad \text{on} \quad \partial\omega_0^{\pm}, \end{cases} \tag{48}$$

where $U \in \mathcal{Z}^{\pm}$ is a given function, $\Psi = \rho_0 (\mathbf{J}^{-1})^2 \nabla \ln(\rho_0 \operatorname{Det}[\mathbf{J}^{-1}])$ and Ψ, $t\Psi_t \in W_q^1(Q_T^+)$.

Theorem 11. *Let $U \in \mathcal{Z}^{\pm}$. Then for almost every $t \in (0,T)$, problem (48) has a solution $P(\cdot,t) \in W_q^2(\omega_0^{\pm})$, and this solution satisfies the estimate*

$$\|P\|_{\mathcal{Y}^{\pm}} \leqq C\, \|U\|_{\mathcal{Z}^{\pm}} \tag{49}$$

with an absolute constant C.

Proof. Let us take for P in ω_0^+ the solution of the Dirichlet problem for the linear uniformly elliptic equation

$$\begin{cases} \mathcal{M}\, P^+ = -a\,\mathrm{Det}\,[\mathbf{J}] \quad \text{in } \omega_0^+, \\ P^+ = 0 \text{ on } \gamma \text{ and } \partial\omega_0^+, \end{cases}$$

and then continue P^+ to ω_0^- by the solution of the problem

$$\begin{cases} \mathcal{M}\, P^- = 0 \quad \text{in } \omega_0^-, \\ P^- = 0 \text{ on } \partial\omega_0^-, \\ \rho_0(\mathbf{J}^{-1})^2\nabla P^- \cdot \mathbf{n} = \rho_0(\mathbf{J}^{-1})^2\nabla P^+ \cdot \mathbf{n} - [\Psi \cdot \mathbf{n}]_\gamma. \end{cases}$$

These problems have solutions which satisfy the estimates (see, for example, Chapter 3, Sections 5, 6, 15 of [19]))

$$\left\| P^\pm(\cdot,t)\right\|^{(2)}_{q,\omega_0^\pm} \leqq C\left(\|\mathrm{Det}[\mathbf{J}]\|^{(2)}_{q,\omega_0^\pm} + \|\Psi\|^{(1)}_{q,\omega_0^\pm}\right) \leqq C'\,\|U(\cdot,t)\|^{(4)}_{q,\omega_0^\pm}.$$

To derive estimate (49) we consider similar problems for the functions $t\,P_t^\pm$. □

Lemma 4. *Under the conditions of Theorem 11*

$$|P(\cdot,t) - P_0|^{(1+\sigma)}_{\omega_0^\pm} \leqq C\,\|P(\cdot,t) - P_0\|_{W^2_q(\omega_0^\pm)} \leqq C'\,t^{\alpha/2}\|U\|_{\mathcal{Z}^\pm}, \quad 0 < \sigma < 1 - \frac{n}{q},$$

where $\alpha \in (0,1)$ *is the exponent from* (45) *and* P_0 *is the solution of the problem*

$$\mathrm{div}\,(\rho_0\,\nabla\,P_0) = -a\,\chi_{\omega_0^+} \quad in \quad \omega(0), \quad P_0 = 0 \quad on \quad \partial\omega_0^\pm, \tag{50}$$

Proof. Problem (50) has a unique solution $P_0 \in W^2_q(\omega(0)) \cap C^{1+\sigma}(\omega(0))$ with $0 < \sigma < 1 - n/q$. Since $\rho_0 \in C^{2+\beta}(\omega(0))$, this solution automatically satisfies the jump condition $[\rho_0\,\nabla(P_0 - \ln u_0)]_\gamma = 0$. Problem (48) is linear and its solution continuously depends on the data. Let us fix $t \in (0, T^*]$ and consider the function $P - P_0$ which satisfies the conditions

