DISCRETE AND CONTINUOUS
DYNAMICAL SYSTEMS
Volume **27**, Number **3**, July **2010**

doi:10.3934/dcds.2010.27.1037

pp. **1037–1058**

ON VERY WEAK SOLUTIONS OF SEMI-LINEAR ELLIPTIC EQUATIONS IN THE FRAMEWORK OF WEIGHTED SPACES WITH RESPECT TO THE DISTANCE TO THE BOUNDARY

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(Communicated by Roger Temam)

Abstract. We prove the existence of an appropriate function (very weak solution) u in the Lorentz space $L^{N',\infty}(\Omega)$, $N' = \dfrac{N}{N-1}$ satisfying $Lu - Vu + g(x,u,\nabla u) = \mu$ in Ω an open bounded set of $\mathbb{R}^N$, and $u = 0$ on $\partial\Omega$ in the sense that

$$(u, L\varphi)_0 - (Vu, \varphi)_0 + \big(g(\cdot, u, \nabla u), \varphi\big)_0 = \mu(\varphi), \quad \forall \varphi \in C_c^2(\Omega).$$

The potential $V \leqslant \lambda < \lambda_1$ is assumed to be in the weighted Lorentz space $L^{N,1}(\Omega,\delta)$, where $\delta(x) = \operatorname{dist}(x, \partial\Omega)$, $\mu \in M^1(\Omega,\delta)$, the set of weighted Radon measures containing $L^1(\Omega,\delta)$, L is an elliptic linear self adjoint second order operator, and λ_1 is the first eigenvalue of L with zero Dirichlet boundary conditions.

If $\mu \in L^1(\Omega,\delta)$ we only assume that for the potential V is in $L^1_{loc}(\Omega)$, $V \leqslant \lambda < \lambda_1$. If $\mu \in M^1(\Omega, \delta^\alpha)$, $\alpha \in [0,1[$, then we prove that the very weak solution $|\nabla u|$ is in the Lorentz space $L^{\frac{N}{N-1+\alpha},\infty}(\Omega)$. We apply those results to the existence of the so called large solutions with a right hand side data in $L^1(\Omega,\delta)$. Finally, we prove some rearrangement comparison results.

1. **Notations and preliminary results.** We shall always consider $\Omega \subset \mathbb{R}^N$, $N \geqslant 2$, a bounded open set of class $C^{2,1}$. For any measurable set $E \subset \mathbb{R}^N$ we shall denote by $|E|$ its Lebesgue measure.

We shall consider a linear operator L :

$$Lu = -\sum_{i,j=1}^{N} \partial_i(a_{ij}(x)\partial_j u).$$

We assume that $a_{ij} = a_{ji} \in C^{0,1}(\overline{\Omega})$, $\forall \xi = (\xi_1, \dots, \xi_N) \in \mathbb{R}^N$, and that L is elliptic in the sense that

$$\sum_{i,j} a_{ij}(x)\xi_i\xi_j \geqslant \alpha_0|\xi|^2 \text{ for some } \alpha_0 > 0.$$

2000 *Mathematics Subject Classification.* 35J25, 35J60, 35P30, 35J67.

Key words and phrases. Very weak solutions; semilinear elliptic equations; distance to the boundary; weighted spaces measure; unbounded potentials.

We introduce the following notations : $\delta(x) = \text{distance}(x, \partial\Omega)$. We recall that :

- $L^1(\Omega, \delta^\alpha) = \Big\{ f : \Omega \to I\!R$ Lebesgue measurable $: \int_\Omega |f(x)|\delta(x)^\alpha \,\mathrm{d}x$ is finite $\Big\}$,
- the *decreasing rearrangement of a measurable function* u is given by

$$u_* : \Omega_* =]0, |\Omega|[\to I\!R, \quad u_*(s) = \inf\{t \in I\!R : |u > t| \leqslant s\}$$
$$u_*(0) = \operatorname*{ess\,sup}_\Omega u, \qquad u_*(|\Omega|) = \operatorname*{ess\,inf}_\Omega u;$$

- the *decreasing radial rearrangement of the function* u is defined, on the ball $\widetilde{\Omega}$ having the same measure as Ω, by

$$\underset{\sim}{u} : \widetilde{\Omega} \to I\!R, \quad \underset{\sim}{u}(x) = u_*(\alpha_N |x|^N) \qquad \text{with } \alpha_N = |B_1(0)|;$$

- the *increasing rearrangement of a measurable function* u is given by

$$u^* : \Omega_* \to I\!R, \quad u^*(s) = u_*(|\Omega| - s),\ s \in]0, |\Omega|[;$$

- the *increasing radial rearrangement of the function* u is defined by

$$\widetilde{u} : \widetilde{\Omega} \to I\!R, \quad \widetilde{u}(x) = u^*(\alpha_N |x|^N).$$

We shall use the following Lorentz spaces (see e.g. [25], [4]),

$$L^{p,q}(\Omega) = \Big\{ v : \Omega \to I\!R \text{ measurable } |v|^q_{L^{p,q}} = \int_0^{|\Omega|} [t^{\frac{1}{p}} |v|_{**}(t)]^q \frac{dt}{t} < +\infty \Big\},$$

for $1 < p < +\infty,\ 1 \leqslant q < +\infty$ and

$$L^{p,\infty}(\Omega) = \Big\{ v : \Omega \to I\!R \text{ measurable } |v|_{L^{p,\infty}} = \sup_{t \leqslant |\Omega|} t^{\frac{1}{p}} |v|_{**}(t) < +\infty \Big\},$$

for $q = +\infty$ where $|v|_{**}(t) = \dfrac{1}{t} \displaystyle\int_0^t |v|_*(s) ds$ for $t \in \Omega_* =]0, |\Omega|[$.

Sometimes, we shall denote by $|v|_{p,q}$ instead of $|v|_{L^{p,q}}$ the norm in $L^{p,q}(\Omega)$.

We recall that $L^{p,q}(\Omega) \subset L^{p,p}(\Omega) = L^p(\Omega)$ for any $p \geqslant q \geqslant 1$.

We denote by $\partial_i = \dfrac{\partial}{\partial x_i}$, $\partial_{ij} = \dfrac{\partial^2}{\partial x_i \partial x_j}$. We define the following Sobolev-Lorentz type spaces

$$W^1(\Omega, |\cdot|_{p,q}) = \Big\{ v \in W^{1,1}(\Omega) : |\nabla v| \in L^{p,q}(\Omega) \Big\},$$

and

$$W^2(\Omega, |\cdot|_{p,q}) = \Big\{ v \in W^{2,1}(\Omega) : \partial_{ij} v \in L^{p,q}(\Omega) \text{ for } (i,j) \in \{1, \dots, N\}^2 \Big\}.$$

We define

$$H^1_c(\Omega) = \Big\{ v \in H^1(\Omega) : \text{ support } v \text{ is compact in } \Omega \Big\}.$$

We shall denote by c various constants depending only on the data. Moreover, the notation $\approx$ stands for equivalence of nonnegative quantities, that is

$$\Lambda_1 \approx \Lambda_2 \iff \exists\, c_1 > 0,\ c_2 > 0 \text{ such that } c_1 \Lambda_2 \leqslant \Lambda_1 \leqslant c_2 \Lambda_2.$$

We recall the following results obtained by the authors in [15] :

Theorem 1.1. *Let* $f \in L^1(\Omega, \delta)$ *and* $N' = \dfrac{N}{N-1}$. *Then there exists a unique function* $v \in L^{N',\infty}(\Omega)$ *satisfying*

$$(DG_L(\Omega)) : \int_\Omega v L\varphi \,\mathrm{d}x = \int_\Omega f\varphi \,\mathrm{d}x, \qquad \forall \varphi \in W^2(\Omega, |\cdot|_{N,1}) \cap H^1_0(\Omega).$$

Moreover, there exists a constant $c(\Omega, L) > 0$ *such that*

$$|v|_{L^{N',\infty}} \leqslant c(\Omega, L)|f|_{L^1(\Omega,\delta)}. \tag{1}$$

Lemma 1.2. *There exist a function* $\varphi_1 \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ *and* $\lambda_1 > 0$, $\forall p \in]1, +\infty[$ *satisfying*

$$\begin{cases} L\varphi_1 = \lambda_1\varphi_1 & \text{in } \Omega, \\ \varphi_1 = 0 & \text{on } \partial\Omega. \end{cases}$$

Moreover, there are two constants $c_1 > 0,\ c_2 > 0$ *such that*

$$c_1\delta(x) \leqslant \varphi_1(x) \leqslant c_2\delta(x) \quad \forall x \in \Omega.$$

Theorem 1.3. *The unique generalized function* v *given in Theorem 1.1 belongs to* $W^{1,q}(\Omega, \delta)$ *for any* $1 \leqslant q < \dfrac{2N}{2N-1}$.

Theorem 1.4. *Let* v *be the unique solution of* $(DG_L(\Omega))$ *given in Theorem 1.1. If* $f \in L^1(\Omega, \delta^\alpha)$ *for some* $\alpha \in [0, 1[$ *then* $|\nabla v| \in L^{\frac{N}{N-1+\alpha},\infty}(\Omega)$. *Moreover, there exists* $c(\alpha, \Omega, L) > 0$ $|\nabla v|_{L^{\frac{N}{N-1+\alpha}},\infty} \leqslant c(\alpha, \Omega, L)|f|_{L^1(\Omega,\delta^\alpha)}$.

In which follows we shall need the following definitions :

$$C_c(\Omega) = \Big\{v : \Omega \to I\!R, \text{ continuous with compact support in } \Omega\Big\}$$

which is a Frechet space and we shall denote by $M(\Omega)$ its dual space. That is the set of all measure μ continuous on the Frechet space $C_c(\Omega)$. When we endow $C_c(\Omega)$ with the norm

$$|u|_\infty = \operatorname*{Max}_{x\in\Omega} |u(x)|,\ u \in C_c(\Omega),$$

then we shall denote by $M^1(\Omega)$ the dual of $\Big(C_c(\Omega), |\cdot|_\infty\Big)$. For $\mu \in M^1(\Omega)$, we denote its norm by

$$|\mu|_* = \sup\Big\{ < \mu, \varphi >,\ |\varphi|_\infty \leqslant 1,\ \varphi \in C_c(\Omega)\Big\}.$$

We shall say that a sequence $(\mu_n)_n \subset M(\Omega)$ converges to μ weakly if

$$< \mu_n, \varphi > \xrightarrow[n\to+\infty]{} < \mu, \varphi >, \quad \forall \varphi \in C_c(\Omega).$$

We shall introduce the following weighted Radon measure space, for $\alpha \in [0, 1]$

$$M^1(\Omega, \delta^\alpha) = \Big\{\mu \in M(\Omega) : \delta^\alpha\mu \in M^1(\Omega)\Big\}.$$

We can define a norm on it by setting

$$||\mu||_{*,\delta^\alpha} = |\delta^\alpha\mu|_*, \quad \mu \in M^1(\Omega, \delta^\alpha).$$

Remark 1. Similar spaces have been considered in [28]. By a straightforward computation one can check that all results stated here are true for $N = 1$.

