

NEW L^1 -GRADIENT TYPE ESTIMATES OF SOLUTIONS TO ONE-DIMENSIONAL QUASILINEAR PARABOLIC SYSTEMS

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We consider a general class of one-dimensional parabolic systems, mainly coupled in the diffusion term, which, in fact, can be of the degenerate type. We derive some new L^1 -gradient type estimates for its solutions which are uniform in the sense that they do not depend on the coefficients nor on the size of the spatial domain. We also give some applications of such estimates to gas dynamics, filtration problems, a p -Laplacian parabolic type equation and some first order systems of Hamilton–Jacobi or conservation laws type.

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1. Introduction

We consider the following class of quasilinear parabolic systems

$$\mathbf{u}_t - (\mathbf{A}(\mathbf{u})\mathbf{u}_x)_x + \mathbf{B}(\mathbf{u}) + \mathbf{C}(\mathbf{u}, \mathbf{u}_x) = \mathbf{f}(t, x), \quad (1)$$

on $Q = (0, T) \times \Omega$, where $\Omega = (0, l)$. Here $\mathbf{u} = (u_1, \dots, u_n)$ is the unknown vector-valued function. To this system we add some boundary conditions as, for instance,

$$\mathbf{u}_x(t, 0) = \mathbf{0}, \mathbf{u}(t, l) = \mathbf{0}, \quad 0 < t < T, \quad (2)$$

and the initial condition

$$\mathbf{u}(0, x) = \mathbf{u}_0(x), \quad x \in \Omega. \quad (3)$$

To simplify the exposition, we assume that the diffusion matrix is diagonal $\mathbf{A}(\mathbf{u}) = (A_{ii}(\mathbf{u}))$. However, each term $A_{ii}(\mathbf{u})$ may depend on the n components of \mathbf{u} and

therefore the system can be considered as *strongly coupled*. We also assume that the vector functions \mathbf{f} and \mathbf{u}_0 are given satisfying certain regularity conditions which will be indicated later.

A special attention is paid to degenerate systems: i.e. to the case in which the corresponding matrix \mathbf{A} is semidefinite positive and in fact vanishes at certain critical values of the vector solution \mathbf{u}_{cr} , i.e. $\mathbf{A}(\mathbf{u}_{cr}) = \mathbf{0}$. The mathematical treatment is considerably more delicate than the *uniformly parabolic case* where the matrix \mathbf{A} is definite positive.

Systems similar to (1) appear very often in many different contexts such as biology, chemistry or filtration in porous media. Our particular motivation corresponds to the case in which system (1) describes the discharge of a laminar hot gas in a stagnant colder atmosphere of the same gas under the assumptions of the boundary layer approximation (see Sec. 3 and [6, 7, 10, 46, 51]). In that case $n = 2$, u_1 is the horizontal component of the velocity and u_2 is the the gas temperature.

When studying the above mentioned special case, the authors became aware of the gradient estimate

$$TV(u_i(t, \cdot)) \leq TV(u_{0i}(\cdot)) + \int_0^t TV(f_i(\tau, \cdot)) d\tau, \quad (4)$$

for $i = 1, \dots, n$, and also that it remains valid for solutions of a large class of systems as (1). Curiously, it seems that this property was not well observed previously in the literature, even for the simpler case of scalar semilinear parabolic equations (the linear heat equation, for instance), neither for simpler systems for which $\mathbf{A}(\mathbf{u})$ is reduced to the identity matrix. Here $TV(\varphi(t, \cdot))$ denotes the total variation of the Radon measure $\varphi_x(t, \cdot)$, i.e.

$$TV(\varphi(t, \cdot)) = \sup \left\{ \int_0^l \varphi(t, x) \phi_x(x) dx : \phi \in C_0^\infty(0, l), |\phi(x)| \leq 1 \text{ for } x \in (0, l) \right\},$$

for a given function $\varphi \in L^\infty(0, T : BV(0, l))$.

This is the reason why instead of presenting a detailed analysis of the specific system describing the gas discharge problem we give here the proof of estimate (4) for the solutions of a wider class of systems of the form (1).

Because of the generality of the considered framework, we shall not discuss here any question related to the existence and regularity of solutions of systems (1)–(3). Certainly, estimate (4) applies to weak solutions (i.e. satisfying the system in distributional sense) which can be obtained as limit of more regular solutions (for instance, classical solutions of some approximate formulations).

In fact, one of the main ingredients in our approach springs from the philosophy that, as in the scalar case, weak solutions of degenerate systems (1) are mainly found by considering the limit of regular solutions of suitable uniformly parabolic auxiliary systems defined through regularization and approximation of its coefficients. So, in this way, some regularity properties of the weak solutions of degenerate equations are obtained by proving them firstly for regular solutions of the approximate systems

and then passing to the limit. That was also the philosophy proposed already by Hopf [37], in 1950, when he studied Burgers equation by considering the limit, as $\mu \rightarrow 0$, in the viscous equation $u_t + uu_x = \mu u_{xx}$.

So, we shall always assume that the considered problem has a weak solution which is *regular enough*, at least by regularizing its coefficients. We recall that some general existence and regularity results for the uniformly parabolic case can be found, for instance, in [41, 42, 2] (where the reader can also find a long list of references) under the essential condition that \mathbf{A} is a positive definite matrix. We remark that in the case in which the term $\mathbf{C}(\mathbf{u}, \mathbf{u}_x)$ involves some sourcing terms with a superlinear growth, the existence of global solutions requires some additional conditions on the data implying the smallness of the L^∞ norm of the solution: see, e.g., [49]. The degeneration of the matrix \mathbf{A} is one of the reasons to understand the solution in a weaker form and the appearance of several *localization properties* of solutions such as the *finite speed of propagation*, *waiting time effect*, etc. (see, e.g., [12]). This creates some significant difficulties in order to get some existence and uniqueness results on weaker classes of solutions. Nevertheless, many results are known to this respect (see, for instance, the treatment of systems of degenerate equations made in [1]).

We point out that the (spatial) gradient estimate (4) is, in some sense “uniform”, since it will not depend on the matrix function \mathbf{A} (neither on the vector functions \mathbf{B} and \mathbf{C}).

In fact, such estimate will be obtained (firstly for regular solutions and then for a more general class of solutions) for a given component, u_i (for some $i = 1, \dots, n$)

$$\int_0^t |u_{ix}(t, x)| dx \leq \int_0^t |u_{ix}(0, x)| dx + \int_0^t \int_0^l |f_{ix}(\tau, x)| dx d\tau, \quad (5)$$

for any $t \in (0, T]$. Obviously, if the required conditions hold for any $i = 1, \dots, n$ then we conclude (at least in the regular case) that

$$\|\mathbf{u}_x\|_{\mathbf{L}^\infty(0, T; \mathbf{L}^1(\Omega))} \leq \|\mathbf{u}_{0x}\|_{\mathbf{L}^1(\Omega)} + \|\mathbf{f}_x\|_{\mathbf{L}^1(0, T; \mathbf{L}^1(\Omega))}$$

(notice that this last inequality does not imply the single one (5)).

This kind of *uniform* gradient estimates remains true under some generalizations (see Sec. 2.2). The extension to higher dimension and to other quasilinear equations including the case of nonlinear gradient diffusion terms, will be the object of a separate work by the authors ([9]). In fact, such estimate has an important meaning in the theory of degenerate parabolic systems since this property implies a compactness criteria on a suitable functional space. This kind of arguments can be used, for instance, to show that (4) remains valid for the case of the pure Cauchy problem associated to (1) (i.e. when the spatial domain Ω is the whole space \mathbb{R}). We also send the reader to Remark 2.5 for some other comments on this property and its study on some exact solutions of the scalar case (as, for instance, the so-called, *Barenblatt solution* of the porous media equation).