$$\begin{cases} \mathrm{div}\,(\rho_0\,\nabla\,(P - P_0)) = F \quad \text{in} \quad \omega_0^\pm, \\ [\rho_0\,\nabla(P - P_0)\cdot\mathbf{n}]_\gamma = \sigma, \quad P - P_0 = 0 \quad \text{on} \quad \partial\omega_0^\pm \end{cases}$$

with

$$\begin{aligned} F &= a\;(1 - \mathrm{Det}[\mathbf{J}])\,\chi_{\omega_0^+} \in L^q(\omega_0^+ \cup \omega_0^-), \\ \sigma &= [\rho_0((\mathbf{J}^{-1})^2 - \mathbf{I})\nabla P\cdot\mathbf{n}]_\gamma + [\rho_0\,(\nabla\ln\rho_0 - \Psi)\cdot\mathbf{n}]_\gamma \\ &\quad + [\rho_0(\mathbf{I} - (\mathbf{J}^{-1})^2)\cdot\Psi\cdot\mathbf{n}]_\gamma. \end{aligned}$$

Applying (45) and Theorem 11 we find that

$$\|P(\cdot,t) - P_0\|_{W^2_q(\omega_0^\pm)} \leqq C\,t^{\alpha/2}\left(\|P^\pm\|_{\mathcal{Y}^\pm} + \|U\|_{\mathcal{Z}^\pm}\right) \leqq C'\,t^\lambda\,\|U\|_{\mathcal{Z}^\pm}$$

and the assertion follows after applying the embedding theorem. □

Corollary 3. *By virtue of* (44)*, it follows by the same arguments that for every* $t_1,\ t_2 \in [0, T^*]$

$$|P(\cdot,t_1) - P(\cdot,t_2)|^{(1+\sigma)}_{\omega_0^\pm} \leqq C\,\|P(\cdot,t_1) - P(\cdot,t_2)\|_{W^2_q(\omega_0^\pm)} \leqq C'\,|t_2 - t_1|^{\alpha/2}\|U\|_{\mathcal{Z}^\pm}.$$

5. Existence of solutions to the problems in Lagrangian and Euler formulations

Theorem 12. *There exists T^* such that for every $T \in (0, T^*)$ problem (PL) has a solution $(U, P) \in \mathcal{Z}^{\pm} \times \mathcal{Y}^{\pm}$, which generates a solution of problem* (10).

Proof. Let us consider the sequences

$$\{U_k\} \in \mathcal{Z}^{+} \cap \mathcal{Z}^{-}, \quad \{P_k\} \in W^2_q(Q^+_T) \cap W^2_q(Q^-_T)$$

defined iteratively: $U_0 = 0$, for every $k \geqq 1$ U_k is a solution of problem (34) with $P = P_{k-1}$, P_k is a solution of problem (48) with $U = U_k$. By Theorems 10 and 11 for all sufficiently small T

$$\|P_k\|_{\mathcal{Y}^{\pm}} + \|U_k\|_{\mathcal{Z}^{\pm}} \leqq \lambda$$

with some absolute constant λ. This estimate, together with (45), means that the sequences $\{U_k\}$ and $\{P_k\}$ contain subsequences (which we assume to coincide with the whole of these sequences) such that

$$\begin{array}{l} U_k \to U \quad \text{as} \quad k \to \infty \quad \text{weakly in} \quad \mathcal{Z}^{\pm}, \\ D^2_{ij} U_k \to D^2_{ij} U \quad \text{as} \quad k \to \infty \quad \text{in} \quad C^{\alpha', \alpha'/2}(\overline{Q^{\pm}_T}), \\ t D^2_{ij} U_{t,k} \to t D^2_{ij} U_t \quad \text{as} \quad k \to \infty \quad \text{in} \quad C^{\alpha', \alpha'/2}(\overline{Q^{\pm}_T}) \quad \text{with some} \quad \alpha' < \alpha, \\ P_k \to P, \underline{t} P_{k,t} \to t P_t \quad \text{as} \quad k \to \infty \quad \text{weakly in} \quad W^2_q(Q^{\pm}_T). \end{array} \tag{51}$$