The following properties follow from the definitions

Property 1. a) $L^1(\Omega, \delta^\alpha) \subset\!\!\!> M^1(\Omega, \delta^\alpha)$.

b) $\forall \mu \in M^1(\Omega, \delta^\alpha)$, there exists a sequence of $(f_n)_n \subset C_c(\Omega)$ such that f_n remains in a bounded set of $L^1(\Omega, \delta^\alpha)$ and $f_n \rightharpoonup \mu$ weakly in $M(\Omega)$. More precisely

$$|f_n|_{L^1(\Omega,\delta^\alpha)} \leqslant ||\mu||_{*,\delta^\alpha}.$$

Proof.

a) If $f \in L^1(\Omega, \delta^\alpha)$, $< \delta^\alpha f, \varphi >= \int_\Omega f \delta^\alpha \varphi \, \mathrm{d}x \leqslant |\varphi|_\infty \cdot \int_\Omega |f| \delta^\alpha \, \mathrm{d}x$

$$\forall \varphi \in C_c(\Omega) \text{ so } \delta^\alpha f \in M^1(\Omega).$$

b) If $\mu \in M^1(\Omega, \delta^\alpha)$, then there is a sequence $(g_n)_{n \geqslant 0} \subset C_c^\infty(\Omega)$, remaining in a bounded set of $L^1(\Omega)$ so that $g_n \rightharpoonup \mu \delta^\alpha$ in $M(\Omega)$ and $|g_n|_{L^1(\Omega)} \leqslant |\delta^\alpha \mu|_*$. Thus $f_n = g_n \delta^{-\alpha}$ remains in a bounded set of $L^1(\Omega, \delta^\alpha)$, $f_n \rightharpoonup \mu$ weakly in $M(\Omega)$ and $(f_n)_n \subset C_c(\Omega)$ with $|f_n|_{L^1(\Omega, \delta^\alpha)} \leqslant ||\mu||_{*, \delta^\alpha}$.

□

2. **Some existence results for linear and semilinear equations with an unbounded potential V and a weighted Radon measure.**

2.1. **The linear case with an unbounded potential.** For convenience and clarity, we shall begin by the following linear equation with an unbounded potential.

Theorem 2.1. *Let $V \in L^1_{loc}(\Omega)$, with $V \leqslant \lambda < \lambda_1$, λ_1 being the first eigenvalue of L. Then, for any $f \in L^1(\Omega, \delta)$, there exists a unique function $u \in L^{N', \infty}(\Omega) \cap W^{1,q}(\Omega, \delta)$, for any $1 \leqslant q < \dfrac{2N}{2N-1}$, satisfying*

$$Vu \in L^1(\Omega, \delta), \text{ and } \int_\Omega u L\varphi \, \mathrm{d}x - \int_\Omega V u \varphi \, \mathrm{d}x = \int_\Omega f \varphi \, \mathrm{d}x \; \forall \, \varphi \in W^2(\Omega, |\cdot|_{N,1}) \cap H^1_c(\Omega).$$

Moreover there are constants $c = c(\Omega, L, V, q) > 0$ such that

1. $|Vu|_{L^1(\Omega, \delta)} \leqslant c |f|_{L^1(\Omega, \delta)}$,
2. $|u|_{L^{N', \infty}(\Omega)} \leqslant c |f|_{L^1(\Omega, \delta)}$,
3. $\displaystyle\int_\Omega |\nabla u|^q \delta(x) \, \mathrm{d}x \leqslant c |f|^{\frac{q}{2}}_{L^1(\Omega, \delta)} \Big(1 + |f|^{N'}_{L^1(\Omega, \delta)}\Big)^{1 - \frac{q}{2}}$.

Finally, if $f \in L^1(\Omega, \delta^\alpha)$ for some $\alpha \in [0, 1[$ and $V \in L^{N,1}(\Omega, \delta^\alpha)$ then

$$u \in W^1_0\Big(\Omega, |\cdot|_{\frac{N}{N-1+\alpha}, \infty}\Big) = \Big\{ v \in W^{1,1}_0(\Omega) : |\nabla v| \in L^{\frac{N}{N-1+\alpha}, \infty}(\Omega) \Big\}$$

and

$$|\nabla u|_{L^{\frac{N}{N-1+\alpha}, \infty}} \leqslant c |f|_{L^1(\Omega, \delta^\alpha)}.$$

Remark 2. In contrast with some previous results (see [28], [10]), the above result shows a regularity in a unweighed space as $L^{q, \infty}$, $q > 1$. Moreover, we may replace $H^1_c(\Omega)$ by $H^1_0(\Omega)$ if $V \in L^1(\Omega, \delta)$.

Proof. Since $V \leqslant \lambda$ thus $V_+ \leqslant \lambda$, where V_+ denotes the positive part of V. We can truncate the negative part of V, V_- : $T_k(V_-) \doteq V_{k-}$ by means of

$$T_k(\sigma) = \min\Big(k, \max(-k, \sigma)\Big), \; \sigma \in I\!R, \; k \geqslant 1,$$

and we consider, $u_k \in W^2(\Omega, |\cdot|_{N,1}) \cap H^1_0(\Omega)$, solution of

$$L u_k + V_{k-} u_k - V_+ u_k = f_k = T_k(f). \tag{2}$$

Multiplying by $\varphi_1 p_\varepsilon(u_k)$ with $p_\varepsilon(s)$ an appropriate regular approximation of the function $\operatorname{sign}(s)$, we get (after passing to the limit in a standard way for a sequence p_ε as $\varepsilon \to 0$ and using the fact that $L\varphi_1 = \lambda_1 \varphi_1$, $\varphi_1 \in H^1_0(\Omega)$),

$$(\lambda_1 - \lambda) \int_\Omega |u_k(x)| \varphi_1(x) \, \mathrm{d}x + \int_\Omega V_{k-} \varphi_1 |u_k| \, \mathrm{d}x \leqslant \int_\Omega |f_k(x)| \varphi_1(x) \, \mathrm{d}x.$$

From Lemma 1.2, there exists a constant $c > 0$ such that

$$\int_\Omega V_{k-}|u_k|\delta\,\mathrm{d}x + \int_\Omega |u_k(x)|\delta(x)\,\mathrm{d}x \leqslant c\int_\Omega |f(x)|\delta(x)\,\mathrm{d}x \qquad \forall\, k \geqslant 1. \tag{3}$$

We deduce from Theorem 1.1, that

$$|u_k|_{L^{N',\infty}} \leqslant c(\Omega, L)|V_+u_k - V_{k-}u_k + f_k|_{L^1(\Omega,\delta).} \tag{4}$$

Using relation (3), relation (4) becomes

$$|u_k|_{L^{N',\infty}} \leqslant c(\Omega, L)|f|_{L^1(\Omega,\delta)}. \tag{5}$$

Therefore, relation (5) and Theorem 1.3, for $q \in \left[1, \dfrac{2N}{2N-1}\right[$, lead to

$$\int_\Omega |\nabla u_k|^q\delta(x)\,\mathrm{d}x \leqslant c_q(\Omega, L)\cdot|f|^{\frac{q}{2}}_{L^1(\Omega,\delta)}\left(1 + |f|^{N'}_{L^1(\Omega,\delta)}\right)^{1-\frac{q}{2}}. \tag{6}$$

□

We shall need the

Lemma 2.2. *The embedding of $W^{1,1}(\Omega,\delta)\cap L^{N',\infty}(\Omega)\doteq X_\delta$ into $L^{q,1}(\Omega)$ is compact for all $q < N'$.*

Proof of Lemma 2.2 Fix $1 < q < N'$, then for any bounded sequence (v_k) in X_δ, there exists a function $v \in L^q(\Omega)$ and a subsequence, still denoted by (v_k) such that $v_k \rightharpoonup v$ weakly in L^q. Knowing that $W^{1,1}(\Omega,\delta)\cap L^q(\Omega) \underset{c}{\hookrightarrow} L^1(\Omega,\delta)$ is compact (see e.g. [24]), we derive that $v_k \to v$ in $L^1(\Omega,\delta)$. By subtracting a subsequence, we can assume that $v_k(x) \xrightarrow[k\to+\infty]{} v(x)$ for a.e. $x \in \Omega$. Since $L^{N',\infty}(\Omega)$ is a continuous embedding in $L^{r,1}(\Omega)$ for all $r < N'$, we get the result. □

Continuation of the proof of Theorem 2.1 Applying the Lemma 2.2, there exists a subsequence (u_k) and a function $u \in W^{1,q}(\Omega,\delta)\cap L^{N',\infty}(\Omega)$ such that

1. $u_k(x) \to u(x)$ a.e. in Ω.
2. $u_k \rightharpoonup u$ weakly in $W^{1,q}(\Omega,\delta)$.
3. $u_k \to u$ strongly in $L^{q,1}(\Omega)$ $\forall\, q < N'$.