We mention that our structural conditions on the coefficients will include some diffusive conservation laws of the form

$$\mathbf{u}_t + \mathbf{D}(\mathbf{u})_x = (\mathbf{A}(\mathbf{u})\mathbf{u}_x)_x + \mathbf{f}(x, t), \quad (6)$$

with

$$\mathbf{D}(\mathbf{u}) = (D_1(u_1), D_2(u_2), \dots, D_n(u_n)). \quad (7)$$

Even in this special case, our method of proof is different from the one used to derive the BV-estimates for solutions of hyperbolic systems of conservation laws. The derivation of such estimates was based on the study of the associate diffusive system and followed the philosophy of the already mentioned work by Hopf: see, e.g., [40, 55, 54]. Indeed, estimates of type (4) are usually obtained for the special case of Cauchy's problem (and under other conditions which usually reduce the system to the scalar case) by using that the associated semigroup is a contraction in $L^1(\mathbb{R})$ and that (under conditions of autonomous coefficients) $\mathbf{u}(t, x + \lambda)$ is also a solution, for any $\lambda \in \mathbb{R}$. In fact, the proof method of (4) can be extended to many other frameworks (case of $L^1(\mathbb{R})$, accretive operators, renormalized solutions, etc.). For such class of systems, our proof method does not require to work necessarily with Cauchy's problem and, in fact, can be applied to the case of a bounded interval $\Omega = (0, l)$ with suitable boundary conditions.

The main idea to prove estimate (5) consists in multiplying the i -equation by $-\partial S_\delta(u_{ix})/\partial x$, where $S_\delta(r) \rightarrow \text{sign}_0(r)$ as $\delta \rightarrow 0$, and to show that other contributions different to the ones arising in (5) converge to zero when $\delta \rightarrow 0$ under suitable conditions on the coefficients. It should be noted that the trick of using a family of test functions converging to the sign function has been very familiar in the realm of elliptic, parabolic and first order hyperbolic equations for several decades in order to prove simpler estimates of the following type

$$\int_0^l |u_i(t, x)| dx \leq \int_0^l |u_i(0, x)| dx + \int_0^t \int_0^l |f_i(\tau, x)| dx d\tau, \quad i = 1, \dots, n. \quad (8)$$

Perhaps, some of the older works related to L^1 -estimates are Miranda [44] and Stampacchia [56] dealing with elliptic equations (Miranda used the test function $\phi(u) = \frac{1}{q+1}|u|^q u$). An important progress in the treatment of elliptic equations with L^1 data was the pioneering work by Brezis and Strauss [23]. The application of the abstract semigroup theory also leads to the use of test functions of the form $\phi(u) = \text{sign}_0(u)$ (see, for instance, Sato [50] and Crandall and Liggett [28]). The application to parabolic and hyperbolic equations of conservation laws type was first presented in Kruzhkov [40] and later extended, by using the semigroup theory, in Crandall [27] and Bénilan [17] (who applied this type of test functions to the porous media equation and other quasilinear parabolic equations). For some results on systems, we refer the reader to [10] and its references.

To the best of our knowledge, this use of the test function $-\partial S_\delta(u_{ix})/\partial x$ is new in the literature. Although a formal integration by parts links our method with

the one already used in the literature consisting of differentiating the equation and applying L^1 -techniques to u_{ix} (see, for instance, [32, 59]), our method has several advantages: it needs less regularity on the coefficients of the diffusion operator, it applies to Dirichlet boundary conditions (avoiding some complicated arguments on the value of the second order operator on the boundary) and it does not involve any constant in the final estimates (4) and (5).

We end the paper by giving, in Sec. 3, some illustrative applications of these results to some special problems. We consider, in particular, the case of the discharge of a hot gas mentioned before, the case of one-dimensional two phase filtration described by the degenerate parabolic equation

$$s_t - (a(s)s_x)_x = V(t)b(s)_x, \tag{9}$$

and the (possibly degenerate) p-Laplacian type problem

$$\begin{cases} u_t - (\Phi(u_x))_x = f(t, x), & t \in (0, T), \quad x \in (0, l), \\ u_x(t, 0) = u_x(t, l) = 0, & t \in (0, T), \\ u(0, x) = u_0(x), & x \in (0, l), \end{cases} \tag{10}$$

where Φ is a continuous increasing real function. In this last case we prove a new second-order estimate

$$\|u_x\|_{L^\infty(0,T;BV(0,l))} \leq \|u_{0xx}\|_{L^1(0,l)} + \|f_{xx}\|_{L^1(0,T;L^1(0,l))}.$$

Finally, we consider the case of some first order Hamilton–Jacobi type system (including some conservation laws)

$$\begin{cases} \mathbf{u}_t + \mathbf{C}(\mathbf{u}, \mathbf{u}_x) = \mathbf{f}(t, x), & t \in (0, T), \quad x \in \mathbb{R}, \\ \mathbf{u}(0, x) = \mathbf{u}_0(x), & x \in \mathbb{R}, \end{cases}$$

and prove that, under suitable conditions,

$$\|\mathbf{u}\|_{\mathbf{L}^\infty(0,T;\mathbf{BV}(\mathbb{R}))} \leq \|\mathbf{u}_{0x}\|_{\mathbf{L}^1(\mathbb{R})} + \|\mathbf{f}_x\|_{\mathbf{L}^1(0,T;\mathbf{L}^1(\mathbb{R}))}. \tag{11}$$

Concerning this last system we point out that, as mentioned before, our proof method is different to the used to derive BV-estimates for some hyperbolic systems and that, in fact, it can be applied to the case of a bounded interval $\Omega = (0, l)$ under suitable boundary conditions. Finally, we recall that estimates of type (11) are very useful to show the convergence of some numerical approximation algorithms like the ones by Lax and Friedrich or the one by Glimm (see, e.g., the exposition made in the monographs [55, 54]).

Here, and in the rest of the paper we use the bold characters for the vector function spaces in \mathbb{R}^n , so, for instance, $\mathbf{L}^1(0, l) = L^1(0, l)^n$, $\mathbf{L}^\infty(0, T : \mathbf{BV}(0, l)) = (L^\infty(0, T : BV(0, l)))^n$, etc.

2. Structural Assumptions and Main Results

In this section we assume, at least, that

$$\mathbf{f} \in \mathbf{L}^1(Q) \quad \text{and} \quad \mathbf{u}_0 \in \mathbf{L}^1(\Omega). \tag{12}$$

We shall assume also that $\mathbf{A} = (A_{ij})$ is a diagonal $C^0(\mathbb{R}^n : \mathbb{R}^{n^2})$ function, $\mathbf{A} = (A_{ii})$, and, additionally, that \mathbf{A} is a semi-definite positive matrix, i.e. such that

$$a_0 |\vec{\xi}|^2 \leq (\mathbf{A}(\mathbf{u})\vec{\xi}, \vec{\xi}) = \sum_{i=1}^n A_{ii} \xi_i^2, \tag{13}$$

for any $(\vec{\xi}, \mathbf{u}) \in \mathbb{R}^{2n}$ and some $a_0 \geq 0$. We recall that the case $a_0 > 0$ corresponds to uniformly parabolic systems (this case leads to existence of regular solutions), and that $a_0 = 0$ corresponds to degenerate systems (in which case the solutions must be understood in a weaker sense).