By the method of construction, each of the pairs (U_k, P_{k-1}) satisfies identity (25) with

$$\mathbf{v}_k = \mathbf{J}_k \nabla \left(\ln (\rho_0 \, \mathrm{Det}[\mathbf{J}_k^{-1}]) - P_{k-1} \right),$$

which allows us to pass to the limit in (25) as $k \to \infty$. It remains to check the fulfillment of item (1) of Definition 4. According to (45) the Jacobian $|\mathbf{J}|$ is bounded away from zero and infinity in $Q^{\pm}_T$ (for small T) and the mapping $\omega^{\pm}_0 \ni \mathbf{y} \mapsto \mathbf{x} \in \omega^{\pm}(t)$ is locally invertible in a neighborhood of every interior point of $Q^{\pm}_T$. It is then sufficient to check that T can be chosen so small that the images of two arbitrary boundary points $\mathbf{y}, \mathbf{z} \in \partial\omega^{\pm}_0$ (or $\mathbf{y}, \mathbf{z} \in \gamma$), $\mathbf{y} \neq \mathbf{z}$, do not coincide on the interval $[0, T^*]$. The arguments are the same for the three possibilities. For example, fix two arbitrary points $\mathbf{y}, \mathbf{z} \in \partial\omega^{+}_0$ and denote by $\mathbf{X}(\mathbf{y}, t)$ and $\mathbf{X}(\mathbf{z}, t)$ their images at the instant t. By the definition $\mathbf{X}(\mathbf{s}, t) = \mathbf{s} + \nabla U(\mathbf{s}, t)$. Further,

$$\begin{aligned} |\mathbf{X}(\mathbf{y}, t) - \mathbf{X}(\mathbf{z}, t)| &= |\mathbf{y} - \mathbf{z} + \nabla\,(U(\mathbf{y}, t) - U(\mathbf{z}, t))| \\ &\geqq |\mathbf{y} - \mathbf{z}| - \int_{L(\mathbf{y},\mathbf{z})} \left| \frac{\mathrm{d}}{\mathrm{d}l} (\nabla U)(\mathbf{s}, t)) \right| \mathrm{d}S, \end{aligned}$$

where $L(\mathbf{y}, \mathbf{z}) \subset \omega_0$ is a curve connecting $\mathbf{y}$ and $\mathbf{z}$. For every surface $\partial\omega^{+}_0 \in C^{0,1}$ the curve $L(\mathbf{y}, \mathbf{z})$ can be chosen Lipschitz-continuous and there exist finite positive constants K_1, K_2 (depending on the geometry of $\partial\omega^{+}_0$) such that

$$K_1 |\mathbf{y} - \mathbf{z}| \leqq \int_{L(\mathbf{y},\mathbf{z})} \mathrm{d}S \leqq K_2 |\mathbf{y} - \mathbf{z}|.$$

Then

$$|\mathbf{X}(\mathbf{y},t)-\mathbf{X}(\mathbf{z},t)| \geqq |\mathbf{y}-\mathbf{z}| - \sum_{i,j=1}^{n} \sup_{\omega_0^+} |D^2_{ij}U| \int_{L(\mathbf{y},\mathbf{z})} \mathrm{d}S$$
$$\geqq |\mathbf{y}-\mathbf{z}| \left(1 - C\, K_2\, T^{\alpha/2}\right)$$

with the constant C from Theorem 12 and $\alpha \in (0,1)$ from (45). Since C is defined through the data of problem (PL), it follows that the trajectories of two arbitrary points separated at the initial instant cannot touch if T^* is chosen appropriately small. □

Remark 4. The formal introduction of Lagrangian coordinates is possible as well in the limit case $a = 0$. However, in this case (8) is the heat equation and the conditions for defining the artificial pressure $p(\mathbf{x},t)$ provide only a trivial solution: $p \equiv 0$. The third boundary condition in (32) (the continuity of the velocity across γ) transforms then into the condition $[\mathbf{J}^{-1}\nabla(\ln(\rho_0\, \mathrm{Det}[\mathbf{J}^{-1}]) \cdot \mathbf{n}]_\gamma$ and the problem for U in $Q_T^+ \cup Q_T^-$ becomes overdefined.