Moreover from (3) to (6) we have

$$\int_\Omega V_-|u|\delta\,\mathrm{d}x + \int_\Omega |u(x)|\delta(x)\,\mathrm{d}x \leqslant c\int_\Omega |f(x)|\delta(x)\,\mathrm{d}x. \tag{7}$$

$$|u|_{L^{N',\infty}(\Omega)} \leqslant c(\Omega,\delta)|f|_{L^1(\Omega,\delta)}. \tag{8}$$

$$\int_\Omega |\nabla u|^q\delta(x)\,\mathrm{d}x \leqslant c|f|^{\frac{q}{2}}_{L^1(\Omega,\delta)}\left(1 + |f|^{N'}_{L^1(\Omega,\delta)}\right)^{1-\frac{q}{2}}. \tag{9}$$

In particular, we have

$$(V_{k-}u_k)(x) \to V_-(x)u(x) \text{ a.e. } x \in \Omega.$$

By arguing in a similar way to [18], we have :

Lemma 2.3. *$V_{k-}u_k\delta$ converges weakly to $V_-u\delta$ in $L^1_{loc}(\Omega)$ and weakly in $L^1(\Omega)$ if $V_- \in L^1(\Omega,\delta)$.*

Proof of Lemma 2.3 Let $t \in I\!R_+$. Consider a sequence of functions

$$\gamma_m \in C^1(I\!R) \cap W^{1,\infty}(I\!R) \text{ such that } \begin{cases} \gamma_m(0) = 0, & \\ \gamma_m'(s) \geqslant 0 & \forall\, s \in I\!R, \\ \gamma_m(s) \to -1 & \text{for } \ s < -t \text{ as } m \to +\infty, \\ \gamma_m(s) \to 1 & \text{for } s > t \text{ as } m \to +\infty, \text{ and} \\ \gamma_m(s) = 0 & \text{on } \ -t \leqslant s \leqslant t. \end{cases}$$

Then choosing $\varphi_1\gamma_m(u_k)$ as a test function in equation (2), we get

$$\begin{aligned} \sum_{i,j} \int_\Omega a_{ij} \frac{\partial u_k}{\partial x_i} \frac{(\partial \varphi_1 \gamma_m(u_k))}{\partial x_j} \,\mathrm{d}x \quad &+ \quad \int_\Omega V_{k-} u_k \varphi_1 \gamma_m(u_k)\,\mathrm{d}x \\ &= \quad \int_\Omega \Big(f_k + V_+ u_k \Big) \varphi_1 \gamma_m(u_k)\,\mathrm{d}x. \end{aligned} \tag{10}$$

Since $\varphi_1 \geqslant 0$ and $\gamma_m' \geqslant 0$, we have

$$\sum_{i,j} \int_\Omega a_{ij} \frac{\partial u_k}{\partial x_i} \frac{(\partial \varphi_1 \gamma_m(u_k))}{\partial x_j} \,\mathrm{d}x \geqslant \sum_{ij} \int_\Omega a_{ij} \frac{\partial u_k}{\partial x_i} \frac{\partial \varphi_1}{\partial x_j} \gamma_m(u_k)\,\mathrm{d}x. \tag{11}$$

Using the equation satisfied by φ_1, one has

$$\sum_{ij} \int_\Omega a_{ij} \frac{\partial u_k}{\partial x_i} \gamma_m(u_k) \frac{\partial \varphi_1}{\partial x_j} \,\mathrm{d}x = \lambda_1 \int_\Omega \,\mathrm{d}x \left(\int_{-t}^{u_k} \gamma_m(\sigma) d\sigma \right) \varphi_1 \,\mathrm{d}x. \tag{12}$$

From (10) to (12), letting $m \to +\infty$,

$$\begin{aligned} \int_{\{|u_k|>t\}} V_{k-} |u_k| \varphi_1 \,\mathrm{d}x \quad &+ \quad \lambda_1 \int_{\{|u_k|>t\}} \varphi_1 \,\mathrm{d}x \\ &\leqslant \quad c \int_{\{|u_k|\geqslant t\}} |f_k| \varphi_1(x)\,\mathrm{d}x + c \int_{\{|u_k|\geqslant t\}} |u_k| \varphi_1 \,\mathrm{d}x. \end{aligned} \tag{13}$$

Thus, we have that for all $t \geqslant 0$

$$\int_{\{|u_k|>t\}} V_{k-} |u_k| \delta(x)\,\mathrm{d}x \leqslant c \left[\int_{\{|u_k|\geqslant t\}} |f_k| \delta(x)\,\mathrm{d}x + \int_{\{|u_k|\geqslant t\}} |u_k| \delta(x)\,\mathrm{d}x \right]. \tag{14}$$

By usual arguments, we deduce that $(V_{k-} u_k \delta)$ satisfies the Dunford-Pettis theorem and the conclusion is proved. □

Continuation of the proof of Theorem 2.1 Since u_k satisfies the identity

$$\int_\Omega u_k L\varphi \,\mathrm{d}x - \int_\Omega V_{k-} u_k \varphi + \int_\Omega V_+ u_k \varphi = \int_\Omega f\varphi \,\mathrm{d}x \qquad \forall\, \varphi \in W^2(\Omega, |\cdot|_{N,1}) \cap H_c^1(\Omega), \tag{15}$$

we can pass to the limit in this equation, using the above convergence on u_k (since $\varphi \in W_0^{1,\infty}(\Omega)$) and in consequence u satisfies the desired identity.

The uniqueness of the solution is obtained by multiplying the equation by $p_+(s)$ (a regular approximation of $s \to \operatorname{sign}_+(s)$) which leads to

$$(\lambda_1 - \lambda) \int_\Omega u_+ \varphi_1 \,\mathrm{d}x \leqslant 0,$$

which implies that $u_+ \equiv 0$ (and analogously $u_- \equiv 0$).

It remains to show that if $f \in L^1(\Omega, \delta^\alpha)$ and $V_- \in L^{N,1}(\Omega, \delta^\alpha)$ then

$$u \in W_0^1\Big(\Omega, |\cdot|_{\frac{N}{N-1+\alpha}}\Big).$$

Since $V_- \in L^{N,1}(\Omega, \delta^\alpha)$, we deduce that $Vu \in L^1(\Omega, \delta^\alpha)$, once that $L^{N,1}(\Omega, \delta^\alpha)$ is the dual and associate space of $L^{N',\infty}(\Omega, \delta^\alpha)$. Indeed,

$$\int_\Omega |Vu|\delta^\alpha \,\mathrm{d}x \leqslant |V|_{L^{N,1}(\Omega,\delta^\alpha)}|u|_{L^{N',\infty}(\Omega,\delta^\alpha)} \leqslant c|u|_{L^{N',\infty}(\Omega)} \leqslant c|f|_{L^1(\Omega,\delta)}.$$

Since u satisfies

$$\int_\Omega uL\varphi \,\mathrm{d}x = \int_\Omega (f+Vu)\varphi \qquad \forall\, \varphi \in W^2(\Omega, |\cdot|_{N-1}) \cap H_0^1(\Omega),$$

we can apply Theorem 1.4, to get

$$|\nabla u|_{L^{\frac{N}{N-1+\alpha}}} \leqslant c_\alpha(\Omega, L)|f+Vu|_{L^1(\Omega,\delta^\alpha)}.$$

Then

$$|\nabla u|_{L^{\frac{N}{N-1+\alpha}}} \leqslant c_{1,\alpha}(\Omega, L)|f|_{L^1(\Omega,\delta^\alpha)}.$$

□

As consequence of the above Theorem 2.1, we can replace f by $\mu \in M^1(\Omega, \delta)$.

Theorem 2.4. *Let $V \in L^{N,1}(\Omega, \delta)$, with $V \leqslant \lambda < \lambda_1$. Then, for any $\mu \in M^1(\Omega, \delta)$, there exists a function $u \in L^{N',\infty}(\Omega) \cap W^{1,q}(\Omega, \delta)$, for any $1 \leqslant q < \dfrac{2N}{2N-1}$, satisfying $Vu \in L^1(\Omega, \delta)$, and*

$$\int_\Omega uL\varphi \,\mathrm{d}x - \int_\Omega Vu\varphi \,\mathrm{d}x = \int_\Omega \varphi d\mu \quad \forall\, \varphi \in W^2(\Omega, |\cdot|_{N,1}) \cap H_c^1(\Omega).$$

Moreover there exist some constants $c = c(\Omega, L, V, q) > 0$ such that

1. $|Vu|_{L^1(\Omega,\delta)} \leqslant c||\mu||_{*,\delta}$,
2. $|u|_{L^{N',\infty}(\Omega)} \leqslant c||\mu||_{*,\delta}$,
3. $\displaystyle\int_\Omega |\nabla u|^q \delta(x)\,\mathrm{d}x \leqslant c||\mu||_{*,\delta}^{\frac{q}{2}}\Big(1+||\mu||_{*,\delta}^{N'}\Big)^{1-\frac{q}{2}}$.

Finally, if $\mu \in M^1(\Omega, \delta^\alpha)$ for some $\alpha \in [0, 1[$ then

$$u \in W_0^1\Big(\Omega, |\cdot|_{\frac{N}{N-1+\alpha},\infty}\Big) = \Big\{v \in W_0^{1,1}(\Omega) : |\nabla v| \in L^{\frac{N}{N-1+\alpha},\infty}(\Omega)\Big\}$$

and

$$|\nabla u|_{L^{\frac{N}{N-1+\alpha},\infty}} \leqslant c||\mu||_{*,\delta^\alpha}.$$

Proof. Since $\mu \in M^1(\Omega, \delta)$, there exists $f_n \in C_c(\Omega)$ such that $(f_n)_n$ is contained in a bounded set of $L^1(\Omega, \delta)$ and $|f_n|_{L^1(\Omega,\delta)} \leqslant ||\mu||_{*,\delta}$ and $f_n \rightharpoonup \mu$ weakly in $M(\Omega)$. Applying Theorem 2.1, we have $u_n \in L^{N',\infty} \cap W^{1,q}(\Omega, \delta)$, $1 \leqslant q < \dfrac{2N}{2N-1}$ and

$$\int_\Omega u_n L\varphi \,\mathrm{d}x - \int_\Omega Vu_n\varphi \,\mathrm{d}x = \int_\Omega f_n\varphi \,\mathrm{d}x \qquad \forall\, \varphi \in W^2(\Omega, |\cdot|_{N,1}) \cap H_c^1(\Omega). \quad (16)$$

Moreover, we have the following uniform boundedness estimates : there exist some constants $c > 0$, and $c_q > 0$ such that

$$|u_n|_{L^{N',\infty}} \leqslant c, \quad (17)$$

$$|\nabla u_n|_{L^q(\Omega,\delta)} \leqslant c_q. \tag{18}$$

From the compactness Lemma 1.2, there is a function $u \in L^{N',\infty}(\Omega) \cap W^{1,q}(\Omega,\delta)$, satisfying the above estimates and a subsequence, still denoted by (u_n), such that

$$\begin{aligned} u_n(x) &\to u(x) \text{ a.e. } x, \\ u_n &\rightharpoonup u \text{ weakly in } W^{1,q}(\Omega,\delta), \text{ and in } L^{N',\infty}(\Omega), \text{ and} \\ u_n &\to u \text{ strongly in } L^{q,1}(\Omega) \qquad \forall\, q < N'. \end{aligned}$$