Since the proof of our main estimate (5) is given first for a single component u_i (for some $i = 1, \dots, n$) of the vector solution \mathbf{u} , we shall assume that the main coupling among the equations of the system comes from the diffusion terms and that, by the contrary, the vector lower order terms $\mathbf{B}(\mathbf{u}) + \mathbf{C}(\mathbf{u}, \mathbf{u}_x)$ are, in some sense, weakly coupled. The precise structural assumptions on $\mathbf{B}(\mathbf{u}) + \mathbf{C}(\mathbf{u}, \mathbf{u}_x)$ appear in the decomposition in a “purely absorption part” and a “convective part”. Regarding the components of the absorption part, \mathbf{B} , we shall assume that

$$B_i(\mathbf{r}) = B_i(r_i), \quad B_i \in C^0(\mathbb{R} : \mathbb{R}) \text{ is nondecreasing,} \quad B_i(0) \leq 0. \tag{14}$$

Regarding the components of the convective part, $\mathbf{C}(\mathbf{u}, \mathbf{u}_x)$, we shall assume that

$$\begin{cases} \mathbf{C} \in C^0(\mathbb{R}^{2n} : \mathbb{R}^n) & \text{and} & |C_i(\mathbf{u}, \mathbf{p})| \leq C_0 |p_i|^{\gamma_i} & \text{for some } \gamma_i > 1/2, & \text{and} \\ \text{some } C_0 > 0, & & \text{for any } (\mathbf{u}, \mathbf{p}) \in \mathbb{R}^n \times \mathbb{R}^n, \end{cases} \tag{15}$$

or

$$\begin{cases} \mathbf{C}(\mathbf{u}, \mathbf{u}_x) = \mathbf{D}(\mathbf{u})_x, & \mathbf{D} \in C^1(\mathbb{R}^n : \mathbb{R}^n) & \text{with} \\ \mathbf{D}(\mathbf{u}) = (D_1(u_1), D_2(u_2), \dots, D_n(u_n)). \end{cases} \tag{16}$$

2.1. On the uniform L^1 -gradient estimate for general systems

The main goal of the paper is to derive a L^1 -gradient *uniform* estimate (i.e. which does not depend on the constants l, a_0, a_1, C_0). For fixed $\delta > 0$ we introduce the functions

$$S_\delta(r) = \frac{r}{\sqrt{\delta^2 + r^2}}, \tag{17}$$

and

$$N_\delta(r) = \int_0^r S_\delta(s) ds, \tag{18}$$

so that

$$S_\delta(r) \rightarrow \text{sign}_0 r \quad \text{and} \quad N_\delta(r) \rightarrow |r|, \quad \text{if } \delta \rightarrow 0, \quad (19)$$

where $\text{sign}_0 r = 1$ if $r > 0$, $\text{sign}_0 r = -1$ if $r < 0$ and $\text{sign}_0 0 = 0$.

We shall start by considering the case of regular coefficients:

$$\begin{cases} \mathbf{A} \in C^1 & \text{and} & \left| \frac{\partial A_{ii}}{\partial u_k}(\mathbf{u}) \right| \leq C, \quad i, k = 1, \dots, n & \text{for any } \mathbf{u} \in \mathbb{R}^n, \\ B_i \in C^1 & \text{and} & \frac{\partial B_i(r_i)}{\partial r_i} \geq 0, & \text{for any } r_i \in \mathbb{R}, \\ \mathbf{C} \in C^1 & \text{(and } \mathbf{D} \in C^2, \mathbf{D}(\mathbf{u}) = (D_1(u_1), D_2(u_2), \dots, D_n(u_n))) \end{cases} \quad (20)$$

Notice that now

$$(\mathbf{A}(\mathbf{u})\mathbf{u}_x)_x \equiv \mathbf{A}\mathbf{u}_{xx} + \left(\frac{\partial \mathbf{A}}{\partial \mathbf{u}} \mathbf{u}_x \right) \mathbf{u}_x,$$

and so, the vector equation (1) can be rewritten as a coupled system of scalar equations

$$u_{it} - [(\mathbf{A}(\mathbf{u})\mathbf{u}_x)_x]_i + [\mathbf{B}(\mathbf{u})]_i + [\mathbf{C}(\mathbf{u}, \mathbf{u}_x)]_i = f_i, \quad i = 1, \dots, n, \quad (21)$$

with

$$[(\mathbf{A}(\mathbf{u})\mathbf{u}_x)_x]_i = A_{ii}u_{ixx} + \sum_{k=1}^n \frac{\partial A_{ii}}{\partial u_k} u_{kx} u_{ix}.$$

Besides the assumed extra regularity, we start by working with *strong* solutions, i.e. regular enough functions satisfying the system of equations and the additional conditions in almost every point. Obviously, the above conditions are satisfied for any classical solution \mathbf{u} of (1)–(3), i.e. such that $\mathbf{u} \in \mathbf{C}_{t,x}^{1,2}([0, T] \times \bar{\Omega})$ and \mathbf{u} satisfies (1)–(3) in every point. Let us recall that such high regularity is not needed for *weak solutions* which satisfies the system in distributional sense.

Concerning the case of the above mentioned class of strong solutions we shall prove the following result.

Theorem 2.1. *Assume (13)–(15) or (16) as well as (20). Assume that there exists a component $i_0 \in \{1, \dots, n\}$ for which*

$$f_{i_0} \in L^1(0, T; W^{1,1}(\Omega)), f_{i_0}(t, l) = 0 \quad \text{for } t \in (0, T) \quad \text{and} \quad u_{0i_0} \in W^{1,1}(\Omega). \quad (22)$$

Let $\mathbf{u} = (u_1, \dots, u_n)$ be a strong solution of the problem (1)–(3). Then

$$\int_0^l |u_{i_0x}(t, x)| dx \leq \int_0^l |u_{0i_0x}(x)| dx + \int_0^t \int_0^l |f_{i_0x}(\tau, x)| dx d\tau, \quad (23)$$

for any $t \in [0, T]$.

One of the most important consequences of the above *uniform* gradient estimate is its extension to the class of weak solutions:

Theorem 2.2. *Let $\mathbf{u} \in C([0, T] : \mathbf{L}^1(\Omega))$ be a weak solution of the system (1)–(3) corresponding to some coefficients \mathbf{A}, \mathbf{B} and \mathbf{C} and data*

$$\mathbf{f} \in \mathbf{L}^1(0, T : \mathbf{W}^{1,1}(\Omega)) \quad \text{with } \mathbf{f}(t, l) = 0 \quad \text{and} \quad \mathbf{u}_0 \in \mathbf{W}^{1,1}(\Omega).$$

We assume that $\mathbf{u} = \lim \mathbf{u}_\varepsilon$, in $C([0, T] : \mathbf{L}^1(\Omega))$, where \mathbf{u}_ε denotes the strong solution of the system corresponding to the approximating coefficients $\mathbf{A}_\varepsilon, \mathbf{B}_\varepsilon$ and \mathbf{C}_ε and the approximating data

$$\mathbf{f}_\varepsilon \in \mathbf{L}^1(0, T : \mathbf{W}^{1,1}(\Omega)), \quad \text{with } \mathbf{f}_\varepsilon(t, l) = 0 \quad \text{for } t \in (0, T), \quad \text{and} \quad \mathbf{u}_{0,\varepsilon} \in \mathbf{W}^{1,1}(\Omega) \tag{24}$$

such that

$$\int_0^l |u_{0\varepsilon ix}(x)| dx + \int_0^t \int_0^l |f_{\varepsilon ix}(\tau, x)| dx d\tau \leq C,$$

for some C independent on ε , for any $i = 1, \dots, n$ and $t \in [0, T]$. We also assume that the conditions of Theorem 2.1 are satisfied by the approximating coefficients $\mathbf{A}_\varepsilon, \mathbf{B}_\varepsilon$ and \mathbf{C}_ε . Then there exists a subfamily of $\{\mathbf{u}_\varepsilon\}$ which converges to \mathbf{u} also in $\mathbf{L}^\infty(0, T : \mathbf{BV}(\Omega))$, in the weak*-topology, and we have the estimate

$$TV(\mathbf{u}(t, \cdot)) \leq TV(\mathbf{u}_0(\cdot)) + \int_0^t TV(\mathbf{f}(\tau, \cdot)) d\tau. \tag{25}$$

We shall split the proof of Theorem 2.1 in several previous lemmas. Its presentation is motivated to exhibit some slight resemblances with the “weak time dependence integration by parts formula” (see, e.g., [1] and its references) and some properties related to the class of accretive operators in $L^1(0, l)$ (see, e.g., [23, 17]).