6. Solution of problem (3); proof of the main theorems

6.1. Proof of Theorem 1

(1) According to Theorems 7 and 12 the pair $(u, \mathcal{C})$ defined by formulas (29) is a solution of problem (10) in the sense of Definition 2.

(2)–(3)–(4) Let S be the image of the surface γ under the mapping $\mathbf{y} \mapsto \mathbf{x}$. The bijectivity of this mapping and the regularity of U yield that for every $\tau > 0$ the set $S \cap \{t > \tau\}$ is an n-dimensional simply connected hypersurface of the class $C^{1+\sigma,(1+\sigma)/2}$ with $\sigma = \sigma(n,q) \in (0,1)$. Consider the domains $\mathcal{C}^\pm$ bounded by the surfaces $\Sigma^\pm$ and S, so that $\mathcal{C} = \mathcal{C}^- \cup S \cup \mathcal{C}^+$. Applying the chain rule and the formulas of interpolation of traces in Sobolev spaces, we find that the function $u(x,t)$ defined by formulas (29) possesses the following regularity:

$$u,\ tu_t \in C^{1+\sigma,(1+\sigma)/2}(\overline{\mathcal{C}^\pm}) \quad \text{with some} \quad \sigma \in (0,1).$$

The function u satisfies (in the weak sense) the heat equation

$$u_t = \Delta u + a\,\chi_{\mathcal{C}^+} \quad \text{in} \quad \mathcal{C}^\pm$$

with the initial and boundary conditions

$$u(\mathbf{x},0) = \rho_0(\mathbf{x}) \equiv u_0(\mathbf{x}) \quad \text{in} \quad \omega_0^\pm,$$
$$u = \mu \quad \text{on} \quad S, \quad u = R \quad \text{on} \quad \Sigma^\pm.$$

To continue u to $D_T \setminus \mathcal{C}$, let us construct such a continuation from $\mathcal{C}^+$ to D_T^+. It follows from local regularity theory for linear parabolic equations that for every subdomain $G^+ \subset \mathcal{C}^+$, separated away from the lateral boundaries Σ^+ and S of $\mathcal{C}^+$, one has $u \in C^{2+\beta,(2+\beta)/2}(\overline{G^+})$. Let us take a smooth surface $\Sigma \subset \mathcal{C}^+$, $\Sigma \cap \Sigma^+ = \emptyset$,

$\Sigma \cap S = \emptyset$, set $\psi = u|_{\Sigma} \in C^{2+\beta,(2+\beta)/2}(S)$, denote by $\mathcal{A}$ an annular cylinder with lateral boundaries $\partial\Omega_e \times [0, T]$ and Σ, and consider the following problem:

$$\begin{cases} w_t = \Delta w + a \quad \text{in} \quad \mathcal{A}, \\ w = \psi \text{ on } \Sigma, \quad w = \phi \quad \text{on} \quad \partial\Omega_e \times [0, T], \\ w(\mathbf{x}, 0) = u_0 \quad \text{in} \quad \overline{\mathcal{A}} \cap \{t = 0\}. \end{cases}$$

This problem has a unique solution $w \in C^{2+\beta,(2+\beta)/2}(\overline{\mathcal{A}})$, that is,

$$D_x^{\kappa} D_t^{s}(w - u)\big|_{\Sigma} = 0 \quad \text{for} \quad 0 \leqq |\kappa| + 2s \leqq 2.$$

Let us define the function

$$W = \begin{cases} u(\mathbf{x}, t) \ \text{ in } D_T^+ \setminus \mathcal{A}, \\ w(\mathbf{x}, t) \text{ in } \mathcal{A}. \end{cases}$$

Then $W \in C^{2,1}(D_T^+) \cap C^0(\overline{D_T^+})$, $W = u$ on S, $W > \mu$ as $t = 0$ and on the lateral boundary $\partial\Omega_e \times [0, T^*]$. If $a > 0$, it follows from the maximum principle for parabolic equations that $W > \mu$ in D_T^+.