Since $V\varphi \in L^{N,1}(\Omega,\delta)$ and $L^{N',\infty}(\Omega) \subset L^{N',\infty}(\Omega,\delta)$ thus

$$\int_\Omega V\varphi u_n \,\mathrm{d}x \xrightarrow[n\to+\infty]{} \int_\Omega V\varphi u \,\mathrm{d}x.$$

Therefore, passing to the limit in (16), one has

$$\int_\Omega uL\varphi \,\mathrm{d}x - \int_\Omega Vu\varphi \,\mathrm{d}x =< \mu, \varphi > \qquad \forall \varphi \in W^2(\Omega, |\cdot|_{N,1}) \cap H^1_c(\Omega).$$

The remainder of the proof is similar to the Theorem 2.1. □

Remark 3. The fact that $u \in W^{1,1}_0(\Omega)$ where $\mu \in M^1(\Omega,\delta^\alpha)$ for some $\alpha \in [0,1]$ implies that the boundary trace is understood in the usual sense, so that

$$\begin{cases} Lu - Vu = \mu \text{ in } \mathcal{D}'(\Omega), \\ u \in W^{1,1}_0(\Omega), \text{ and} \\ Vu \in L^1(\Omega,\delta). \end{cases}$$

2.2. **The semilinear case.** In this section, we shall consider the case in which a perturbation term $g(x,u,\nabla u)$ is added to the linear operator $Lu - Vu$. Our main result deals with the case in which $f \in L^1(\Omega,\delta)$. We consider a non linear map satisfying :

H1) $g : \Omega \times I\!R \times I\!R^N \to I\!R$ is a Caratheodory function, i.e.

$$\begin{aligned} x &\to g(x,s,\xi) \text{ is measurable}, \forall\, s \in R,\ \forall\, \xi \in I\!R^N, \\ (s,\xi) &\to g(x,s,\xi) \text{ is continuous, for a.e. } x \in \Omega, \end{aligned}$$

H2) $sg(x,s,\xi) \geqslant 0,\ \forall\, \xi \in I\!R^N,$ for a.e. $x,$ and for all $s \in I\!R,$

H3) $\displaystyle\sup_{\{\xi \in I\!R^N,\ |s| \leqslant t\}} |g(\cdot,s,\xi)|\delta(\cdot) \in L^1_{loc}(\Omega),$ for all $t \geqslant 0.$

Theorem 2.5. *Under the assumptions of Theorem 2.1, and assumptions H1) to H3), there exists a function $u \in L^{N',\infty}(\Omega) \cap W^{1,q}(\Omega,\delta)$, $1 \leqslant q < \dfrac{2N}{2N-1}$, satisfying*

$$g(x,u,\nabla u) \in L^1(\Omega,\delta),\ Vu \in L^1(\Omega,\delta),$$

and

$$\int_\Omega uL\varphi \,\mathrm{d}x - \int_\Omega Vu\varphi + \int_\Omega \varphi g(x,u,\nabla u)\,\mathrm{d}x = \int_\Omega f\varphi \,\mathrm{d}x,\ \forall\, \varphi \in W^2(\Omega, |\cdot|_{N,1}) \cap H^1_0(\Omega).$$

Moreover, there exists some constants $c = c(\Omega, L, V, q) > 0$ such that

1. $|Vu|_{L^1(\Omega,\delta)} \leqslant c|f|_{L^1(\Omega,\delta)},$
2. $|u|_{L^{N',\infty}(\Omega)} \leqslant c|f|_{L^1(\Omega,\delta)},$
3. $\displaystyle\int_\Omega |\nabla u|^q \delta(x)\,\mathrm{d}x \leqslant c|f|^{\frac{q}{2}}_{L^1(\Omega,\delta)} \Big(1 + |f|^{N'}_{L^1(\Omega,\delta)}\Big)^{1-\frac{q}{2}},$

4. *if* $f \in L^1(\Omega, \delta^\alpha)$ *for some* $\alpha \in [0,1[$ *and* $V \in L^{N,1}(\Omega, \delta^\alpha)$ *then*

$$u \in W_0^1\left(\Omega, |\cdot|_{\frac{N}{N-1+\alpha},\infty}\right) = \left\{v \in W_0^{1,1}(\Omega) : |\nabla v| \in L^{\frac{N}{N-1+\alpha},\infty}(\Omega)\right\}$$

and

$$|\nabla u|_{L^{\frac{N}{N-1+\alpha},\infty}} \leqslant c|f|_{L^1(\Omega,\delta^\alpha)}.$$

Finally, if $g(x,s,\xi) = g(x,s)$ *(i.e.* g *is independent of* ξ*) and the function* $s \mapsto g(x,s)$ *is nondecreassing, we get the estimate*

$$|u - \widehat{u}|_{L^1(\Omega)} + |g(x,u) - g(x,\widehat{u})|_{L^1(\Omega,\delta)} \leqslant c|f - \widehat{f}|_{L^1(\Omega,\delta)}, \tag{19}$$

for u, $\widehat{u}$ *very weak solutions corresponding to the data* f *and* $\widehat{f}$ *respectively. In particular, for this type of function* $g(x,s)$, *we have the uniqueness of the very weak solution* u *of the problem.*

Proof. Consider $g_k = T_k(g)$ with T_k given in the proof of Theorem 2.1. Then, by well-known results, $\forall s,\ \xi$, and a.e. $x \in \Omega$, $\operatorname{sign}(s)T_k(g(x,s,\xi)) = |T_k(g)|(x,s,t)|$ and there exists $u_k \in H_0^1(\Omega) \cap H^2(\Omega)$ such that

$$Lu_k + V_{k-}u_k - V_+u_k + g_k(x,u_k,\nabla u_k) = T_k(f). \tag{20}$$

In order to prove inequality (3), we consider $\varphi_1 p_\varepsilon(u_k)$, where $p_\varepsilon(u_\varepsilon)$ is an appropriate regular approximation of the sign function $\operatorname{sign}(s)$. Then we deduce

$$\begin{aligned}(\lambda_1 - \lambda)\int_\Omega |u_k|(x)\varphi_1(x)\,\mathrm{d}x \quad &+ \quad \int_\Omega V_{k-}\varphi_1|u_k| + \int_\Omega \varphi_1\big|g_k(x,u_k,\nabla u_k)\big|\,\mathrm{d}x \\ &\leqslant \quad c\int_\Omega |f|\varphi_1\,\mathrm{d}x.\end{aligned}$$

Thus

$$\int_\Omega |u_k(x)|\delta(x)\,\mathrm{d}x + \int_\Omega V_{k-}|u_k|\delta(x)\,\mathrm{d}x + \int_\Omega \big|g_k(x,u_k,\nabla u_k)\big|\delta(x)\,\mathrm{d}x \leqslant c\int_\Omega |f|\delta(x)\,\mathrm{d}x. \tag{21}$$

As in the proof of Theorem 2.1 we conclude that (u_k) satisfies the same relations (5) and (6), moreover there exists a function $u \in W^{1,q}(\Omega,\delta) \cap L^{N',\infty}(\Omega)$ such that relations (7), (8), (9) hold and

$$V_{k-}u_k(x) \to V_-(x)u(x) \text{ a.e. } x.$$

Next we shall prove the strong convergence of a gradients subsequence :

Lemma 2.6.

$$\nabla u_k(x) \to \nabla u(x) \text{ a.e. } x,$$

for a subsequence still denoted by (u_k).

Proof of Lemma 2.6 We shall adapt to our framework some ideas already used in the literature ([7], [26], [16]). We consider

$$S_\varepsilon(\sigma) = \begin{cases} \sigma & \text{if } |\sigma| \leqslant \varepsilon, \\ \varepsilon & \text{if } \sigma > \varepsilon, \\ -\varepsilon & \text{if } \sigma < -\varepsilon. \end{cases}$$

For all $n \geqslant 1,\ j \geqslant 1$, we take $S_\varepsilon(u_k - u_n)\varphi_1$ as a test function, for the difference between the two equations satisfied by u_k and u_n. From the ellipticity condition we get

$$\begin{aligned}\alpha_0 \int_\Omega |\nabla S_\varepsilon(u_k-u_n)|^2 \varphi_1 \,\mathrm{d}x \quad &+ \quad \int_\Omega \sum_{i,j} a_{ij} \frac{\partial \varphi_1}{\partial x_i} \frac{\partial}{\partial x_j} \int_0^{u_k-u_n} S_\varepsilon(\sigma)\mathrm{d}\sigma \,\mathrm{d}x \\ &\leqslant \quad \varepsilon \int_\Omega \Big[V_{k-}|u_k| + \lambda |u_k| + |g_k| \\ &\qquad + V_{n-}|u_n| + \lambda|u_n| + |g_n| \Big] \varphi_1 \,\mathrm{d}x + 2\varepsilon \int_\Omega |f|\varphi_1 \,\mathrm{d}x \\ &\leqslant \quad c\varepsilon \int_\Omega |f|\delta(x)\,\mathrm{d}x. \end{aligned} \tag{22}$$

On the other hand, we have :

$$\int_\Omega \sum_{i,j} a_{ij} \frac{\partial}{\partial x_j}\left(\int_0^{u_k-u_n} S_\varepsilon(\sigma)d\sigma\right)\frac{\partial \varphi_1}{\partial x_i}\,\mathrm{d}x = \lambda_1 \int_\Omega \varphi_1 \left(\int_0^{u_k-u_n} S_\varepsilon(\sigma)d\sigma\right)\mathrm{d}x. \tag{23}$$

From (22) to (23), using relation (21), we have

$$\int_{\{|u_k-u_n|\leqslant\varepsilon\}} |\nabla(u_k-u_n)|^2\delta(x)\,\mathrm{d}x \leqslant c\varepsilon\int_\Omega |f|\delta(x)\,\mathrm{d}x. \tag{24}$$

On the other hand, by Hölder inequality, and the boundedness of $|\nabla u_k|$ in $L^q(\Omega,\delta)$, for any $1 < q < \dfrac{2N}{2N-1}$,

$$\int_{\{|u_k-u_n|>\varepsilon\}} |\nabla(u_k-u_n)|\delta(x)\,\mathrm{d}x \leqslant c_q \mathrm{meas}\Big\{x : |u_k(x)-u_n(x)| > \varepsilon\Big\}^{1-\frac{1}{q}}. \tag{25}$$