Lemma 2.1. *Assume $\frac{\partial S_\delta(u_{ix})}{\partial x} \in L^p(0, T : L^p(0, l))$, $u_{it} \in L^{p'}(0, T : L^{p'}(0, l)) \cap C([0, T] \times ([0, \underline{\mu}] \cup (\bar{\mu}, l]))$ for some $0 < \underline{\mu} \leq \bar{\mu} < l$ and some $p \in [1, +\infty]$, $(\frac{1}{p} + \frac{1}{p'} = 1)$ with $u_i(t, l) = 0, u_{ix}(t, 0) = 0$ for a.e. $t \in (0, T)$ and $u_{0i} \in W^{1,1}(\Omega)$. Then $u_{ixt} S_\delta(u_{ix}) \in L^1(0, T : L^1(0, l))$, $N_\delta(u_{ix}) \in C([0, T] : L^1(0, l))$ and*

$$\int_0^t \int_0^l u_{it} \left(-\frac{\partial S_\delta(u_{ix})}{\partial x} \right) dx d\tau = \int_0^l N_\delta(u_{ix}(t, x)) dx - \int_0^l N_\delta(u_{ix}(0, x)) dx. \tag{26}$$

Proof. Notice that under the assumed regularity, $u_{it}(t, l) = 0$ for any $t \in (0, T)$. Thus

$$\begin{aligned} \int_0^l u_{ixt} S_\delta(u_{ix}) dx &= \int_0^l u_{it} \left(-\frac{\partial S_\delta(u_{ix})}{\partial x} \right) dx + u_{it} S_\delta(u_{ix}) \Big|_{x=0}^{x=l} \\ &= \frac{d}{dt} \int_0^l \left[\int_0^{u_{ix}} S_\delta(s) ds \right] dx = \frac{d}{dt} \int_0^l N_\delta(u_{ix}) dx, \end{aligned} \tag{27}$$

which leads to the result. □

Lemma 2.2. *Assume*

$$A_{ii}u_{ixx} \frac{\partial S_\delta(u_{ix})}{\partial x} \in L^1(0, T : L^1(0, l)),$$

$$\frac{\partial A_{ii}}{\partial u_k} u_{kx} u_{ix} \frac{\partial S_\delta(u_{ix})}{\partial x} \in L^1(0, T : L^1(0, l)).$$

Then

$$\lim_{\delta \rightarrow 0} \int_0^t \int_0^l [(-\mathbf{A}(\mathbf{u})\mathbf{u}_x)_x]_i \left(-\frac{\partial S_\delta(u_{ix})}{\partial x} \right) dx d\tau \geq 0. \quad (28)$$

Moreover,

$$\lim_{\delta \rightarrow 0} \int_0^t \int_0^l [\mathbf{B}(\mathbf{u})]_i \left(-\frac{\partial S_\delta(u_{ix})}{\partial x} \right) dx d\tau \geq 0, \quad (29)$$

and

$$\lim_{\delta \rightarrow 0} \int_0^t \int_0^l [\mathbf{C}(\mathbf{u}, \mathbf{u}_x)]_i \left(-\frac{\partial S_\delta(u_{ix})}{\partial x} \right) dx d\tau \geq 0. \quad (30)$$

Proof. Obviously

$$\frac{dS_\delta}{dr}(r) = \frac{\delta^2}{(\delta^2 + r^2)^{\frac{3}{2}}} \quad \text{and} \quad \frac{\partial S_\delta(u_{ix})}{\partial x} = \frac{\delta^2 u_{ixx}}{(\delta^2 + u_{ix}^2)^{\frac{3}{2}}}.$$

Then

$$\int_0^t \int_0^l [(-\mathbf{A}(\mathbf{u})\mathbf{u}_x)_x]_i \frac{\partial S_\delta(u_{ix})}{\partial x} dx d\tau := I_1 + I_2$$

with

$$I_1 := \int_0^t \int_0^l A_{ii} \frac{\delta^2 (u_{ixx})^2}{(\delta^2 + u_{ix}^2)^{\frac{3}{2}}} dx d\tau \geq a_0 \int_0^t \int_0^l \frac{\delta^2 (u_{ixx})^2}{(\delta^2 + u_{ix}^2)^{\frac{3}{2}}} dx d\tau := I_0,$$

and

$$I_2 := - \int_0^t \int_0^l \frac{\partial A_{ii}}{\partial u_k} u_{kx} u_{ix} \frac{\delta^2 u_{ixx}}{(\delta^2 + u_{ix}^2)^{\frac{3}{2}}} dx d\tau.$$

Using Cauchy's inequality we can estimate I_2 as follows:

$$|I_2| \leq \int_0^t \int_0^l \left[\frac{a_0}{3} \frac{\delta^2 (u_{ixx})^2}{(\delta^2 + u_{ix}^2)^{\frac{3}{2}}} + \frac{3\delta C^2}{4a_0} \left(\sum_{k=1}^n |u_{kx}| \right)^2 \frac{\delta}{(\delta^2 + u_{ix}^2)^{\frac{1}{2}}} \frac{u_{ix}^2}{(\delta^2 + u_{ix}^2)} \right] dx d\tau$$

$$\leq \frac{1}{3} I_0 + \delta \frac{3C^2}{4a_0} \int_0^t \int_0^l \left(\sum_{k=1}^n |u_{kx}| \right)^2 dx d\tau.$$

This proves (28) (recall that \mathbf{u} is regular enough and that the parameter δ only arises in the definition of the test function). By integrating by parts and using (14),

we have

$$\int_0^t \int_0^l \frac{\partial B_i(u_i)}{\partial r_i} \frac{u_{ix}^2}{(\delta^2 + u_{ix}^2)} dx d\tau \geq 0,$$

which shows (29). To prove (30) we notice that if we define

$$I_3 := \int_0^t \int_0^l [\mathbf{C}(u, u_x)]_i \left(-\frac{\partial S_\delta(u_{ix})}{\partial x} \right) dx d\tau$$

and if (15) holds then we get

$$\begin{aligned} |I_3| &\leq \left| \int_0^t \int_0^l C_0 \frac{|u_{ix}|^{\gamma_i} \delta^2 u_{ixx}}{(\delta^2 + u_{ix}^2)^{\frac{3}{2}}} dx d\tau \right| \leq \int_0^t \int_0^l \frac{\delta |u_{ixx}|}{(\delta^2 + u_{ix}^2)^{\frac{3}{4}}} \frac{\delta C_0 |u_{ix}|^{\gamma_i}}{(\delta^2 + u_{ix}^2)^{\frac{3}{4}}} dx d\tau \\ &\leq \int_0^t \int_0^l \left(\frac{a_0}{3} \frac{\delta^2 u_{ixx}^2}{(\delta^2 + u_{ix}^2)^{\frac{3}{2}}} + \frac{3}{4a_0} C_0^2 \frac{\delta^2 |u_{ix}|^{2\gamma_i}}{(\delta^2 + u_{ix}^2)^{\frac{3}{2}}} \right) dx d\tau \\ &\leq \frac{1}{3} I_0 + \frac{3\delta^{2\gamma_i-1}}{4a_0} C_0^2 l T + \delta \frac{3C^2}{4a_0} \left(\int_0^t \int_0^l \left(\sum_{k=1}^n |u_{kx}| \right)^2 dx d\tau + \frac{C(T, l)}{C^2} \right). \end{aligned}$$