The continuation from $\mathcal{C}^-$ to the domain $D_T^- \setminus \mathcal{C}^-$, where the forcing term $a\, \chi_{\mathcal{C}^+}$ equals zero, is constructed in like manner.

6.2. *Proof of Theorems 2 and 3*

Theorems 2 and 3 are immediate byproducts of Theorems 1 and 12 with the function p defined as a solution of the problem

$$\begin{cases} \operatorname{div}(u \nabla p) + a\, \chi_{\omega^+(t)} = 0 \quad \text{in} \quad \omega^{\pm}(t), \\ [\nabla p \cdot \mathbf{n}]\,|_{\Gamma_\mu(t)} = [\nabla \ln u \cdot \mathbf{n}]\,|_{\Gamma_\mu(t)}, \\ p = 0 \quad \text{on} \quad \partial\omega^{\pm}(t), \end{cases} \tag{52}$$

Let us consider the solution of problem (3) $u(\mathbf{x}, t)$, generated by the solution of problem in Lagrangian formulation, as a solution of the heat equation with the given forcing term $a\, h_u(\mathbf{x}, t) \in L^\infty(D_T)$. Since $u_0 \in C^{2+\beta}(\Omega)$, then $u \in W_q^{2,1}(\mathcal{C})$ with every $1 < q < \infty$. This means that ∇u is continuous on $\Gamma_\mu(t)$, $[\nabla u \cdot \mathbf{n}]_{\Gamma_\mu(t)} = 0$, and problems (17), (52) are equivalent for $u \in W_q^{2,1}(\mathcal{C})$.

6.3. *Proof of Theorem 4*

The first assertion follows from (17) and Corollary 3. Let us denote by $u(\mathbf{x}, t)$ the solution of problem (3) generated by the constructed solution of the problem in Lagrangian setting. The generic regularity of Γ_μ follows from the representation (15) (b) and the inclusions (13):

$$\mathbf{x}|_{\Gamma_\mu} = (\mathbf{y} + \nabla U)|_{\mathbf{y} \in \gamma}, \quad U \in W_q^4(Q_T^{\pm}), \quad U_t \in W_q^4(Q_T^{\pm} \cap \{t > \tau\}).$$

It follows from the proof of Theorem 3 that for the constructed solution $\nabla u \in C_{\mathbf{x},t}^{\delta,\delta/2}(\mathcal{C})$ with some $\delta \in (0, 1)$. Since p is defined as the solution of problem

(17), we also have that $p \in W^2_q(\omega(t)) \cap C^{1+\sigma}(\omega(t))$ with $\sigma \in (0, 1-n/q)$. By Corollary 3 $\nabla_{\mathbf{x}} p = \nabla_{\mathbf{y}} P \cdot \mathbf{J}^{-1}$ is Hölder continuous with respect to t, whence Hölder continuity of $\mathbf{v}$ in $\overline{\mathcal{C}}$ and the convergence $|\mathbf{v}(\mathbf{x}(\mathbf{y}, t), t) - \mathbf{v}_0(\mathbf{y})|^{(\alpha)}_{\omega_0^{\pm}} \to 0$ as $t \to 0$.

Nonuniqueness of the constructed solution is a byproduct of the fact that for every γ satisfying the conditions of Theorem 1 there exists a solution with smooth interface Γ_μ which continuously tends to γ as $t \to 0$.

To prove assertion (4) it suffices to notice that the uniqueness of u yields the uniqueness of $\mathbf{v}(\mathbf{x}, t)$ for $\mathbf{x} \in \Gamma_\mu(t)$.