Thus, relations (24) and (25) lead to

$$\int_\Omega |\nabla(u_k-u_n)|\,\mathrm{d}x \leqslant c\varepsilon + c_q \mathrm{meas}\Big\{x\in\Omega : |u_k(x)-u_n(x)| > \varepsilon\Big\}^{1-\frac{1}{q}}. \tag{26}$$

Since $u_k(x) \to u(x)$ a.e. x as $k \to +\infty$, (26) implies that (∇u_k) is a Cauchy sequence in $L^1(\Omega,\delta)$. Therefore,

$$\nabla u_k \to \nabla u \text{ in } L^1(\Omega,\delta).$$

Since ∇u_k remains in a bounded set of $L^r(\Omega,\delta)$ for all $r \in \left[1, \dfrac{2N}{2N-1}\right[$ (see relation (6)), by an (Hölder) interpolation argument (or by the Vitali's convergence theorem), we deduce that the convergence of the gradient remains true in

$$L^q(\Omega,\delta), \text{ for any q }, 1 < q < \frac{2N}{2N-1}.$$

Subtracting a subsequence, still denoted by (u_k), we have

$$\nabla u_k(x) \to \nabla u(x) \text{ a.e. } x\in\Omega.$$

□

End of the proof of Theorem 2.5 In particular,

$$g_k(x,u_k,\nabla u_k) \to g(x,u(x),\nabla u(x)) \ \textit{a.e. in } \Omega.$$

Moreover, as for relation (14) in the proof as Lemma 2.3, choosing as a test function $\varphi_1\gamma_m(u_k)$

$$\begin{aligned}\int_{\{|u_k|>t\}} V_{k-}|u_k|\delta(x)\,\mathrm{d}x \quad &+ \quad \int_{\{|u_k|>t\}} g_k(x,u_k,\nabla u_k)|\delta(x)\,\mathrm{d}x \\ &\leqslant \quad c\left[\int_{\{|u_k|\geqslant t\}} |f_k|\delta(x)\,\mathrm{d}x + \int_{\{|u_k|\geqslant t\}} |u_k|\delta(x)\,\mathrm{d}x\right]. \quad (27)\end{aligned}$$

As in the proof of Lemma 2.3 we have

Lemma 2.7. *We have the following weak convergence in $L^1_{loc}(\Omega)$:*

$$V_{k-}u_k\delta \text{ converges to } V_-u\delta,$$

and

$$g_k(x,u_k,\nabla u_k)\delta \text{ converges to } g(x,u,\nabla u)\delta.$$

As a consequence of the above convergence, $\forall\,\varphi \in W^2(\Omega;|\cdot|_{N,1}) \cap H^1_c(\Omega)$, we have

$$\int_\Omega V_{k-}u_k\varphi\,\mathrm{d}x \underset{k}{\rightarrow} \int_\Omega V_-u\varphi\,\mathrm{d}x,$$

and

$$\int_\Omega g_k(x,u_k,\nabla u_k)\varphi\,\mathrm{d}x \rightarrow \int_\Omega g(x,u,\nabla u)\varphi\,\mathrm{d}x.$$

From the equation satisfied by u_k, we derive the conclusions of Theorem 2.5 and the regularity properties. When we assume $g(x,s,\xi) \equiv g(x,s)$ nondecreasing, the proof of the estimate (19) holds by multiplying the difference of the equations satisfied by u and $\widehat{u}$ by $p(u-\widehat{u})\varphi_1$, with $p(s)$ an appropriate regular approximation of $\operatorname{sign}(s)$ and by using the assumption that $V \leqslant \lambda < \lambda_1$.
This end the proof of Theorem 2.5 □

We can state an analogous result replacing f by $\mu \in M^1(\Omega,\delta)$ but replacing H3) by
$H3_\mu)$:

$$|g(x,s,\xi)| \leqslant a(x) + \beta|\xi|\delta(x) \text{ for a.e. } x,\ \forall\, s \in I\!R,\ \forall\,\xi \in I\!R^N,\ \beta \geqslant 0, \text{ for some } a \in L^1(\Omega,\delta).$$

Theorem 2.8. *Let $\mu \in M^1(\Omega,\delta)$. Under the assumptions of Theorem 2.5 and H1), H2), H3$_\mu$), there exists a function $u \in L^{N',\infty}(\Omega) \cap W^{1,q}(\Omega,\delta)$, $1 \leqslant q < \dfrac{2N}{2N-1}$, satisfying $Vu \in L^1(\Omega,\delta)$, $g(x,u,\nabla u) \in L^1(\Omega,\delta)$, and $\forall\,\varphi \in W^2(\Omega,|\cdot|_{N,1}) \cap H^1_c(\Omega)$,*

$$\int_\Omega uL\varphi\,\mathrm{d}x - \int_\Omega Vu\varphi\,\mathrm{d}x + \int_\Omega \varphi g(x,u,\nabla u)\,\mathrm{d}x = \int_\Omega \varphi d\mu.$$

Moreover there exists some constants $c = c(\Omega,L,V), q > 0$ such that

1. $|Vu|_{L^1(\Omega,\delta)} \leqslant c||\mu||_{*,\delta}$,
2. $|u|_{L^{N',\infty}(\Omega)} \leqslant c||\mu||_{*,\delta}$,
3. $\displaystyle\int_\Omega |\nabla u|^q\delta(x)\,\mathrm{d}x \leqslant c||\mu||^{\frac{q}{2}}_{*,\delta}\left(1+||\mu||^{N'}_{*,\delta}\right)^{1-\frac{q}{2}}$.
4. *If $\mu \in M^1(\Omega,\delta^\alpha)$ for some $\alpha \in [0,1[$ and $V \in L^{N,1}(\Omega,\delta^\alpha)$ then*

$$u \in W^1_0\left(\Omega,|\cdot|_{\frac{N}{N-1+\alpha},\infty}\right) = \left\{v \in W^{1,1}_0(\Omega) : |\nabla v| \in L^{\frac{N}{N-1+\alpha},\infty}(\Omega)\right\}$$

and

$$|\nabla u|_{L^{\frac{N}{N-1+\alpha},\infty}} \leqslant c||\mu||_{*,\delta^\alpha}.$$

The proof is analogous to the proof of in Theorems 2.5, 2.1 and 2.4. □

Remark 4. After the pioneering work [3], it is well known that when $\mu \in M^1(\Omega,\delta)$ and $g(x,s,\xi)$ is independent on ξ and x, and $V \equiv 0$, a sharper growth assumption on $g(s)$ (involving the dimension N of the spatial domain) must be impossed (instead of $H3_\mu$)) in order to get the existence of a very weak solution (see also Theorem 3.10 and other results in section 3 of [27]).

Remark 5. In many applications (see, e.g. [5]) the term involves a "natural growth" gradient of the form

$$g(x,u,\nabla u) = h(u)|\nabla u|^2. \tag{28}$$

Although assumption $H3_\mu$) fails for this special case, we can adapt the conclusion of Theorem 2.8 to this choice of g to arrive to a relaxed conclusion. In particular, we can show that

$$g_\varepsilon(x,u,\nabla u) = \frac{h(u)}{1+\varepsilon(h(u))}\cdot\frac{|\nabla u|^2}{1+\varepsilon|\nabla u|^2},$$

has a solution u_ε which converges in some suitable sense, to a solution of the semi-linear equation under the growth (28).

2.3. **Large solutions of semi-linear equations with $L^1(\Omega,\delta)$ as data.** In this section, we shall consider a function $g : \Omega\times I\!R\times I\!R^N \to I\!R$ which is a Caratheodory function and a superlinear nondecreasing function, with respect to the second variable that is $t \to g(x,t,\xi)$ is nondecreasing, satisfying $g(x,0,\xi) \geqslant 0$ and that there are three constants $m > 1$, c_1 and c_2 such that $c_1 > (m+1)c_2$, and

$$|g(x,t,\xi)| \geqslant c_1|t|^m - c_2 \ \forall\, t \in I\!R. \tag{29}$$

We also assume that $H3$) holds for g.

In all this subsection, we shall always assume that $f \in L^1(\Omega,\delta)$. We are interested in the so-called "large solution" for the associated semi-linear equation, that can be written formally as

$$(LSP)\begin{cases} -\Delta u + g(x,u,\nabla u) = f(x) & \text{in } \Omega,\\ u(x) \xrightarrow[x\to\partial\Omega]{} +\infty. \end{cases}$$

To give a rigorous sense to the problem (LSP), we shall introduce the following definition :

Definition 2.9. Let $M \geqslant 0$ and $f \in L^1(\Omega,\delta)$. A function $u \in L^1(\Omega)\cap W^{1,1}_{loc}(\Omega)$ is a very weak solution for the nonhomogenous Dirichlet problem

$$(DG)_{HM}\begin{cases} -\Delta u + g(x,u,\nabla u) = f & \text{in } \Omega,\\ u = M & \text{on } \partial\Omega, \end{cases}$$

if and only if $u - M = w$ satisfies the following (DG) problem, that is for all $\varphi \in W^2(\Omega,|\cdot|_{N,1})\cap H^1_0(\Omega)$, $w \in W^{1,1}(\Omega,\delta)$, $g(\cdot,w+M,\nabla w)\delta \in L^1(\Omega)$,

$$-\int_\Omega w\Delta\varphi\,\mathrm{d}x + \int_\Omega \varphi g(x,w+M,\nabla w)\,\mathrm{d}x = \int_\Omega f\varphi\,\mathrm{d}x.$$

Notice that this is equivalent to have

$$-\int_\Omega (u-M)\Delta\varphi\,\mathrm{d}x \quad + \quad \int_\Omega \varphi g(x,u,\nabla u)\,\mathrm{d}x$$
$$= \quad \int_\Omega f\varphi\,\mathrm{d}x,\ \ g(x,u,\nabla u)\delta \in L^1(\Omega),\ \ u \in W^{1,1}(\Omega,\delta).$$

Corollary 1. (of Theorem 2.5). *Under the above assumptions on g, for any $M \geqslant 0$ and $f \in L^1(\Omega,\delta)$, there exists a unique very weak solution $u_M \in L^1(\Omega)\cap W^{1,q}(\Omega,\delta)$ of $(DG)_{HM}$, $1 \leqslant q < \dfrac{2N}{2N-1}$. Moreover, $u_M \leqslant u_N$ if $M \leqslant N$.*