In the case in which (16) holds

$$\begin{aligned} I_3 &= \int_0^t \int_0^l \frac{dD_i}{du_i} \frac{u_{ix} \delta^2 u_{ixx}}{(\delta^2 + u_{ix}^2)^{\frac{3}{2}}} dx d\tau \\ &= -\delta^2 \int_0^t \int_0^l \frac{dD_i}{du_i} \frac{\partial}{\partial x_i} \left(\frac{1}{(\delta^2 + u_{ix}^2)^{\frac{1}{2}}} \right) dx d\tau \\ &= \delta^2 \left[\int_0^t \int_0^l \frac{d^2 D_i}{d^2 u_i} \left(\frac{u_{xi}}{(\delta^2 + u_{ix}^2)^{\frac{1}{2}}} \right) dx d\tau - \frac{dD_i}{du_i} \frac{1}{(\delta^2 + u_{ix}^2)^{\frac{1}{2}}} \Big|_{x=0}^{x=l} \right]. \end{aligned}$$

Passing to the limit we get (30). □

Proof of Theorem 2.1. We denote the component i_0 simply by i for the sake of notation. By using that \mathbf{u} satisfies the system in a strong sense, and applying the above lemmas, we get

$$\int_0^l N_\delta(u_{ix}(t, x)) dx \leq \int_0^l N_\delta(u_{ix}(0, x)) dx + I_4, \tag{31}$$

with

$$|I_4| = \left| \int_0^t \int_0^l f_i \left(-\frac{\partial S_\delta(u_{ix})}{\partial x} \right) dx d\tau \right| = \left| \int_0^t \int_0^l f_{ix} S_\delta(u_{ix}) dx d\tau \right| \leq \int_0^t \int_0^l |f_{ix}| dx d\tau,$$

where we used that $f_i(t, l) = 0$ for $t \in (0, T)$. Joining (31) and the above estimates, we reach the inequality

$$\int_0^l N_\delta(u_{ix}(t, x))dx \leq \int_0^l N_\delta(u_{0ix}(x))dx + \int_0^t \int_0^l |f_{ix}(\tau, x)|dx d\tau. \tag{32}$$

Passing to the limit as $\delta \rightarrow 0$, we get the desired estimate. □

Proof of Theorem 2.2. From the assumptions we can apply Theorem 2.1 to all components of \mathbf{u}_ε and, in particular, we get that

$$\int_0^l |u_{\varepsilon ix}(t, x)|dx \leq C, \tag{33}$$

for any $i = 1, \dots, n$ and $t \in [0, T]$, with C independent of ε . Since $\mathbf{L}^\infty(0, T : \mathbf{BV}(\Omega))$ is the dual of a separable space (see, e.g., [4, p. 299]) we can apply Banach–Alaoglu–Bourbaki’s compactness theorem (see, e.g., [21]) which implies the result. □

Remark 2.1. The statements of the above theorems remain valid under different boundary conditions. For instance, we can replace the boundary conditions (2) and the assumption (22) by one of the following alternative boundary conditions (and alternative assumptions): either

$$u_{i0x}(t, l) = u_{i0x}(t, 0) = 0, \quad 0 < t < T, \tag{34}$$

and we assume merely

$$f_{i0} \in L^1(0, T : W^{1,1}(\Omega)) \quad \text{and} \quad u_{0i0} \in W^{1,1}(\Omega), \tag{35}$$

or

$$u_{i0}(t, 0) = u_{i0}(t, l) = 0, \quad 0 < t < T, \tag{36}$$

and we assume

$$f_{i0} \in L^1(0, T : W_0^{1,1}(\Omega)) \quad \text{and} \quad u_{0i0} \in W^{1,1}(\Omega). \tag{37}$$

Many possibilities can be considered: for instance, we can assume that the boundary conditions (2) hold for a subset of the components but that there are other subsets of the components for which (34) or (36) are prescribed. Finally, we send the reader to the paper [10] for a treatment of the case in which the coefficients are nonautonomous (and suitable assumptions are made on the dependence of $\mathbf{A}(t, x, \mathbf{u})$, $\mathbf{B}(t, x, \mathbf{u})$ and $\mathbf{C}(t, x, \mathbf{u}, \mathbf{u}_x)$ on t and x). A nonhomogeneous Dirichlet boundary condition and the associated stationary problem are also considered in the mentioned reference.

Remark 2.2. In many cases, by using some supplementary assumptions and arguments, it can be shown that, in fact, the limit function \mathbf{u} of Theorem 2.2 is more regular and $\mathbf{u} \in \mathbf{L}^\infty(0, T : \mathbf{W}^{1,1}(\Omega))$. See, for instance, [1] for some systems and [36] for some scalar parabolic equations. The regularity of weak solutions (defined initially in the BV space) has been a question under a constant research: see, e.g., Serrin [52] and Brezis [22] for an old conjecture and its recent proof concerning

the case of elliptic equations. Many other references in the literature deal with the gradient regularity of weak solutions. In many cases $\mathbf{u}_x \in \mathbf{L}^\infty(0, T : \mathbf{L}^1(\Omega))$ and the conclusion (25) can be properly written in the terms

$$\int_0^l |\mathbf{u}_x(t, x)| dx \leq \int_0^l |\mathbf{u}_{0x}(x)| dx + \int_0^t \int_0^l |\mathbf{f}_x(\tau, x)| dx d\tau.$$

Remark 2.3. The estimates in Theorems 2.1 and 2.2 (and the cases indicated in the above remarks) do not depend on l and remain valid for solutions of Cauchy’s problem

$$\begin{cases} \mathbf{u}_t - (\mathbf{A}(\mathbf{u})\mathbf{u}_x)_x + \mathbf{B}(\mathbf{u}) + \mathbf{C}(\mathbf{u}, \mathbf{u}_x) = \mathbf{f}(t, x), & t \in (0, T), \quad x \in \mathbb{R}, \\ \mathbf{u}(0, x) = \mathbf{u}_0(x), & x \in \mathbb{R}, \end{cases}$$

assuming that

$$\mathbf{f} \in \mathbf{L}^1(0, T : \mathbf{W}^{1,1}(\mathbb{R})) \quad \text{and} \quad \mathbf{u}_0 \in \mathbf{W}^{1,1}(\mathbb{R}).$$

The solution of the last problem can be obtained, for instance, as the limit of the following problem

$$\begin{cases} \mathbf{u}_t - (\mathbf{A}(\mathbf{u})\mathbf{u}_x)_x + \mathbf{B}(\mathbf{u}) + \mathbf{C}(\mathbf{u}, \mathbf{u}_x) = \mathbf{f}_l(t, x) & t \in (0, T), \quad x \in (-l, l), \\ \mathbf{u}(t, \pm l) = \mathbf{0} & t \in (0, T), \\ \mathbf{u}(0, x) = \mathbf{u}_{0,l}(x) & x \in (-l, l), \end{cases}$$

as $l \rightarrow \infty$, once we approximate $\mathbf{f}(t, x)$ and $\mathbf{u}_0(x)$ by $\mathbf{f}_l(t, x)$ and $\mathbf{u}_{0,l}(x)$ satisfying the natural corresponding assumptions. For other qualitative properties independent on l , see [10] and the general exposition made in [26].