7. Generalizations

Let us now consider problem (1) for Eq. (6): the goal is to find a function $u(\mathbf{x}, t)$ satisfying the conditions

$$\begin{cases} u_t - \Delta u + \mathcal{G}(\mathbf{x}, u) - f(\mathbf{x}, t) \\ \quad = Q\, S(\mathbf{x}, t)\, h_u(\mathbf{x}, t) \in Q\, S(\mathbf{x}, t)\mathbb{H}(u-\mu) \quad \text{in} \quad D_T, \\ u = \phi \quad \text{on} \quad S_T, \\ u(\mathbf{x}, 0) = u_0(\mathbf{x}) \quad \text{in} \quad \Omega \end{cases} \tag{53}$$

under the foregoing assumptions on the domain Ω and the data ϕ, u_0. The function $\mathcal{G}$, S and f are given functions of their arguments, $Q = \text{const}$, $\mathbb{H}(\cdot)$ is the Heaviside maximal monotone graph, and the function

$$h_u : D_T \mapsto [0, 1], \qquad h_u \in \mathbb{H}(u-\mu) \quad \text{almost everywhere in} \quad D_T$$

has to be defined together with the solution u.

Let us assume that problem (53) admits a continuous positive solution u. Choosing a domain $\mathcal{C}$ according to conditions (9) we may formulate the auxiliary problem of motion of a fluid volume $\omega(t)$ with constant mass. The density of the fluid equals u and we assume that the velocity is given by the formula

$$\mathbf{v} = -\nabla \ln u + \nabla p,$$

where p is a new unknown representing the artificial pressure. Formally repeating the arguments of Section 2, we find that the pressure p has to satisfy the conditions

$$\forall \phi \in H^1(\omega(t)) \qquad \int_{\omega(t)} [u \nabla p \cdot \nabla \phi - Q\, S\, h_u\, \phi + (\mathcal{G} - f)\, \phi]\, \mathrm{d}\mathbf{x} = 0.$$

It follows that p, considered as a function of the Euler variables $(\mathbf{x}, t)$, is a weak solution of the *linear* elliptic problem

$$\begin{cases} \operatorname{div}\,(u \nabla p) + Q\, S\, h_u - \mathcal{G} = f \quad \text{in} \quad \omega^{\pm}(t), \\ [\nabla p \cdot \mathbf{n}]|_{\Gamma_\mu(t)} = [\nabla \ln u \cdot \mathbf{n}]|_{\Gamma_\mu(t)}, \\ \nabla p \cdot \mathbf{n} = (\mathbf{v} + \nabla \ln u) \cdot \mathbf{n} \quad \text{on} \quad \partial\omega^{\pm}(t). \end{cases} \tag{54}$$

The first boundary condition means that the velocity $\mathbf{v}$ is zero on $\Gamma_\mu(t)$, the second condition expresses the assumption that the boundary of the volume $\omega(t)$ is constituted by the same particles at every instant $t > 0$. Introducing the system of Lagrangian coordinates, we arrive at the following problem: to find the potential $U(\mathbf{y}, t)$ and $P(\mathbf{y}, t)$ (the artificial pressure) which satisfy the system of equations

$$\begin{aligned} &\mathbf{J}\nabla U_t + \nabla\left(\ln\left(u_0 \mathrm{Det}[\mathbf{J}^{-1}]\right)\right) \in \mathbb{J}_\pm, \\ &\mathrm{Det}[\mathbf{J}^{-1}]\,\mathrm{div}\left(u_0\left(\mathbf{J}^{-1}\right)^2 \nabla P\right) = f(\mathbf{y}+\nabla U, t) - \mathcal{G}(\mathbf{y}+\nabla U, u_0\mathrm{Det}[\mathbf{J}^{-1}]) \\ &\qquad\qquad -Q\,S(\mathbf{y}+\nabla U, t)\chi_{Q_T^+}\left(1-\chi_{Q_T^-}\right) \end{aligned}$$

under the initial and boundary conditions (32). Once this problem is solved, the solution of the original problem (53) can be constructed exactly as in the case of the model (3). The conclusions about the regularity and behavior of the interface $\Gamma_\mu(t)$ follow automatically.

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