Proof. The existence and uniqueness of a very weak solution is a consequence of the above Theorem 2.5, since we have the following sign condition $\sigma\Big[g(x,\sigma+M,\xi) - g(x,M,\xi)\Big] \geqslant 0 \quad \forall \sigma \in I\!R$. Then, the function $w = u_M - M$ satisfies

$$-\int_\Omega w\Delta\varphi + \int_\Omega \varphi g(x,w+M,\nabla w)\,\mathrm{d}x = \int_\Omega \varphi f\,\mathrm{d}x \qquad \forall\,\varphi \in W^2(\Omega,|\cdot|_{N,1})\cap H_0^1(\Omega).$$

Moreover, for a.e x, $w(x) = \lim\limits_{k\to+\infty} w_k(x)$, $w_k \in W^2(\Omega,\cdot|_{N,1})\cap H_0^1(\Omega)$

$$-\Delta w_k + g(x,w_k+M,\nabla w_k) = T_k(f) = f_k \tag{30}$$

and

$$\delta g(x,w_k+M,\nabla w_k) \rightharpoonup \delta g(x,w+M,\nabla w) \text{ weakly in } L^1(\Omega).$$

To prove that $u_M \leqslant u_N$ if $M \leqslant N$, we consider also $v_k \in W^2(\Omega,\cdot|_{N,1})\cap H_0^1(\Omega)$

$$-\Delta v_k + g(x,v_k+N,\nabla v_k) = T_k(f) = f_k. \tag{31}$$

By a standard comparison argument, we have

$$w_k + M \leqslant v_k + N.$$

Letting k goes to infinity we get the result. □

Theorem 2.10. *Under the above assumptions on g and f, the sequence $(u_M)_{M\geqslant 0}$ remains in a bounded set of $W^{1,q}_{loc}(\Omega)$, for all $1 \leqslant q < \dfrac{2m}{m+1}$.*
Moreover, for a.e $x \in \Omega$ $\lim\limits_{M\to+\infty} u_M(x) = u(x)$ exists in $]-\infty,+\infty[$ (all the sequence converges), and $g(x,u,\nabla u) \in L^1_{loc}(\Omega)$, $u \in L^m_{loc}(\Omega)\cap W^{1,q}_{loc}(\Omega)$. Finally, u satisfies

$$-\int_\Omega u\Delta\varphi\,\mathrm{d}x + \int_\Omega g(x,u,\nabla u)\varphi\,\mathrm{d}x = \int_\Omega \varphi f\,\mathrm{d}x \qquad \forall\,\varphi \in C_c^2(\Omega).$$

Proof. Since the sequence $(u_M)_M$ is nondecreasing, there is a function u such that:

$$\lim_{M\to+\infty} u_M(x) = u(x) \text{ exists in }]-\infty,+\infty]$$

for almost all x (in fact all the sequence converges). Thus, we just need to show that $u(x)$ is almost everywhere finite. For this, we have to show the local regularity.

Let $x_0 \in \Omega$, $B(x_0,2R)$ be the ball centered at x_0 of radius $R > 0$ contained in Ω.

If g depends only on u, then the results can be derived from [8] (see also [12]). In the more general cases of $g(x,s,\xi)$, we can proceed otherwise. We start by assuring that for γ a fixed number satisfying $\dfrac{1}{m} < \gamma < 1$, there is a constant c

depending only on R, x_0, Ω, and a function $\theta \in C_c^\infty(B(x_0, 2R))$ such that $|\Delta\theta(x)| \leqslant c\theta(x)^\gamma, \quad \forall\, x \in \Omega$, and $\operatorname*{Min}_{B(x_0,\frac{R}{2})} \theta(x) > 0$. This was shown by many ways (apply, for instance Theorem 1.5 of [13]).

Consider also $w_k \in W^{2,p}(\Omega) \cap H_0^1(\Omega)$ solution of the equation (30) such that

$$\lim_{k\to+\infty} w_k(x) = u_M(x) - M.$$

We have the following estimates :

First consider

$$S_\varepsilon(\sigma) = \begin{cases} \operatorname{sign}(\sigma) & if\ |\sigma| \geqslant \varepsilon, \\ \dfrac{\sigma}{\varepsilon} & if\ |\sigma| \leqslant \varepsilon, \end{cases}$$

and choose $\theta S_\varepsilon(w_k + M)$ as a test function in equation (30), we deduce as before (letting $\varepsilon \to 0$)

$$\int_\Omega |g(x, w_k + M, \nabla w_k)|\theta \,\mathrm{d}x \leqslant \int_\Omega |f|\theta \,\mathrm{d}x + \int_\Omega |\Delta\theta||w_k + M| \,\mathrm{d}x. \tag{32}$$

Using Young's inequality, we have $\forall\, \eta > 0,\ \exists c_\eta(\theta) > 0$

$$\int_\Omega |\Delta\theta||v_k + M| \,\mathrm{d}x \leqslant c \int_\Omega \theta^\gamma |w_k + M| \,\mathrm{d}x \leqslant \eta \int_\Omega \theta |w_k + M|^m \,\mathrm{d}x + c_\eta(\theta). \tag{33}$$

Since $|g(x, w_k + M, \nabla w_k)| \geqslant c_1 |w_k + M|^m - c_2$ in Ω, we deduce from (33) and (32), that there exists a constant $c_0 > 0$ such that

$$\int_\Omega |w_k + M|^m \theta \,\mathrm{d}x \leqslant c_0 \left(\int_\Omega |f|\theta \,\mathrm{d}x + 1 \right). \tag{34}$$

Therefore, we have

$$\int_{B(x_0,\frac{R}{2})} |w_k + M|^m \,\mathrm{d}x \leqslant c(R, f), \qquad \forall\, k \geqslant 0. \tag{35}$$

From which we derive

$$\int_\omega |u_M|^m \,\mathrm{d}x \leqslant c(\omega, f), \qquad \forall\, M \geqslant 0 \tag{36}$$

for any open set ω relatively compact in Ω. Coming back the above relation (32), we deduce

$$\int_\Omega |g(x, u_M, \nabla u_M)|\theta \,\mathrm{d}x \leqslant \int_\Omega |f|\theta \,\mathrm{d}x + c \int_\Omega \theta^\gamma |u_M| \,\mathrm{d}x. \tag{37}$$

This implies, that

$$\int_\omega |g(x, u_M, \nabla u_M)| \,\mathrm{d}x \leqslant c(\omega, f) \quad \forall\, M \geqslant 0. \tag{38}$$

Considering the function

$$\phi_\eta(\sigma) = \int_0^\sigma \frac{dt}{(1 + t^2)^{\frac{1+\eta}{2}}},\ \eta > 0,$$

taking $\phi_k(w_k + M)$ as a test function we deduce that

$$\int_\Omega \frac{|\nabla(w_k + M)|^2}{(1 + (w_k + M)^2)^{\frac{1+\eta}{2}}} \theta \,\mathrm{d}x \leqslant c(\theta, f). \tag{39}$$

Thus

$$\int_{B(x_0,\frac{R}{2})} \frac{|\nabla(w_k+M)|^2}{(1+(w_k+M)^2)^{\frac{1+\eta}{2}}}\,\mathrm{d}x \leqslant c(R,f), \tag{40}$$

using (35) and (40), we have choosing correctly η

$$\int_{B(x_0,\frac{R}{2})} |\nabla(w_k+M)|^q\,\mathrm{d}x \leqslant c(q,R,f) \qquad \forall\, 1 \leqslant q < \frac{2m}{m+1}. \tag{41}$$

We conclude

$$\int_{\omega} |\nabla u_M|^q\,\mathrm{d}x \leqslant c(q,\omega,f) \qquad \forall M \geqslant 0,\ \forall \omega \subset\subset \Omega. \tag{42}$$

From relations (36) and (42), we conclude that u_M remains in $W^{1,q}_{loc}(\Omega)$ as $M \to +\infty$.

By standard compactness results, we have a subsequence $(u_M)_{M\geqslant 0}$, and a function $u \in W^{1,q}_{loc}(\Omega)$ such that $u_M(x) \xrightarrow[M\to+\infty]{} u(x)$ a.e and in $L^1_{loc}(\Omega)$ with relations (36) (38), (42) we have $u \in L^m_{loc}(\Omega) \cap W^{1,q}_{loc}(\Omega)$, $g(x,u,\nabla u) \in L^1_{loc}(\Omega)$. But, since u is unique all the sequence converges in the above spaces. As before, choosing as a test function $\theta\gamma_m(w_k+M)$ (γ_m as in the proof of Lemma 2.3), one can show that $g(x,u_M,\nabla u_M) \rightharpoonup g(x,u,\nabla u)$ weakly in $L^1(\omega)$ for any open set $\omega \subset\subset \Omega$.

We can pass to the limit in the equation

$$-\int_\Omega u_M \Delta\varphi\,\mathrm{d}x + \int_\Omega \varphi g(x,u_M,\nabla u_M)\,\mathrm{d}x = \int_\Omega f\varphi, \qquad \forall \varphi \in C^2_c(\Omega).$$

□

Definition 2.11. The function u obtained in Theorem 2.10 will be called the "minimal large solution of (LSP)".

Remark 6. If $f \in W^{1,\infty}_{loc}(\Omega)$ and satisfies some suitable geometric condition, then it can be shown in [14] (see also [27], [2] for $f=0$), that for any "other solution" v of (LSP), one has

$$u \leqslant v.$$

As in [14] an upper bound for u on all the domain Ω can be used here under additional assumptions. For instance, when $g(x,t,\xi) = |t|^{m-1}t$, for some $m>1$, and f satisfies additionally some growth conditions of the type

$$f(x)\delta(x)^p \to c \text{ as } \delta(x) \to 0,$$

for some $c>0$, for some $p > \dfrac{2m}{(m-1)}$ and if $\partial\Omega \in C^2$ then the minimal large solution is the unique large solution (see [14]). The above conditions can be considerably relaxed if we assume $f \equiv 0$.

Remark 7. The superlinear condition (28) on the dependence on u of the term $g(x,u,\nabla u)$ can be replaced by other conditions in which the superlinear growth is assumed on the ∇u-dependence and the mere monotonicity on the u-dependence (see [21] and [11]).