Remark 2.4. Inequality (25) can be obtained by a different method, for $\Omega = \mathbb{R}$, when we know that the solutions satisfy a L^1 -contraction principle, i.e.

$$\int_{\mathbb{R}} |\mathbf{u}(t, x) - \mathbf{v}(t, x)| dx \leq \int_{\mathbb{R}} |\mathbf{u}_0(x) - \mathbf{v}_0(x)| dx + \int_0^t \int_{\mathbb{R}} |\mathbf{f}(\tau, x) - \mathbf{g}(\tau, x)| dx d\tau, \tag{38}$$

for any $t \in [0, T]$, where $\mathbf{v}(t, x)$ denotes the solution corresponding to the initial datum $\mathbf{v}_0(x)$ and source term $\mathbf{g}(t, x)$, that and that $\mathbf{u}(t, x + h)$ is also a solution, for any $h \in \mathbb{R}$. Indeed, it is enough to take $\mathbf{v}(t, x) = \mathbf{u}(t, x + h)$ in (38) and use the fact that

$$TV(\mathbf{u}(t, \cdot)) = \lim_{h \rightarrow 0} \frac{1}{h} \int_{\mathbb{R}} |\mathbf{u}(t, x) - \mathbf{u}(t, x + h)| dx$$

(see, [34]). This argument is typical of scalar (diffusive or not) conservation laws (see [40, 58] and other references in [54, 55]) but it can be applied for other kinds of equations leading to accretive operators in $\mathbf{L}^1(\mathbb{R})$ ([18]) or for classes of solutions satisfying the problem in some weaker sense: renormalized solutions ([20, 3]) and entropy solutions ([24, 59, 4]).

Remark 2.5. Although the first mathematical treatment of degenerate parabolic equations goes back to 1958, or even earlier (see, e.g., [45]) and it has been very extensively considered by many authors since then up to present days (see, e.g., the survey [38], the monographs [12, 57] and their long lists of references) it seems very curious to us that, as far as we know, quite simple uniform estimates as the one obtained in Theorem 2.2 were not included before in the literature. Perhaps, one of the reasons for this was the extraordinary importance given to the self-similar solution corresponding to the scalar equation (the so-called porous media equation)

$$u_t = (u^m)_{xx} \quad \text{in } Q = (0, T) \times \mathbb{R}. \tag{39}$$

Indeed, the so-called Barenblatt solution is given explicitly by the expression

$$u(t, x) = \frac{1}{t^{\frac{1}{m+1}}} \left(1 - \frac{(m-1)}{2m(m+1)} \frac{|x|^2}{t^{2/(m+1)}} \right)_+^{1/(m-1)}. \tag{40}$$

For many different purposes many authors devoted an important part of their research trying to get optimal gradient estimates of the form

$$|(u^m(t, x))_x| \leq Mu(t, x), \tag{41}$$

for any $(t, x) \in (0, T) \times \mathbb{R}$, which, in fact, become optimal for a suitable constant M , as it can be checked by means of (40) (see, e.g., [19] and its references dealing with Bernstein’s type arguments to prove (41)). Our point of view is slightly different since, this time, our motivation starts on systems of coupled degenerate equations involving some possible anisotropic coefficients which, in some cases, make impossible to find special solutions of self-similar type. Moreover, even for the case of scalar equations, it is clear that regularity (41) or $(u^{m-1})_x \in L^\infty(Q)$ is, in some sense, independent of the estimate obtained in Theorem 2.2. To see that, notice that if we denote by $[-l, +l]$ the support of $u(t, \cdot)$, for t fixed, we can write

$$\int_{-l}^l |u_x(t, x)| dx = \frac{1}{m-1} \int_{-l}^l \frac{1}{u(t, x)^{m-2}} |(u^{m-1}(t, x))_x| dx.$$

If we assume $(u^{m-1})_x \in L^\infty(Q)$ then the integrability of $u_x(t, x)$ is related to the integrability condition

$$\int_{-l}^l \frac{1}{u(t, x)^{m-2}} dx < +\infty. \tag{42}$$

But, at least in the case of the Barenblatt solution, $u(t, x)$ behaves like $|x \pm l|^{2/(m-1)}$ near the boundary of its support and we see that condition (42) holds if and only if $m \in (1, 3)$. We also point out that, even when the Bernstein technique is used to get sharper estimates of the type $(u^{m-1})_x \in L^\infty(Q)$ the obtained estimates are not of contraction type, in contrast with the estimate obtained in Theorem 2.2 (see, for instance, the collection of estimates obtained in [19]).

We end this section by giving a variant of Theorem 2.1 in which we get the uniform gradient estimate with an extra term.

Proposition 2.1. *Assume the same conditions as in Theorem 2.1, but assuming $\gamma_i \geq 1/2$ in (15). Then*

$$\int_0^l |u_{ix}(x, t)| dx \leq \int_0^l |u_{ix}(x, 0)| dx + \int_0^t \int_0^l |f_{ix}(x, \tau)| dx d\tau + \lambda_i \frac{C_{0i}^2}{a_0} l t, \tag{43}$$

for any $t \in [0, T]$, where $\lambda_i = 0$ if $\gamma_i > 1/2$ and $\lambda_i = 3/4$ if $\gamma_i = 1/2$.

Proof. The only change with respect to the proof of Theorem 2.1 concerns the estimate of I_3 for $\gamma_i = 1/2$. Now, by Young’s inequality we get that

$$|I_3| \leq \frac{1}{3} \int_0^t \int_0^l A_{ii} \frac{\delta^2 u_{ixx}^2}{(\delta^2 + u_{ix}^2)^{\frac{3}{2}}} dx d\tau + \frac{3}{4} \int_0^t \int_0^l \frac{C_{0i}^2}{A_{ii}} dx d\tau,$$

and so the conclusion holds by using (13) and passing to the limit as before. \square

3. Some Applications

3.1. A diffusively coupled nonlinear system in gas dynamics

As mentioned in the Introduction, one of the motivations of this work started with the preparation of our articles [6, 7] where we considered the following system of (possibly degenerate) equations: given the domain $Q := \{(\psi, x) \in \mathbb{R}^2 : 0 < \psi < l, 0 < x < X\}$, for some arbitrarily fixed $l, X > 0$, and given the constants $\sigma \in (0, 1), Pr > 0, G \geq 0, \varepsilon \geq 0$ and $\delta \geq 0$, find a solution (u, T) of the system

$$\begin{cases} u_x - (A_{11}(u, T)u_\psi)_\psi + B_1(u, T) = 0, \\ T_x - (A_{22}(u, T)T_\psi)_\psi = 0, \end{cases} \tag{44}$$

with the boundary conditions

$$\begin{cases} u_\psi(0, x) = 0, & T_\psi(0, x) = 0 & 0 < x < X, \\ u(l, x) = \mu, & T(l, x) = \varepsilon & 0 < x < X, \end{cases} \tag{45}$$

and the “initial” conditions

$$\begin{cases} u(\psi, 0) = u_0(\psi), & 0 \leq \psi \leq l, \\ T(\psi, 0) = T_0(\psi), & 0 \leq \psi \leq l, \end{cases} \tag{46}$$

where

$$A_{11}(u, T) = T^{\sigma-1}u, \quad A_{22}(u, T) = (Pr)^{-1}T^{\sigma-1}u, \quad B_1(u, T) = -\frac{G}{u}(T - \varepsilon), \tag{47}$$

and we assume that, at least,

$$u_0, T_0 \in W^{1,1}(0, l), \quad u_0(\psi) \geq \mu \quad \text{and} \quad 1 \geq T_0(\psi) \geq \varepsilon \quad \text{on} \quad 0 \leq \psi \leq l. \tag{48}$$

Here $Pr > 0$ is the Prandtl number and $G > 0$ is the Froude number. System (44) is of parabolic type. Therefore condition (46) looks like an initial condition if we

understand the variable x as a “fictitious” time. Problems (44)–(46) was obtained, using von Mises’ transformation, from a mathematical model which describes the discharge of a laminar hot gas into the stagnant colder atmosphere of the same gas (see [46–48, 51]).