3. **Rearrangement comparison results.** Let us assume that $V \geqslant 0$ and that

$$\begin{cases} g(x,s) = q(x)\beta(s) \text{ with } \beta \text{ a continuous nondecreasing} \\ \text{real function with } \beta(0)=0 \text{ and } q \in L^1_{loc}(\Omega),\ q(x) \geqslant 0 \text{ a.e. } x \in \Omega. \end{cases} \tag{43}$$

In several applications, function $q(x)$ grows as a certain power (positive or negative) of the distance to the boundary $\delta(x)$ and then previous rearrangement comparison results in the literature (see [13] and its references) cannot be directly applied. The main goal of this section is to present here an extension of them in the framework of the space $L^1(\Omega, \delta)$.

Given $q \in L^1_{loc}(\Omega)$, $q(x) \geqslant 0$ *a.e.* $x \in \Omega$, we consider the radially symmetric problem

$$SP(\widetilde{\Omega} : \widetilde{q}\beta, F) \equiv \begin{cases} -\Delta U - \underset{\sim}{V}(|x|)U + \widetilde{q}(|x|)\beta(U) = F(|x|) \text{ in } \widetilde{\Omega}, \\ U = 0 \text{ on } \partial\widetilde{\Omega}. \end{cases}$$

Here we are assuming that $F \in L^1_{loc}(\widetilde{\Omega})$ is given function and that we have

$$F(x) \geqslant \underset{\sim}{f}(x) \geqslant 0 \text{ a.e. } x \in \widetilde{\Omega}, \tag{44}$$

where $\widetilde{\Omega} = B_R(0)$ is the N-dimensional ball centered at the origin and with measure $|\Omega|$. Notice that the assumption $F \in L^1(\widetilde{\Omega}, \delta_{\widetilde{\Omega}})$ and the condition that $\underset{\sim}{F} \in L^1(\widetilde{\Omega}, \delta_{\widetilde{\Omega}})$ implies that necessarely $F \in L^1(\widetilde{\Omega})$ (see Remark 2 of [15]) and so, the only relevant weighted integrability concerns the term $\beta(U^g)$, when U^g is the unique solution of $SP(\widetilde{\Omega} : \widetilde{q}\beta, F)$.

Theorem 3.1. *Assume that $V \geqslant 0$ satisfies $V \leqslant \lambda < \lambda_1$, that $g(x,u)$ satisfies (43) and assume (44) for some $F \in L^1(\widetilde{\Omega})$. Then, if u is the (unique) very weak solution given in Theorem 4, we have*

$$\int_{B_r(0)} \widetilde{q}(|x|)\beta(\underset{\sim}{u}(|x|))\delta_{\widetilde{\Omega}}(x)\,\mathrm{d}x \leqslant \int_{B_r(0)} \widetilde{q}(|x|)\beta(\underset{\sim}{U}^g(|x|))\delta_{\widetilde{\Omega}}(x)\,\mathrm{d}x \textit{ for any } r \in (0,R). \tag{45}$$

In particular, if $q(x) \equiv 1$ then

$$||\beta(u)||_{L^1(\Omega,\delta)} \leqslant ||\beta(U^g)||_{L^1(\widetilde{\Omega},\delta_{\widetilde{\Omega}})}. \tag{46}$$

Proof. As in the proof of Theorem 2.5, we approximate functions f, g, and F_n by u_n and U^g_n to the solutions of the associated problem $(SP)_n$ and $SP(\widetilde{\Omega} : (\widetilde{q}\beta)_n, F_n)$ respectively. In that case, by applying Theorem 1.26 in [13] we get that

$$\int_{B_r(0)} \widetilde{q}(|x|)\beta(\underset{\sim}{u_n}(|x|))\,\mathrm{d}x \leqslant \int_{B_r(0)} \widetilde{q}(|x|)\beta(\underset{\sim}{U_n}^g(|x|))\,\mathrm{d}x \text{ for any } r \in (0,R). \tag{47}$$

Note the absence of the weight $\delta_{\widetilde{\Omega}}$ in (47). But from the fact that functions

$$\beta(\underset{\sim}{u_n}(|x|)),\ \beta_n(\underset{\sim}{U_n}^g(|x|)) \text{ and } \delta_{\widetilde{\Omega}}(|x|)$$

are decreasing with the radii $|x|$ we get that (47) implies the same inequality with the weight $\delta_{\widetilde{\Omega}}$ for any $r \in (0,R]$,

$$\int_{B_r(0)} \widetilde{q}(|x|)\beta(\underset{\sim}{u_n}(|x|))\delta_{\widetilde{\Omega}}(x)\,\mathrm{d}x \leqslant \int_{B_r(0)} \widetilde{q}(|x|)\beta(\underset{\sim}{U_n}^g(|x|))\delta_{\widetilde{\Omega}}(x)\,\mathrm{d}x \text{ for any } r \in (0,R). \tag{48}$$

Since we have convergence in $L^1(\Omega,\delta)$ and $L^1(\widetilde{\Omega},\delta_{\widetilde{\Omega}})$ of both terms (by Theorems 2.5), we arrive to the inequality (45) by passing to the limit in (48). Estimate (46) follows from equiintegrability properties (see e.g. Theorem 1.25 in [13]) and the inequality $\delta_{\underset{\sim}{\Omega}}(x) \leqslant \delta_{\widetilde{\Omega}}(x)$ for any $x \in \widetilde{\Omega}$ (see [6]). □

Remark 8. In fact, estimate (46) can be extended to

$$\int_\Omega \Phi\Big(\beta(u)\Big)\delta_\Omega \,\mathrm{d}x \leqslant \int_{\widetilde{\Omega}} \Phi\Big(\beta(U^g)\Big)\delta_{\widetilde{\Omega}}\,\mathrm{d}x, \tag{49}$$

for any convex nondecreasing real function Φ. This follows from a general property due to Hardy, Littlewood and Polya (see, e.g. Lemma 1.33 in [13]). In particular we get that

$$||\beta(u)||_{L^p(\Omega,\delta)} \leqslant ||\beta(U^g)||_{L^p(\widetilde{\Omega},\delta_{\widetilde{\Omega}})} \tag{50}$$

for any $1 \leqslant p \leqslant +\infty$. The application to the estimate of the measure of the "dead core" in terms of the measure of the associated "dead core" on the ball $\widetilde{\Omega}$, typical of the case in which $\beta(u)$ is not Lipschitz continuous near $u = 0$, follows the same argument than Theorem 1.28 in [13] where f and F are assumed to be in $L^\infty(\Omega)$ and $L^\infty(\widetilde{\Omega})$ respectively.

Remark 9. Many variants to the conclusions of Theorem 3.1 are possible. For instance, instead of the condition $q(x) \equiv 1$ we can assume that

$$\underline{C}(\delta_{\underset{\sim}{\Omega}}(x))^{-\alpha} \leqslant \widetilde{q}(x) \leqslant \overline{C}(\delta_{\widetilde{\Omega}}(x))^{-\gamma} \text{ a.e. } x \in \widetilde{\Omega}, \tag{51}$$

for some $0 < \underline{C} \leqslant \overline{C}$ and α, $\gamma \in (0,1)$. Then estimate (50) must be replaced by

$$\int_\Omega |\beta(u)|\delta_\Omega^{(1-\alpha)}(x)\,\mathrm{d}x \leqslant \frac{\overline{C}}{\underline{C}}\int_{\widetilde{\Omega}} |\beta(U^g)|\delta_{\widetilde{\Omega}}^{(1-\gamma)}(x)\,\mathrm{d}x. \tag{52}$$

We point out that semilinear problems with $q(x)$ a non-constant function have been attracted the attention of many specialists (see, e.g. the long list of references in the survey [20]). When we apply this arguments to dead core type problems the "balance condition on the data and Ω" (see [13] Section 1.2) must be modified with respect to the case of $q(x) \equiv 1$ (see e.g. [19]). The case $\beta(u) = u$ arises in the study of Schrödinger type problems (see, e.g. [1] and its references).

Remark 10. Many other generalizations are possible: for instance, the conclusions of Theorem 9 can be usually adapted to the case in which $f(x)$ is changing sign or when $\Omega = I\!R^N$. Some applications to the study of the associated parabolic problems will be developed in some other paper by the authors. We also point out that the comparison (45) can be made sharper with respect to the weights $\delta_{\underset{\sim}{\Omega}}$ and $\delta_{\widetilde{\Omega}}$ when f and F are more general than functions L^1_{loc} but we approximate them by a sequence of functions f_n and F_n in L^1_{loc}. For instance, it can be shown that if

$$g(x,u) = |u|^{p-1}u \text{ for some } p > 0, \tag{53}$$

and $f \in L^1_{loc}(\Omega)$ such that $f\delta^\alpha \in L^1(\Omega)$ for some $\alpha \geqslant \dfrac{(p+1)}{(p-1)}$, then it is possible to define the notion of very weak solution $u \in L^1(\Omega)$ of the associated (SP) and that this solution (being the unique solution) satisfies that $|u|^p\delta^\alpha \in L^1(\Omega)$ (notice that $L^1(\Omega,\delta) \subset L^1(\Omega,\delta^\alpha)$ for $\alpha \geqslant 1$). The details will be given in a future work.