The existence and uniqueness of classical solutions of the nondegenerate case (corresponding to the assumptions $\mu > 0$ and $\varepsilon > 0$) were proved in [6, 7]. These solutions satisfy the *a priori* inequalities

$$0 < \mu \leq u \leq C(G, X), \quad 0 < \varepsilon \leq T \leq 1 \quad (\text{with } C(0, X) = 1). \quad (49)$$

Note if $G > 0$ we cannot apply directly Theorem 2.1 since $B_1(u, T)$ does not satisfy condition (14). Nevertheless, arguing as in the proof of Theorem 2.1 we can derive similar L^1 -gradient estimates.

Proposition 3.1. *Assume (47) with $G \geq 0, \mu > 0$ and $\varepsilon > 0$. Then if (u, T) is a strong solution of (44)–(46) we have*

$$\int_0^l |T_\psi(\psi, x)| d\psi \leq \int_0^l |T_{0\psi}(\psi)| d\psi. \quad (50)$$

Moreover, if $G = 0$ then

$$\int_0^l |u_\psi(\psi, x)| d\psi \leq \int_0^l |u_{0\psi}(\psi)| d\psi.$$

Finally, if $G > 0$, then

$$\int_0^l |u_\psi(\psi, x)| d\psi \leq \int_0^l |u_{0\psi}(\psi)| d\psi + (\exp(G/\mu^2) - 1) \int_0^l |T_{0\psi}(\psi)| d\psi. \quad (51)$$

Proof. The two first estimates result from a direct application of Theorem 2.1. If $G > 0, \mu > 0$ and $\varepsilon > 0$ by multiplying the first equation of (44) by $\frac{\partial S_\delta(u_\psi)}{\partial \psi}$ and integrating over Q we get the identity

$$\int_0^l N_\delta(u_\psi(\psi, x)) d\psi + I_1 = \int_0^l N_\delta(u_\psi(\psi, 0)) d\psi + I_2 + I_3,$$

where

$$I_1 := \int_0^x \int_0^l A_{11} \frac{\delta^2 u_{\psi\psi}^2}{(\delta^2 + u_\psi^2)^{\frac{3}{2}}} d\psi d\xi,$$

$$I_2 := - \int_0^x \int_0^l \left[\frac{\partial A_{11}}{\partial T} T_\psi + \frac{\partial A_{11}}{\partial u} u_\psi \right] u_\psi \frac{\delta^2 u_{\psi\psi}^2}{(\delta^2 + u_\psi^2)^{\frac{3}{2}}} d\psi d\xi,$$

and

$$I_3 := - \int_0^x \int_0^l \frac{G}{u} (T - \varepsilon) \frac{\partial S_\delta(u_\psi)}{\partial \psi} d\psi d\xi.$$

Integrating by parts and using the boundary conditions (45) we can rewrite I_3 in the form

$$I_3 = \int_0^x \int_0^l S_\delta(u_\psi) G \left[-\frac{T-\varepsilon}{u^2} u_\psi + \frac{1}{u} T_\psi \right] d\psi d\xi.$$

Using (49) and assuming $0 < \mu < 1$, we evaluate I_3 in the following way

$$|I_3| \leq \frac{G}{\mu^2} \int_0^x \int_0^l (|u_\psi| + |T_\psi|) d\psi d\xi.$$

Repeating the arguments of Theorem 2.1, we get the inequality

$$\int_0^l |u_\psi(\psi, x)| d\psi \leq \int_0^l |u_{0\psi}(\psi)| d\psi + \frac{G}{\mu^2} \int_0^x \int_0^l (|u_\psi| + |T_\psi|) d\psi d\xi.$$

Applying Gronwall's inequality and (50), we get (51). □

Remark 3.1. Similar estimates hold for the limit cases $\varepsilon = 0$ and $\mu = 0$ but now stated in terms of the total variation of the associate weak solutions (u, T) (see [8]).

3.2. An one-dimensional two phase filtration equation

It is well known (see, e.g., [5, 11–14, 35]), that the one-dimensional two-phase filtration in a porous medium model can be described by the degenerate parabolic equation

$$s_t = (a(s)s_x + V(t)b(s))_x, \quad (t, x) \in Q = (0, T) \times (0, l), \tag{52}$$

with the boundary and initial conditions

$$s(0, x) = s_0(x), \quad x \in (0, l), \tag{53}$$

$$s(t, 0) = \mu > 0, \quad s(t, l) = (1 - \varepsilon) > 0, \quad t \in (0, T). \tag{54}$$

Here $a, b \in C^1[0, 1]$ and $V \in C[0, T]$ are given functions such that

$$0 < a_0(\mu, \varepsilon) \leq a(s) \leq 1, \quad s \in [\mu, 1 - \varepsilon] \subset (0, 1), \quad a(s) = 0, \\ \text{for } s = 0 \quad \text{and} \quad s = 1, \tag{55}$$

and

$$|V(t)| \leq C < \infty \quad \text{for any } t \in [0, T]. \tag{56}$$

We assume, at least, that

$$s_0 \in W^{1,1}(0, l), \quad s_0(x) \in [\mu, 1 - \varepsilon] \quad \text{for any } x \in (0, l). \tag{57}$$

The existence and uniqueness of classical solutions of the nondegenerate problem (corresponding to the assumptions $\mu > 0$ and $\varepsilon > 0$) were proved in [14, 13]. Note that, again, we cannot apply directly Theorem 2.1 due to the degeneracy of $a(s)$

and the presence of the term $V(t)b(s)$. Nevertheless, we can adapt the arguments of Theorem 2.1 to prove:

Proposition 3.2. *Assume (55)–(57). Let $s(t, x)$ be the classical solution of (52)–(54). Then, for any $t \in (0, T)$ we have*

$$\int_0^l |s_x(t, x)| dx \leq \int_0^l |s_{0x}(x)| dx.$$

Proof. Multiplying (52) by $\frac{\partial S_\delta(s_x)}{\partial x}$ and integrating over Q , we get the identity

$$\int_0^l N_\delta(s_x(t, x)) dx + I_1 = \int_0^l N_\delta(s_{0x}(x)) dx + I_2 + I_3,$$

where

$$I_1 := \int_0^t \int_0^l a \frac{\delta^2 s_{xx}^2}{(\delta^2 + u_{ix}^2)^{\frac{3}{2}}} dx d\tau, \quad I_2 := - \int_0^t \int_0^l a'_s \frac{s_x^2 \delta^2 s_{xx}}{(\delta^2 + s_x^2)^{\frac{3}{2}}} dx d\tau,$$

and

$$I_3 := - \int_0^t \int_0^l V b'_s \frac{s_x \delta^2 s_{xx}}{(\delta^2 + s_x^2)^{\frac{3}{2}}} dx d\tau.$$

But we have the estimates

$$\begin{aligned} |I_2| &\leq \frac{1}{2} \int_0^t \int_0^l a \frac{\delta^2 s_{xx}^2}{(\delta^2 + s_x^2)^{\frac{3}{2}}} dx d\tau + \frac{1}{2} \int_0^t \int_0^l \frac{a'^2}{a} \frac{\delta^2 s_x^4}{(\delta^2 + s_x^2)^{\frac{3}{2}}} dx d\tau \\ &\leq \frac{I_1}{2} + \frac{\delta}{2} \int_0^t \int_0^l \frac{a'^2}{a} s_x^2 dx d\tau, \end{aligned}$$

and

$$|I_3| \leq \frac{I_1}{2} + \frac{1}{2} \int_0^t \int_0^l V^2 \frac{b_s'^2}{a} \frac{\delta^2 s_x^2}{(\delta^2 + s_x^2)^{\frac{3}{2}}} dx d\tau \leq \frac{I_1}{2} + \frac{\delta}{2} \int_0^t \int_0^l V^2 \frac{b_s'^2}{a} dx d\tau,$$

which lead to

$$\int_0^l N_\delta(s_x(t, x)) dx \leq \int_0^l N_\delta(s_{0x}(x)) dx + \frac{\delta}{2} \int_0^t \int_0^l \left(\frac{a'^2}{a} s_x^2 + V^2 \frac{b_s'^2}{a} \right) dx d\tau.$$

Moreover, it is not difficult to get an estimate of the last integral in the right hand side of the above inequality. Indeed, let

$$I := \frac{1}{2} \int_0^t \int_0^l \left(\frac{a'^2}{a} s_x^2 + V^2 \frac{b_s'^2}{a} \right) dx d\tau. \tag{58}$$

Then

$$I \leq \frac{\max_{s \in [0,1]} [a'^2, b'^2] + \max_{t \in [0,T]} V^2}{a_0(\mu, \varepsilon)} \int_0^t \int_0^l (s_x^2 + 1) dx d\tau. \tag{59}$$

However, s is a classical solution (recall that $\mu > 0, \varepsilon > 0$). Then

$$\int_0^t \int_0^l (s_x^2 + 1) dx d\tau \leq C(T, \mu, \varepsilon) \tag{60}$$

and thus

$$I \leq \frac{1}{2} \frac{\max_{s \in [0,1]} [a'^2, b'^2] + \max_{t \in [0,T]} V^2}{a_0(\mu, \varepsilon)} C(T, \mu, \varepsilon) \leq \tilde{C}(T, \mu, \varepsilon), \tag{61}$$

for a constant $\tilde{C}(T, \mu, \varepsilon)$ which does not depend on δ . Passing to the limit, as $\delta \rightarrow 0$, we get the desired estimate. \square

3.3. A p -Laplacian parabolic type equation

The arguments of Theorems 2.1 and 2.2 can be used to get some new second order estimates for solutions to the problem of p -Laplacian type

$$\begin{cases} u_t - (\Phi(u_x))_x = f(t, x) & t \in (0, T), \quad x \in (-l, l), \\ u_x(t, \pm l) = 0 & t \in (0, T), \\ u(0, x) = u_0(x) & x \in (-l, l), \end{cases} \tag{62}$$

where Φ is a continuous increasing real function. The existence of a unique weak solution, under this general condition on Φ , is a well known result (see, e.g., [29]).

Proposition 3.3. *Let u be the weak solution of (62) and assume that $u_0 \in W^{2,1}(-l, l)$ and $f \in L^1(0, T : W^{2,1}(-l, l))$. Then, $u_x \in L^\infty(0, T : BV(-l, l))$ and*

$$\|u_x\|_{L^\infty(0, T : BV(-l, l))} \leq \|u_{0xx}\|_{L^1(-l, l)} + \|f_{xx}\|_{L^1(0, T : L^1(-l, l))}. \tag{63}$$

Proof. It suffices to use the fact that the function $v = u_x$ is the unique weak solution of the problem

$$\begin{cases} v_t - \Phi(v)_{xx} = f_x(t, x) & t \in (0, T), \quad x \in (-l, l), \\ v(t, \pm l) = 0 & t \in [0, T], \\ v(0, x) = u_{0x}(x) & x \in (-l, l), \end{cases} \tag{64}$$

and then use the variant of Theorem 2.2 given in Remark 2.1 after a standard approximation of solutions of (64) by classical solutions of some associate problems (see, e.g., [36]). \square

Remark 3.2. When $\Phi(r) = |r|^{p-2}r$, for some $p > 1$, we get the one-dimensional p -Laplacian operator. Although there are many regularity results for p -Laplacian parabolic equations (see, e.g., [42, 29]) it seems that the regularity $L^\infty(0, T : W^{2,1}(-l, l))$ was not proved before in the literature. Notice that the regularity and the estimate given by (63) are new even in the class of “strong solutions” (satisfying that $u_t, (\Phi(u_x))_x \in L^1(Q)$): see, e.g., [4] and its references. Some references on the regularity ($|u_x|^{p-2}u_x \in L^q$) can be found, for instance, in [30].

Remark 3.3. We point out that, although we shall not develop it here, the arguments of Theorems 2.1 and 2.2 can be applied, with some slight modifications, to

the solutions of (62) leading, in this way, to uniform L^1 -gradient estimates (see a close point of view in [32, 59]). We also point out that other different uniform gradient estimates, obtained by Bernstein type arguments, are known in the literature (see, e.g., [31, 33, 53] and their references).

3.4. Hamilton–Jacobi and conservation laws

Consider the first order partial differential equation

$$\begin{cases} \mathbf{u}_t + \mathbf{C}(\mathbf{u}, \mathbf{u}_x) = \mathbf{f}(t, x), & t \in (0, T), \quad x \in \mathbb{R}, \\ \mathbf{u}(0, x) = \mathbf{u}_0(x), & x \in \mathbb{R}, \end{cases} \quad (65)$$

with

$$\mathbf{f} \in \mathbf{L}^1(0, T : \mathbf{W}^{1,1}(\mathbb{R})) \quad \text{and} \quad \mathbf{u}_0 \in \mathbf{W}^{1,1}(\mathbb{R}).$$

We also assume that:

$$\begin{cases} \mathbf{u} \text{ is the limit, when } \varepsilon \rightarrow 0, \text{ of solutions } \mathbf{u}_\varepsilon \text{ of} \\ \mathbf{u}_t - \varepsilon \mathbf{u}_{xx} + \mathbf{C}(\mathbf{u}, \mathbf{u}_x) = \mathbf{f}(t, x) & t \in (0, T), \quad x \in \mathbb{R}, \\ \mathbf{u}(0, x) = \mathbf{u}_0(x) & x \in \mathbb{R}. \end{cases} \quad (66)$$

Then as a direct consequence of Theorem 2.2 and Remark 2.4, we get

Proposition 3.4. *Let us assume (66) and conditions (15) or (16). Then*

$$TV(\mathbf{u}(t, \cdot)) \leq TV(\mathbf{u}_0(\cdot)) + \int_0^t TV(\mathbf{f}(\tau, \cdot)) d\tau.$$

Remark 3.4. As noticed in the Introduction, assumption (66) was first proved in [37] for Burgers’ equation. Since then, many other results are available in the literature (see, e.g., the monographs [43, 55, 16, 54]). We also point out that this was obtained in [15, 25] for the case of some scalar conservation laws equations on a bounded domain when a suitable boundary condition is added to the equation jointly to the initial condition. For the case of Hamilton–Jacobi equations on a bounded domain see [39, 43, 16].

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