We end this section with a comparison result which this time involves the symmetric increasing rearrangement of functions (see the definition in Section 5). To consider several further applications, it is useful to shift the formulation of the semilinear problem (SP) to the case of non homogeneous boundary conditions :

$$(SP:M)\begin{cases}-\Delta u - V(x)u + g(x,u) = f(x) \text{ in } \Omega,\\ u = M \text{ on } \partial\Omega,\end{cases}$$

with M a given positive constant. For a very weak solution of the semilinear problem $(SP:M)$ we mean the following:

$$u \in L^1(\Omega),\ g(x,u)\delta \in L^1(\Omega),\ V(x)u\delta \in L^1(\Omega), \tag{54}$$

and

$$\begin{aligned}\int_\Omega u(-\Delta\varphi)\,\mathrm{d}x - \int_\Omega Vu\varphi\,\mathrm{d}x + \int_\Omega g(x,u)\varphi\,\mathrm{d}x\\ = \int_\Omega f(x)\varphi\,\mathrm{d}x - \int_{\partial\Omega} M\frac{\partial\varphi}{\partial n}\,\mathrm{d}x,\end{aligned}$$

for any $\varphi \in W^{2,\infty}(\Omega)\cap W_0^{1,\infty}(\Omega)$. We have

Theorem 3.2. *Given $M \geqslant 0$, let $g(x,u)$ satisfying $H1)$ and $H2)$ and such that*

$$\begin{aligned}&g(x,s) \geqslant g(x,M) \text{ for a.e. } x\in\Omega,\ \text{if } s>M,\\ &g(x,s) \leqslant g(x,M) \text{ for a.e. } x\in\Omega,\ \text{if } s<M.\end{aligned} \tag{55}$$

Let $V \in L^1_{loc}(\Omega)$ satisfying $V \leqslant \lambda < \lambda_1$ and assume that

$$f(\cdot) + V(\cdot)M - g(\cdot,M) \in L^1(\Omega,\delta). \tag{56}$$

Then, there exists, at least, one solution u of $(SP:M)$ and this solution is unique if the function $s \mapsto g(x,s)$ is nondecreasing. Moreover, let us assume that there exists $F \in L^1(\widetilde{\Omega},\delta_{\widetilde{\Omega}})$ and $G_M \in L^1(\widetilde{\Omega},\delta_{\widetilde{\Omega}})$ such that

$$0 \leqslant F(x) \leqslant \widetilde{f}(x) \text{ a.e. } x\in\widetilde{\Omega}, \tag{57}$$

$$G_M(x) \geqslant -\underset{\sim}{\Phi(\cdot)}(x) \text{ a. e. } x\in\widetilde{\Omega} \qquad \text{with } \Phi(\cdot) = -g(\cdot,u(\cdot)) - V_-(\cdot)u(\cdot). \tag{58}$$

Then

$$W_M(x) \leqslant \widetilde{u}(x) \text{ for a.e. } x\in\widetilde{\Omega}, \tag{59}$$

where $W_M(x)$ is the (unique) very weak solution of the problem

$$LP(\widetilde{\Omega},G_M,F,M)\begin{cases}-\Delta W - \widetilde{V}_+(|x|)W = F(x) - G_M(x) \text{ in } \widetilde{\Omega},\\ W = M \text{ on } \partial\widetilde{\Omega}.\end{cases}$$

Proof. The existence (and the uniqueness part) of a very weak solution is a direct consequence of Theorem 2.5 since function $w := M-u$ satisfies that

$$\begin{cases}-\Delta w - V_+(x)w + \widehat{g}(x,w) = \widehat{f}(x) \text{ in } \Omega,\\ w = 0 \text{ on } \partial\Omega,\end{cases} \tag{60}$$

with

$$\widehat{g}(x,w) := g(x,M) - g(x,M-w) - V_-(x)(M-w),$$

and

$$\widehat{f}(x) = -f(x) - V_+(x)M + g(x,M).$$

Note that function $\widehat{g}(x,w)$ satisfies condition H2) thanks to the assumption (55). As in the proof of Theorems 9 and Theorem 1.25 in [13], the comparison (59) follows from the comparison $\widehat{W}_M(x) \leqslant \widetilde{u}(x)$ with $\widehat{W}_M$ solution of $LP(\widetilde{\Omega}, G_M, \widetilde{f}, M)$ since, by the usual comparison and (57), we get that $\widehat{W}_M \geqslant W_M$. By considering the approximation of problems $(SP : M)_n$ and $LP(\widetilde{\Omega}, G_M, (\widetilde{f}_n), M)$ associated to $f_n := \min(n, f)$, the comparison with the increasing rearrangement

$$\widehat{W}_n(x) \leqslant \widetilde{(u_n)}(x) \text{ a.e. } x \in \widetilde{\Omega}, \tag{61}$$

is an easy adaptation of the proof of the comparison (60) for the case of the decreasing rearrangement. The only modification (the occurrence of function G_M in the problem $LP(\widetilde{\Omega}, G_M, F, M)$) comes from the fact that for any $t \in I\!R$, due to (58), we have that

$$-\int_{\{u\leqslant t\}} (g(x,u) + V_-(x)u)\,\mathrm{d}x = -\int_{\{u\leqslant t\}} (g(x,u) + V_-(x)u)\chi_{\{u\leqslant t\}}\,\mathrm{d}x$$

$$\geqslant \int_{\widetilde{\Omega}} \underset{\sim}{\Phi(x)}\,\chi_{\{\widetilde{u}\leqslant t\}}\,\mathrm{d}x \geqslant -\int_{\{\widetilde{u}\leqslant t\}} G_M(x)\,\mathrm{d}x$$

with $\Phi(x) = (-g(x,u) - V_-u)$, the rest of details follows as in the Theorem 1.26 of [13].

Finally, the convergence arguments of the proof of Theorem 2.5 also applies in this case and so $\widetilde{W}_n \to \widetilde{W}$ and $u_n \to u$ in $L^1(\widetilde{\Omega})$ and $L^1(\Omega)$ respectively, which ends the proof. □

Remark 11. Note that if $M = 0$ condition (55) coincides with the assumption H2). Moreover if, for instance $g(x,u) = q(x)\beta(x)$ with $q \in L^1(\Omega, \delta)$, $q \geqslant 0$ and β is a continuous non decreasing function then condition (55) holds. We also point out that if $M > 0$ assumption (55) does not imply condition H2).

Remark 12. When $0 \leqslant u(x) \leqslant M$ a.e. $x \in \Omega$ and g is increasing in u, condition (58) is satisfied if we assume that

$$G_M(x) \geqslant \underset{\sim}{g(x,M)} + \underset{\sim}{V_-(x)}M\,. \tag{62}$$

Note that, in contrast to Theorem 3.1, now, in general, $G_M \neq 0$ if $M > 0$ (this fact seems to be unadvertised before in literature).

As in the case of comparison in terms of the decreasing rearrangement (recall Theorem 3.2), it is possible to improve the comparison (59) by replacing the role of function $G_M(x)$ by a nonlinear function of W_M. As before, we shall state it only for the special case of $g(x,s) = q(x)\beta(s)$ satisfying (56) and for $V \geqslant 0$. Let $M > 0$ and let $F,\ q,\ V \in L^1(\widetilde{\Omega}, \delta_{\widetilde{\Omega}})$, $V \geqslant 0$. Let $U_M^q(x)$ be the (unique) very weak solution of the problem

$$SM(\widetilde{\Omega}, \underset{\sim}{q}\,\beta, F, M) \equiv \begin{cases} -\Delta U - \widetilde{V}(|x|)U + \underset{\sim}{q}(|x|)\beta(U) = F(x) \text{ in } \widetilde{\Omega}, \\ u = M \text{ on } \delta\widetilde{\Omega}. \end{cases}$$

Theorem 3.3. *Assume $V \geqslant 0$ satisfying $V \leqslant \lambda < \lambda_1$. Then*

$$\int_{\widetilde{B_r}(0)} \underset{\sim}{q}(|x|)\beta(\widetilde{u}(|x|))\delta_{\widetilde{\Omega}}(x)\,\mathrm{d}x \geqslant \int_{\widetilde{B_r}(0)} \underset{\sim}{q}(|x|)\beta(\widetilde{U}_M^q(|x|))\delta_{\widetilde{\Omega}}(x)\,\mathrm{d}x \textit{ for any } r \in (0,R). \tag{63}$$

In particular, if $q(x) \equiv 1$ then

$$\operatorname*{ess\,inf}_{\widetilde{\Omega}} U_M^g \leqslant \operatorname*{ess\,inf}_{\Omega} u. \tag{64}$$

Proof. The mass comparison (63) is obtained as in the proofs of Theorem 3.2 and Theorem 3.1 (note that now we are taking $G_M(x) = \underset{\sim}{q}(|x|)\beta(\widetilde{u}(x))$). The comparison (64) uses the second order differential problem satisfied by $\widehat{U}_M^q(|x|)$, solution of $SM(\widetilde{\Omega}, \underset{\sim}{q}\,\beta, \widetilde{f}, M)$ (for which $\widetilde{u}$ is a subsolution). This was done in Theorem 3.2 of [2] for the associated H^1-approximations of $\widetilde{U}_M^q$ and u respectively (see also Theorem 1.28 of [13] for an alternative proof). Passing to the limit, when $n \to +\infty$, we get the conclusion. □

Remark 13. In the case in which $(SP : M)$ arises in chemical engineering the inequality (63) means that the "chemical effectiveness" of a ball $(\widetilde{\Omega})$ has the lower value among the chemical effectiveness for any other domain Ω having the same measure than $\widetilde{\Omega}$ (see [2] and [13] Remark 1.17).

Corollary 2. (of Theorem 2.10).
Under the same assumptions of Theorem 2.10, let u and W_∞^g be the minimal large solutions of (LSP) corresponding to Ω and f, and $\widetilde{\Omega}$ and $\widetilde{f}$ respectively. Then

$$\operatorname*{ess\,inf}_{\widetilde{\Omega}} W_\infty^g \leqslant \operatorname*{ess\,inf}_{\Omega} u.$$

Proof. Let $u = \lim\limits_{K\to+\infty} u_K$ and let $W_\infty^g = \lim\limits_{K\to+\infty} W_K^g$, where W_K^g is the solution of the problem of Corollary 1 of Theorem 2.5, but replacing Ω and f by $\widetilde{\Omega}$ and $\widetilde{f}$. Then, with a suitable choice of the truncated approximation $f_{g(K)}$ and by the comparison obtained in Theorem 3.3 we get that, for any $K > 0$,

$$\int_0^s (W_K^g)^*(\sigma)d\sigma \leqslant \int_0^s (u_K)^*(t)dt \quad \forall\, s > 0,$$

from which we have

$$\operatorname*{ess\,inf}_{\widetilde{\Omega}} W_K^g \leqslant \operatorname*{ess\,inf}_{\Omega} u_K,$$

and so the conclusion holds by passing to the limit when $K \to +\infty$. □

Remark 14. The conclusion of Corollary 2 was proved previously in [23] for the special case of $f \equiv 0$ and $g(x,u) = e^u$. It was shown there that, in this special case

$$log\left(\frac{8\pi}{|\Omega|}\right) \leqslant \operatorname*{ess\,inf}_{\widetilde{\Omega}} W_\infty^g \qquad if\ N = 2.$$

Acknowledgments. The research of the first author was partially supported by the project ref. MTM2008-06208 of the DGISPI (Spain),the Research Group MOMAT (Ref. 910480) supported by UCM and Reseach Training Networks of the European Commission (Grant Agreement Number 238702) Fronts and Interfaces in Science and Technology.

This work was completed when the second author was invited by Universidad Politécnica de Madrid, Departamento de Matemática Aplicada a la Edificación. He would like to thank Professor J.F. Padial and the other members of the department for their kindness.

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Received August 2009; revised December 2009.

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